A TEXT-BOOK OF GENERAL PHYSICS

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A TEXT-BOOK OF GENERAL PHYSICS

AN INTRODUCTION TO MECHANICS PROPERTIES OF MATTER AND WAVE MOTION

BY

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PREFACE

UNDER the title of General Physics this book deals with the principles of Newtonian mechanics and their application to the study of solids, liquids, and gases. It is intended for those taking Physics at Advanced and Scholarship levels, and its outlook is experimental rather than mathematical.

Many students say that they find physics difficult because of the mathematics. This is usually self-delusion, and is certainly an inversion of the truth—which is that they find the mathematical aspects difficult because of the physics. Mastery of the physical principles of mechanics is something to be won slowly as experience develops; the mistake is in supposing that it all ought to come easily, and in being disappointed when one finds it impossible to start mechanics at the point where Galileo and Newton left off. My chief aims in this book have been to encourage a proper attitude of mind towards the subject, to guide the student's progress along the right lines, and to lead him to think carefully about the essential physics of a problem before attempting its solution.

The chapters on the laws of motion and gravitation contain simplified versions of Newton's own treatment, based on Prof. Cajori's translation of the *Principia Mathematica*. Even when expressed in the language and notation of the present time, Newton's sayings are not always easy to follow; but this is surely because he is using concepts based on his own experiments, and the reader has so much more limited a background for their interpretation. The laws of motion may be hard to understand thoroughly, but those who make the effort are well rewarded; and to them mechanics is revealed as a limited number of principles founded on experiment—instead of a vast array of *ad hoc* rules and formulae for the solution of artificial problems.

The law of conservation of energy is taken as the starting-point from which to discuss impact, friction, and simple harmonic motion. It is true that the principle of work for an ideal machine, and one or two expressions useful only under ideal conditions, can be deduced

from the laws of motion. But it seemed better to treat the principle of work as established with an experimental status equal to that of the laws of motion themselves; and indeed this is the only possible way of treating the law of conservation of energy in a physics course.

Most of the theory needed for the study of sound at this stage is included in the two chapters on simple harmonic motion and wave motion. Wave motion is best treated as mechanics, and as a topic of general importance on its own, rather than as incidental to experimental work in sound or physical optics, where a very rudimentary theory often suffices for the task immediately in hand. I have tried to explain clearly the important detailed features of wave propagation, treating the compression wave and the displacement wave separately in some places, and also dealt fully with the establishment of stationary wave systems. The selection of other topics for discussion here was determined by their general interest from the point of view of acoustics, and by their relevance to the material of earlier chapters.

My thanks are due to Sir Lawrence Bragg for the bubble-raft picture reproduced on p. 209, and to Prof. E. N. da C. Andrade for allowing me to reproduce the dust pattern shown on p. 350; to the Cambridge University Press for allowing me to use quotations from Cajori's translation of the *Principia Mathematica*, and from Whittaker's *Analytical Dynamics*; to Messrs. Blackie and Son, Ltd., for the use of illustrations from Alexander Wood's *Text-book of Acoustics*; and to Messrs. Macmillan for permission to use the mathematical tables by Frank Castle, included at the end of the book. I must also acknowledge my indebtedness to R. W. Pohl's *Physical Principles of Mechanics and Acoustics* (Blackie and Son, Ltd.) for an admirably clear discussion of circular motion on which my own treatment is inevitably based, and to G. F. C. Searle's *Experimental Elasticity* and *Experimental Physics* (Cambridge University Press) as sources of reference on elasticity and surface tension.

Permission to use copyright examination questions was very kindly given by the Joint Matriculation Board (*N.*); the Oxford and Cambridge Schools Examination Board (*O. & C.*); the Oxford Delegacy for Local Examinations (*O.*); and the representatives of the Colleges at Oxford (*O.S.*) and Cambridge (*C.S.*) whose Open

Scholarship papers have been used. These questions have been selected for their relevance and variety rather than for intrinsic difficulty, and while most of them are from Higher School Certificate and Scholarship examinations, a certain number of School Certificate questions have been used where they seemed appropriate. Partial solutions are in some cases given with the answers at the end of the book; while every care has been taken in checking the answers I cannot suppose that they are all faultless, and I shall be grateful to receive any necessary corrections.

Finally, it is a great pleasure to acknowledge the generous assistance of Mr. A. J. V. Gale in seeing the book through the press, and to thank him most sincerely for the encouragement he has given me at all stages of the work.

G. R. NOAKES

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CHAPTER 1

FUNDAMENTAL DEFINITIONS

Units and Measurements

Fundamental units. To measure a quantity, its magnitude is compared with that of a standard of the same kind. This standard is called the **unit** of that particular quantity. If U be the magnitude of the unit, M the magnitude of the quantity, then its measure N is M/U, or

$$M = NU.$$

The measure of a quantity is only a *number*; the magnitude is not completely stated unless the unit is also stated, for the value of N for a given M depends on the size of U. If U_1 and U_2 be two units of different sizes, and N_1 and N_2 the measures of a given quantity of magnitude M in the two units, then $M = N_1U_1$, and $M = N_2U_2$, so $N_1U_1 = N_2U_2$, or

$$N_1/N_2 = U_2/U_1.$$

For each quantity we wish to measure, a unit must be defined. This is, of course, obvious; but there are two ways of setting about it. One is to choose for each quantity a convenient and reproducible unit, having no regard to the units of any other quantities; for example, the pressure exerted by a column of mercury of stated height is a useful and commonly used unit of pressure. The other is to consider carefully the physical relationship with other quantities, and to use this to define the unit. In this case there is then no question of free choice of units; and we construct a **consistent system** of units. For example, if pressure is defined as force per unit area, the unit of pressure must be that given by unit force acting over unit area. The unit of pressure is thus derived automatically from those of the " simpler " quantities force and area. Thus, once we decide to construct a consistent system, only a limited number of standard units will be required. For all purely mechanical purposes, three fundamental units suffice; these are the units of **mass**, of **length**, and of **time**.

The unit of length, the **centimetre**, is defined as 1/100 of the distance, at the melting-point of ice, between two marks on a bar of platinum-iridium known as the **International Prototype Metre.**

The unit of mass, the **gram** (sometimes spelled **gramme**) is defined as 1/1000 of the mass of the **International Prototype Kilogram**, a cylinder of platinum-iridium.

The unit of time, the **second**, is defined as 1/86,164·1 of a sidereal day, which is the period of rotation of the earth with respect to the fixed stars. It is often defined as 1/86,400 of a mean solar day, which is the average value of the interval between two successive returns of the sun to the meridian.

The consistent system based on these three units is called the **centimetre-gram-second (C.G.S.) system.**

Other systems in use, based consistently on units of mass, length, and time, are the **metre-kilogram-second (M.K.S.) system**, and the British system based on the **pound**, the **foot**, and the **second**, the pound and the foot being defined with reference to metal standards.

Objection may be raised to the selection of mass as one of the three fundamental units, since it is very difficult to frame a definition of mass itself. Newton's name for mass, "quantity of matter", as measured by the product of volume and density, is usually considered imperfect as a definition. If we say that mass is a form of energy, or is the origin of a gravitational field, it is impossible to explain these statements in the opening chapter of an elementary text-book. And a good grasp of simple mechanics is required before the idea of mass as inertia, or passive resistance to acceleration by a force, can really be appreciated. Yet everyone will agree that by the *mass of a body* he understands something that he assumes to be an unchanging property *of that body*, unaltered and unalterable by any practicable physical change. This assumption implies that the preservation of a standard is relatively easy, which commends mass as a fundamental unit.

Derived units. The physical relation, usually in the form of a *defining equation*, connecting a quantity either directly or indirectly with the quantities mass, length and time, must be stated before the unit on the C.G.S. system can be derived. In some cases, the statement of the equation here must precede its explanation in the text;

this treatment is essential to avoid a diffuse discussion of units scattered throughout the book.

Unit of area. The area of a rectangular surface is given by the equation

$$area = length \times breadth.$$

The C.G.S. unit of area is that of a rectangle the length and breadth of which are each one centimetre; that is, a square of side 1 cm. This is **one square centimetre** (written 1 sq. cm., or 1 $cm.^2$).

Unit of volume. The volume of a rectangular block is given by the equation

$$volume = length \times breadth \times depth.$$

The unit of volume is thus that of a block, length, breadth, and depth of which are each one centimetre, that is, a cube of side 1 cm. This is **one cubic centimetre** (written 1 c.c., or 1 $cm.^3$).

The litre, the volume of 1 kilogram of pure water at its temperature of maximum density (4° C.), is a derived unit based on the kilogram, but is not a C.G.S. unit. It is the unit of volume on the *metric system*, which is not the same thing. It is now established that one litre is equal to 1000·028 $cm.^3$. One millilitre (ml.) is the accepted and correct *metric unit* for the graduation of chemical glassware; it is not another name for cubic centimetre; though, of course, for most purposes the litre is taken as 1000 c.c.

Unit of density. Density is defined by the equation

$$density = \frac{mass}{volume}.$$

As the unit of mass is the gram and the unit of volume the cubic centimetre, the unit of density is **one gram per** $cm.^3$, or 1 gm. $cm.^{-3}$ on the C.G.S. system.

Unit of speed and velocity. The speed of a moving body is its rate of change of position with time. The term velocity is used when the direction is also specified; that is, velocity is speed *in a given direction*. A body has unit speed when it traverses unit distance at constant speed in unit time; thus on the C.G.S. system, the unit is a speed of **one centimetre per second** (or 1 cm. $sec.^{-1}$), while the M.K.S. unit is one metre per second.

Unit of acceleration. Acceleration is rate of change of velocity. Unit acceleration occurs when unit velocity change takes place, at constant rate, in unit time. The C.G.S. unit is thus an acceleration of **one-centimetre-per-second change per second**, or 1 cm. sec.$^{-2}$, and the M.K.S. unit 1 m. sec.$^{-2}$.

Unit of force. Force—that is, any push or pull or weight—is a fundamental conception based on our own bodily sensations. My hand experiences a downward pull when I hold a stone in it; I assume that the same downward force is experienced by anything else supporting the same stone. If I stretch a spring between my hands, I can feel them being pulled together; and I assume that the spring stretched between any two supports will likewise tend to pull them together. When a motor-car in which I am sitting accelerates, I feel a more or less gentle push from behind; and I assume that any body that is accelerating must experience a force in the direction of the acceleration. Force, like mass, really defies definition in simpler terms because its effects are continually part of our experience.

A good case might thus be presented for choosing the unit of force to be one of the fundamental units, and in the *engineering* systems (ft., lb.-wt., sec., and metre, kgm.-wt., sec.) this is done. But it is difficult to preserve the unit in a convenient way, for we regard a force as acting in a definite direction along a definite line (possibly for a limited period rather than existing permanently); and nobody has ever seen or constructed a force, for we observe the effect produced by a force and not the force itself. Also, dynamical considerations show that we can derive the unit of force from those of mass, length, and time—all of which are satisfactorily established in reproducible form. The position really is, that, although it is hard to explain precisely what we understand by mass, it is easy to make and keep a mass which can be labelled as a "standard"; and although the idea of force is clear enough, we cannot readily preserve a standard force.

Two possible defining equations exist, by means of which the unit of force might be derived. The first of these, and that in fact employed, is based on the dynamical principles known as Newton's Laws of Motion (p. 67). From the Second Law of Motion, it can

be stated that the acceleration of a body, its mass, and the force producing the acceleration are related by the expression

$$force \propto mass \times acceleration.$$

Accepting it now, we can see that it does give a meaning to the mass of a body ; for the greater the mass of a body, the smaller the acceleration it experiences under a given force. Hence the *mass of a body is something permanently possessed by that body*, which can be measured by its inertia, or passive resistance to acceleration by a force. The idea of mass in this connection really depends on the more fundamental idea of force ; but convenience dictates that the unit of force shall be based on the more accessible unit of mass.

If a force F produces acceleration a in a body of mass m, then whatever units are used, the expression

$$F \propto ma$$

is true. This can be written

$$F = kma,$$

where k is a number the size of which depends on the units chosen.

Let unit force be that which gives unit acceleration to unit mass, then

$$1 = k \times 1 \times 1, \quad \text{so} \quad k = 1 ;$$

thus the defining equation for the unit of force is

$$F = ma.$$

The C.G.S. unit of force is that force which gives a mass of 1 gm. an acceleration of 1 cm. sec.$^{-2}$. This unit is named the **dyne**. The M.K.S. unit of force which gives a mass of 1 kgm. an acceleration of 1 metre sec.$^{-2}$ is named the **newton**, and is 10^5 dynes.

It is found that the force experienced by a body of mass m due to its own weight (which is the vertical force it experiences due to the earth's pull upon it) gives it a downward acceleration of about 981 cm. sec.$^{-2}$ in Britain. The value of this acceleration varies with the latitude, being about 983 cm. sec.$^{-2}$ at the poles and about 979 cm. sec.$^{-2}$ at the equator, and its variation from place to place is also slightly affected by other factors ; in general, when we wish to speak of the acceleration of a freely falling body without specifying the exact numerical value at the place concerned, the symbol g is used.

Thus, the weight of a mass of 1 gm. acting alone on that mass gives it an acceleration of g cm. sec.$^{-2}$; as 1 dyne gives a mass of 1 gm. an acceleration of 1 cm. sec.$^{-2}$, the weight of 1 gm. is a force of g dynes. Similarly, the weight of a mass of m gm. is a force of mg dynes.

Weights provide very convenient means of applying and measuring forces, and the *weight of the mass of one gram*, known as 1 gram-weight (written 1 gm. wt.), is a useful practical unit of force for measurements restricted to one given place. As g varies from place to place, the value of the gram-weight is not constant ; the gram-weight is called a *gravitational unit*, whereas the dyne is an *absolute unit*, since it is derived from fundamental units.

The British system units are the *poundal*, the absolute unit of force, which gives a mass of 1 lb. an acceleration of 1 ft. sec.$^{-2}$; and the pound-weight (lb. wt.), which is the pull of the earth on a mass of 1 lb. 1 lb. wt. $=g$ poundals, where g is approximately 32·2 ft. sec.$^{-2}$.

The other equation which might have been used to derive the unit of force is given by Newton's Law of Universal Gravitation, which states that the attractive force F between uniform spherical masses m_1 and m_2, the centres of which are separated by a distance d, is given by

$$F \propto \frac{m_1 m_2}{d^2}.$$

It would thus appear possible to define unit force as that experienced between two such masses, each of 1 gm., separated by a distance of 1 cm. This force is in actual fact less than one ten-millionth of a dyne, and there would be little point in selecting a unit so small as to be unmeasurable. If, however, we take the constant of proportionality as unity, consider m_1 as 1 gm., m_2 as 6×10^{27} gm. (the mass of the earth) and d as about $6{\cdot}4 \times 10^8$ cm. (the radius of the earth), the gravitational force acting on a 1 gm. mass is about $1{\cdot}47 \times 10^{10}$ such units, which are thus equivalent to 1 gram-weight. We can thus think of the gram-weight as being about $1{\cdot}47 \times 10^{10}$ " natural " gravitational units of force.

Unit of pressure. Pressure is defined as force per unit area, the force supposed acting at right angles to the area. Unit pressure is thus that experienced when unit force (1 dyne) acts over unit area (1 cm.^{2}) and is expressed in dynes per square centimetre (1 dyne cm.$^{-2}$).

Unit of work. When a force moves the body (or the point on a body) to which it is applied, it is said to do work. Work is measured by the product of the force and the distance moved in the direction of the force ; so the defining equation is

$$work = force \times distance.$$

Unit work is done when unit force traverses unit distance in its own direction. The C.G.S. unit of work, called the **erg**, is the work done when a force of 1 dyne moves its point of application 1 cm. in its own direction.

The M.K.S. unit of work, called the **joule**, is the work done when a force of 1 newton moves its point of application 1 metre in its own direction ; 1 joule is 10^7 ergs.

Other units of work commonly used are : the **kilowatt-hour** (kWh.) : 1 kWh. = 3,600,000 joules = $3{\cdot}6 \times 10^{13}$ ergs ; the **electron-volt** (eV.) : 1 eV. = $1{\cdot}6 \times 10^{-12}$ ergs ; and the British unit of work, the **foot-pound**, which is the work done when a force of one pound-weight moves its point of application one foot in its own direction.

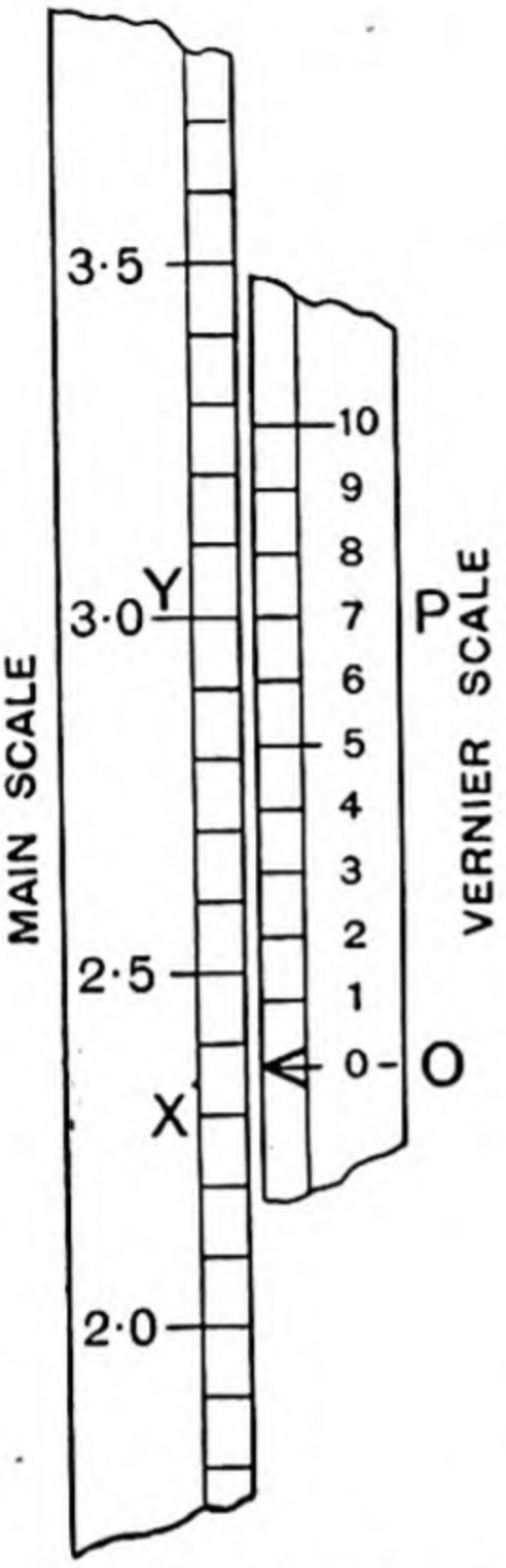
Unit of energy. The energy of a body, or system of bodies, is the capacity for doing work that the body or system possesses. Various forms of energy, and their transformations, are discussed later ; some are not commonly measured by mechanical means, but all forms of energy can if required be measured in work units.

Unit of power. Power is the rate of doing work, and the C.G.S. unit of power is thus a rate of working of 1 erg per second. This is a small unit, and the usual practical unit of power is the watt. **One watt** is a rate of working of one joule per second and is thus the M.K.S. unit of power. A larger unit, the **kilowatt**, is a rate of working of 1,000 joules per second.

The British system unit of power is the **horsepower** (H.P.), which is a rate of working of 550 ft. lb. per second. 1 H.P. is nearly equal to 746 watts.

Measurements

Measurement of length. The proper application of a scale graduated in centimetres and millimetres enables one to measure lengths certainly to the nearest half-millimetre, with a proportional



Scale graduated in centimetres and millimetres, with vernier reading to 0·1 mm. The setting indicates 2·37 cm.

error of less than one part in a hundred if the length to be measured exceeds 5 cm. Estimation to one-tenth of the smallest graduation on any kind of scale should always be attempted; its satisfactory execution depends on practice and skill.

Fractions of a scale division can be read with certainty using a **vernier**. The simplest type, to read to one-tenth of a scale division (that is, to 1/10 mm. on the usual type of scale), consists of a small sliding scale 9 mm. long, divided into 10 equal parts. This vernier scale slides by the side of the main scale, and the vernier zero, *O*, is the reference mark which is made to coincide with one extremity of the length to be measured. Each vernier division is 0·9 mm. long, and so is 0·1 mm. shorter than a millimetre; two vernier divisions are 0·2 mm. shorter than 2 mm., three vernier divisions are 0·3 mm. shorter than 3 mm., and so on.

To read the vernier, first note the position of the vernier zero, *O*, on the main scale, reading to the nearest millimetre *on the low side*. In the diagram, the reading of the division *X*, which is 2·3 cm., is taken, although *O* is clearly nearer the 2·4 cm. mark. Then look along the vernier and note the number of the vernier division which coincides with some scale division (no matter which); this number gives the number of tenths of a millimetre to be added to the main scale reading. In the figure, the seventh vernier division, *P*, coincides with the scale division *Y*; *OP* is 0·7 mm. shorter than the 7 mm. of *XY*, so the gap *XO* is 0·7 mm., or 0·07 cm., and the true position of *O* on the *main* scale is (2·3 + 0·07) cm., or 2·37 cm.

The general rule in dealing with instruments equipped with verniers (which are rarely graduated in this simple way) is to take nothing for granted. Examine the scales carefully and warily, and

ignore all legends until you are quite certain that you understand the graduations of both. Then find the difference between one vernier division and one scale division ; this is the " least count " of the vernier—in the example above it was 0·1 mm., and the vernier could be read to 0·1 mm. Then, when the vernier is in use, the number of the vernier division coinciding with some scale division (no matter which) gives the number of " least counts " to be added to the main scale reading which has been taken to the nearest scale division on the low side. It is best to count the divisions, record the main-scale and vernier readings separately, and to proceed rather like a small child counting on his fingers, until you have had considerable practice with any particular instrument.

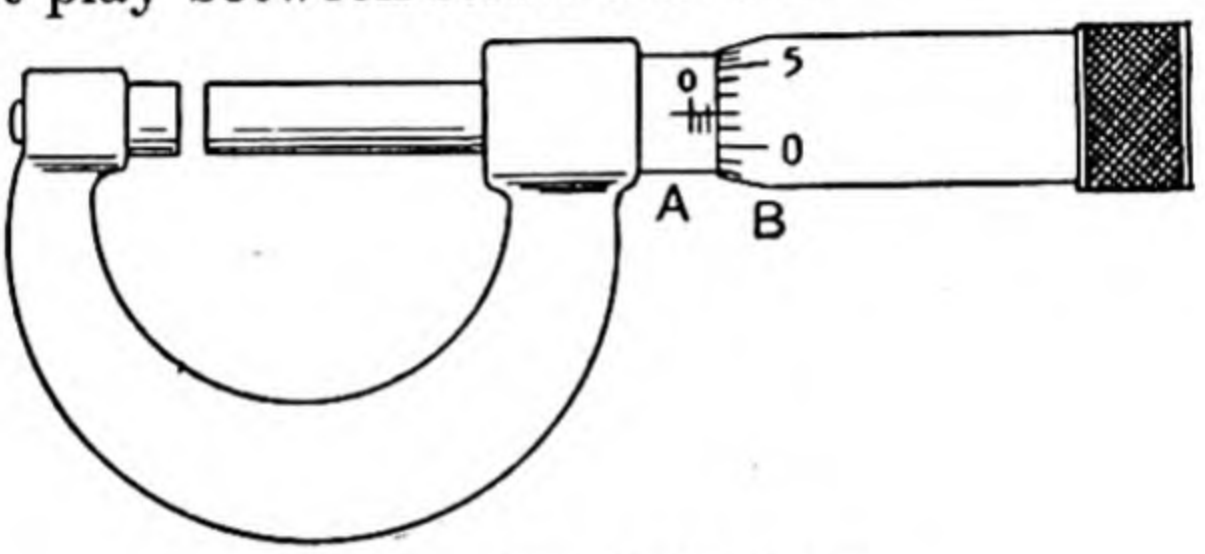
The vernier fitted to the standard type of Fortin barometer is a good example on which to master the general principles. The main scale is divided into twentieths of an inch ; the vernier is 24 scale divisions long, and bears 25 divisions. Each vernier division is 24/25 of a scale division long, and the least count is $(1 - 24/25) = 1/25$ of a scale division, which is $1/25 \times 1/20 = 1/500$ inch. As in all cases, after the setting has been made, the position of the vernier zero on the main scale is read to the nearest 1/20 inch on the low side ; looking along the vernier, the number of the vernier division coinciding with some scale division (no matter which) is noted. This gives the number of 1/500ths of an inch to be added to the main-scale reading.

For measuring short lengths, of the order of 1 cm. or less, the **micrometer screw** is usually employed. It simply depends on the principle that, as one turn of a screw sends it forward by one pitch (the distance between successive turns), then a small fraction of a turn sends it forward by the same small fraction of one pitch.

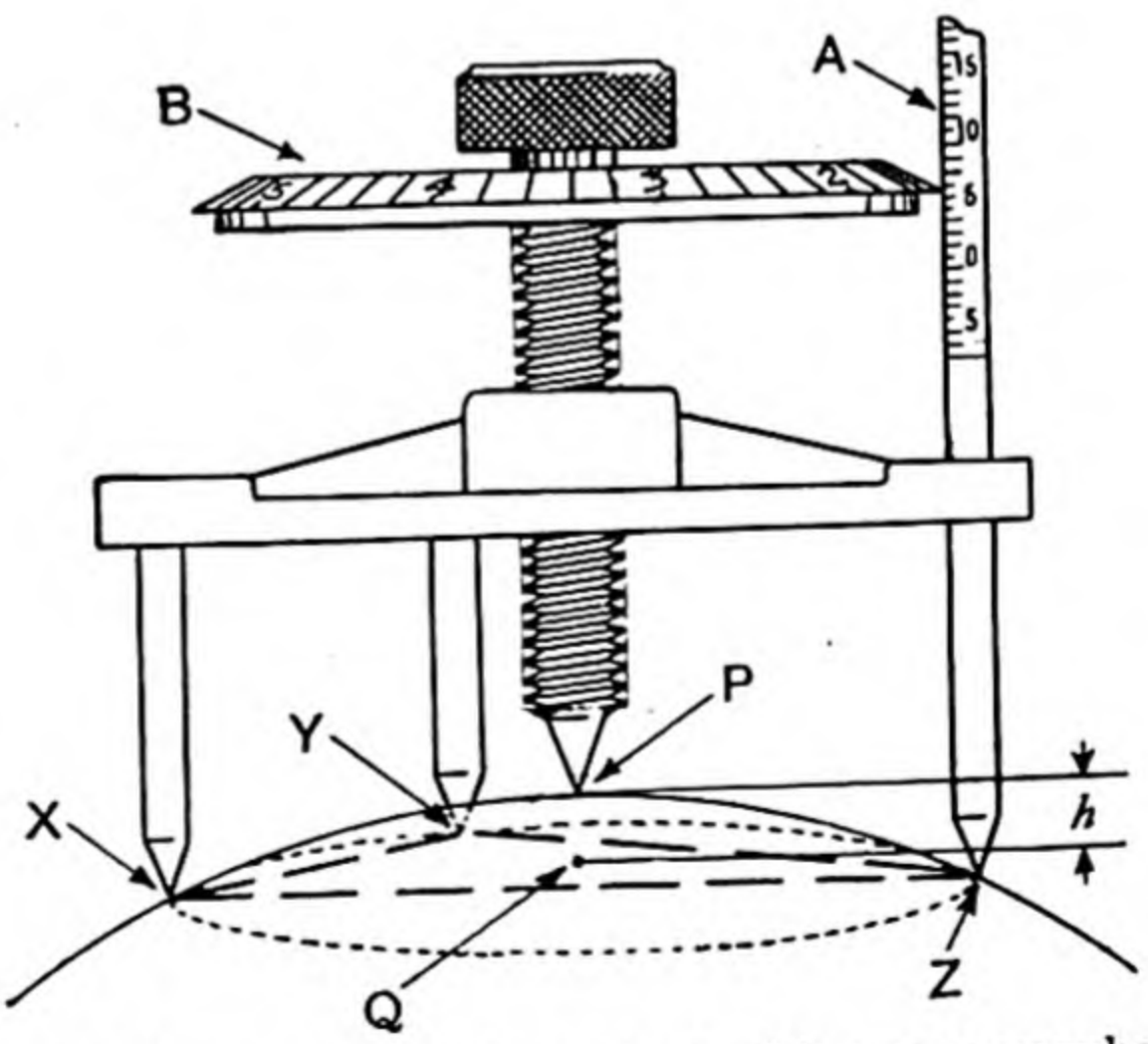
In the micrometer screw gauge used for measuring the diameters of wires and other narrow objects, an accurately cut screw, usually of $\frac{1}{2}$ mm. pitch, turns in a fixed nut on which a scale *A* records whole turns. (Sometimes the scale *A* records whole *millimetres* only, though the screw has a $\frac{1}{2}$ mm. pitch ; in this case care is needed to decide whether the required reading is on the first or second half-millimetre.) The screw head bears a scale *B* graduated to read fractions of a turn ; usually there are 50 divisions on *B*, each division indicating 1/50 of $\frac{1}{2}$ mm., or 1/100 mm. The reference marks are the circular edge of scale *B*, and the horizontal base-line of scale *A*.

The position on scale *A* of the edge of *B* gives the reading to the nearest whole turn ; the position on scale *B* of the base-line of scale *A* gives the number of 1/100 mm. to be added to this.

Several precautions are needed in use. The friction clutch should always be used, so that the same pressure is exerted every time by the screw. Before taking a reading, the jaws should be closed, and the zero reading noted ; if this reading is positive, its value is *subtracted* from the observation ; if it is negative (as the screw has overshot the mark) its value is *added* to the observation. Care should be taken never to ease the screw back after screwing up for a reading ; if it is believed to be screwed up too tightly, open it and screw up afresh ; the reason for this is to avoid error due to back-lash, or slight play between screw and nut.



Micrometer screw gauge.



Spherometer, with legs and screw in contact with a convex spherical surface.

In the **spherometer**, a micrometer screw is used to find the radius of curvature of a spherical surface. The screw is mounted as shown ; the nut is supported on three legs the points of which, X, Y, Z, form the vertices of an equilateral triangle with sides of measured length L. The zero reading is taken with X, Y, Z, and P all in contact with an accurately plane surface. If the instrument is then placed on a spherical surface and the micrometer screw adjusted until point P and X, Y and Z are all in contact with the surface, then it can be shown by geometry that

$$r = \frac{L^2}{6h} + \frac{h}{2},$$

where r is the radius of the sphere of which the surface is part, and h is the distance of P above or below the plane XYZ.

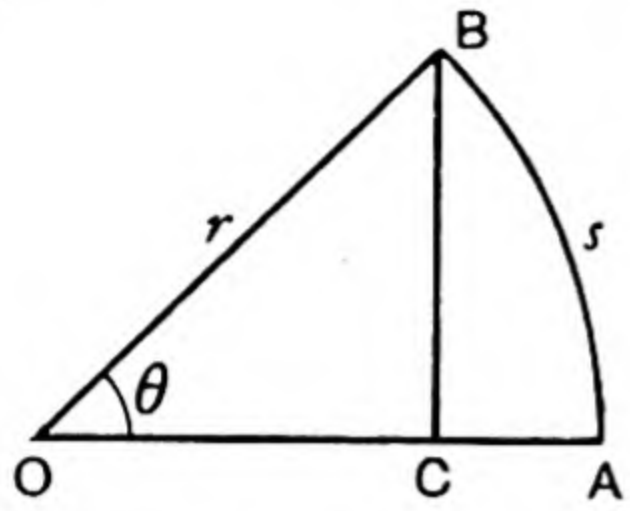
Mensuration formulae : The following formulae, not already given, should be known. They are used for calculating areas and volumes from linear measurements :

1. **Triangle :** $area = \frac{1}{2}(base \times perpendicular\ height)$.
2. **Trapezium :** $area = \frac{1}{2}(sum\ of\ parallel\ sides \times perpendicular\ height)$.
3. **Circle**, of radius r cm. :
   circumference : $2\pi r$ cm.
   area of the surface : πr^2 sq. cm.
4. **Cylinder**, of radius r cm., vertical height h cm. :
   area of the curved surface : $2\pi rh$ sq. cm
   volume : $\pi r^2 h$ c.c.
5. **Sphere**, of radius r cm. :
   surface area : $4\pi r^2$ sq. cm.
   volume : $\frac{4}{3}\pi r^3$ c.c.

Measurement of angles. Angles are commonly measured either in sexagesimal measure, when the angle turned through by a radius OA in describing a complete circle (or four right angles) is 360 degrees (°), each degree being divided into 60 minutes (′) and each minute into 60 seconds (″) ; or in circular measure, when the complete revolution of four right angles is 2π radians (c).

So $$360° \equiv 2\pi^c,$$

and $$1 \text{ radian} = \frac{360}{2\pi} = 57° \text{ approximately.}$$



Measurement of angles.

An arc of a circle of length s subtends at the centre of the circle an angle $\theta = s/r$ radians. In the diagram, θ^c = arc AB/BO. The trigonometrical ratios for θ are:

$$\sin\theta = BC/BO\ ;$$
$$\cos\theta = CO/BO\ ;$$
$$\tan\theta = BC/OC.$$

If θ is small, BC differs little from AB, and CO is nearly equal to BO; so, *for small values of* θ,

$$\sin\theta \text{ is nearly equal to } \theta^c,$$
$$\tan\theta \text{ is nearly equal to } \theta^c,$$
and $$\cos\theta \text{ is nearly equal to } 1.$$

It can easily be verified from tables that for angles less than 6° (or about $0{\cdot}1^c$), the discrepancies are less than one part in a thousand, and for angles up to $0{\cdot}2^c$ are still less than 1 per cent.

Solid angles, as at the vertex of a cone, are measured by extending the circular-measure system into three dimensions. An area S of the surface of a sphere of radius r subtends a solid angle S/r^2 at the centre of the sphere. The unit, called the *stereradian*, is the solid angle which unit area of a sphere of unit radius subtends at the centre of the sphere. The complete surface of a sphere subtends a solid angle of 4π stereradians at the centre.

Dimensions

Physical quantities can be divided into two classes: (*a*) those the measures of which depend on the magnitude of the fundamental units, and are different in different unit systems; and (*b*) those the measures of which do not depend on the system of units in use. The former are said to be *dimensional quantities*, or to possess dimensions; the latter are *dimensionless quantities*.

Dimensional quantities. These are in principle measured by direct or indirect application of the fundamental units, and their measures are expressed in derived units. The powers to which the fundamental units appear in the derived units are called the *dimensions* of the quantity, and are written in the following way:

Area. As unit area is *unit length* × *unit length*, the dimensions of area are L^2.

Volume. As unit volume is *unit length* × *unit length* × *unit length*, the dimensions of volume are L^3.

Velocity. As unit velocity is *unit length*/*unit time*, the dimensions of velocity are L/T, or LT^{-1}.

Similarly, the dimensions of other quantities can be written down :

Acceleration has dimensions LT^{-2},

Force (= *mass* × *acceleration*) has dimensions MLT^{-2} ,

Pressure (= *force*/*area*) has dimensions MLT^{-2}/L^2 or $ML^{-1}T^{-2}$,

Work (= *force* × *distance*) has dimensions

$$MLT^{-2} \times L, \text{ or } ML^2T^{-2},$$

Power (= *rate of working*) has dimensions ML^2T^{-2}/T, or ML^2T^{-3}.

Dimensionless quantities. Examples of such quantities are **angles** and their **trigonometrical ratios** (expressible as a ratio between two lengths, and thus quite independent of the unit of length) ; **specific gravity** (a ratio between two densities, which is not only independent of the units in which density is being measured, but is actually usually measured without any direct density determinations at all) ; the **efficiency** of a machine (the ratio between two quantities of work, which will always be the same whatever the units in which the work is measured). In each ratio, of course, the same unit must be used in measuring the two quantities being compared. All pure numbers, logarithms, and indices are dimensionless.

The full usefulness of this classification will appear later, and whenever a new quantity is introduced, its dimensions will be considered.

Weighing

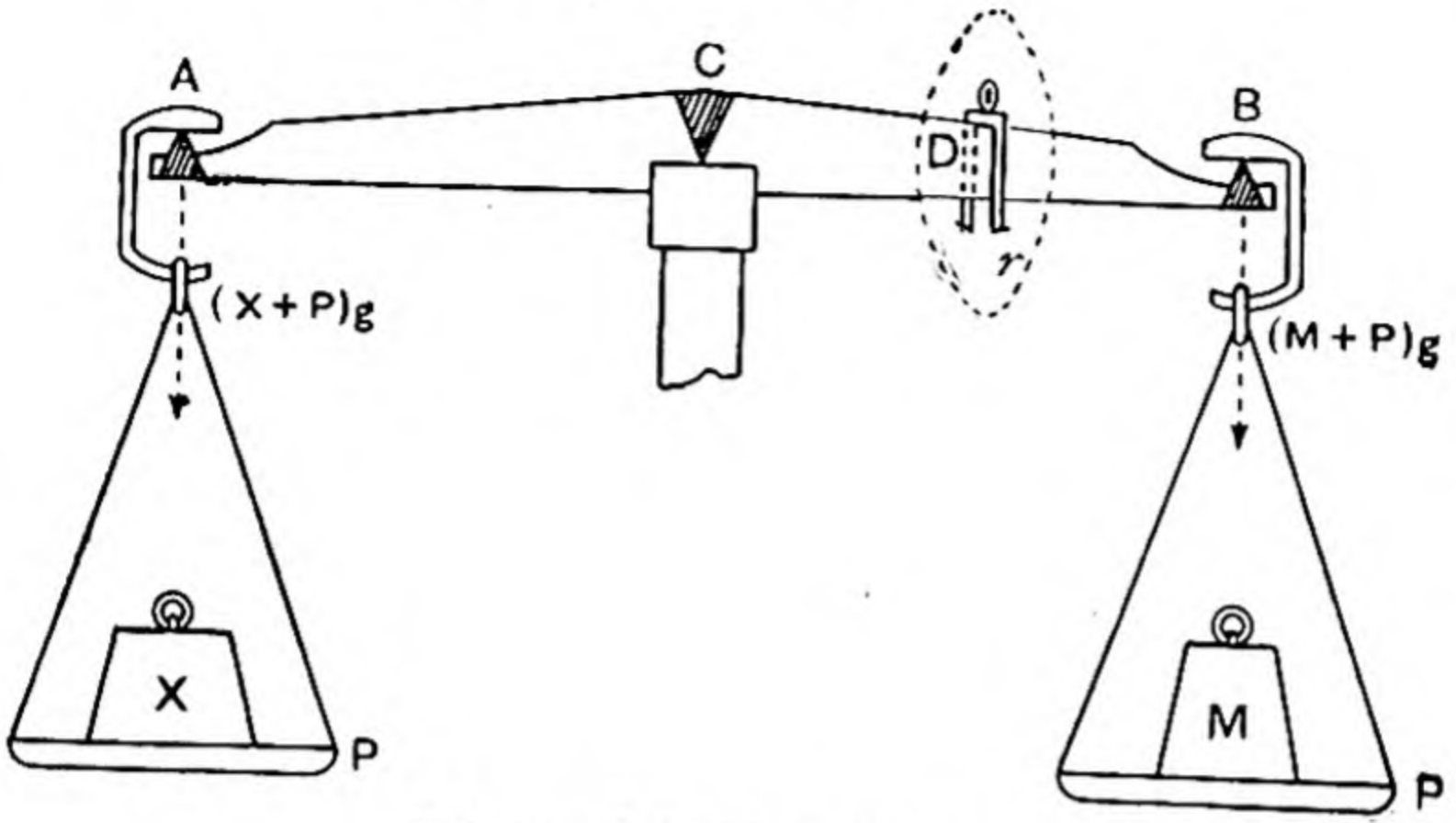
The balance. The mass of a body is something usually regarded as a permanent property of that body. We can speak of it as " the quantity of matter possessed by the body ", which (although it is not a perfect definition) emphasises this idea.

The weight of a body is a *force*—the vertical pull of the earth experienced by the body; it is something acting on the body, apparently from without, and is not an essential property of the body. This weight is different for a given body at different places on the earth's surface—about 0·3 per cent. greater at the poles than at the equator. Transporting a body above the earth's surface to a height of 4,000 miles—that is, doubling its distance from the centre of the earth—would reduce the *weight* to one quarter its value at the earth's surface; at the distance of the moon from the earth, the *weight* would be about 1/3600 of its value at the earth's surface; but in every case the *mass* of the body would be the same.

The laws of dynamics and of gravitation state, in effect, that, at a fixed place on the earth's surface, bodies *possessing* equal masses *experience* equal weights, and that the weights experienced by different bodies are proportional to the masses they possess. The simplest and most convenient way of determining the mass possessed by a body is to compare the weight it experiences at a given place with the weight experienced at the same place by a body possessing unit mass, or a known multiple or fraction of unit mass. This process is called **weighing**.

This is not the only way in which dynamical principles can be applied to compare masses. If this point is realised now, there should be no confusion between *mass* (the property being measured) and *weight* (the particular effect that happens to be selected to do the measurement with). For example, the mass of a gun can be compared with that of a shot by observing the muzzle velocity of the shot and the velocity of recoil of the gun (p. 82); the mass of the sun can be compared with that of the earth if we know the length of the year and the month, and the radii of the orbits of the earth and the moon (p. 375); in neither of these cases is weight involved at all.

To emphasise still further the point we are trying to make, consider an analogy from electricity, namely the potentiometer method of comparing resistances. Here, Ohm's Law shows that, when carrying the same current, wires *possessing* equal resistances *experience* equal potential drops across them, and that bodies possessing different resistances experience potential drops proportional to those resistances. Now, nobody ever supposes that the potential drops *are* the resistances, nor has it ever been suggested that the



Scheme of the beam balance.

name " potentiometer " might imply this ; the reason is, of course, that Ohm's Law is fully understood and explained before the experiment is attempted. In comparing masses by weighing, the issue is prejudiced by the impracticability of explaining fully the dynamical laws concerned before the balance is first encountered, and by the use of the term " weight " to mean " mass " exclusively in all branches of science but physics.

The common beam balance (see diagram) has a light beam carrying three knife-edges A, B, C ; pans of equal mass P are supported from A and B, and the beam is pivoted on a plane at C ; the lengths AC and BC are equal ; the beam carries a light pointer (not shown), the end of which moves over a scale. The body to be weighed, of unknown mass X, is placed in one pan ; standard masses M, made up of suitable multiples of the unit mass, are placed in the other pan.

Thus, acting downwards on the beam at A is a force of $(X+P)g$ dynes ; and acting downwards at B a force of $(M+P)g$ dynes.

The beam rests horizontally when the moments (p. 42) of these two forces about the pivot C are equal, that is, when

$$(X+P)g \,.\, AC = (M+P)g \,.\, BC$$

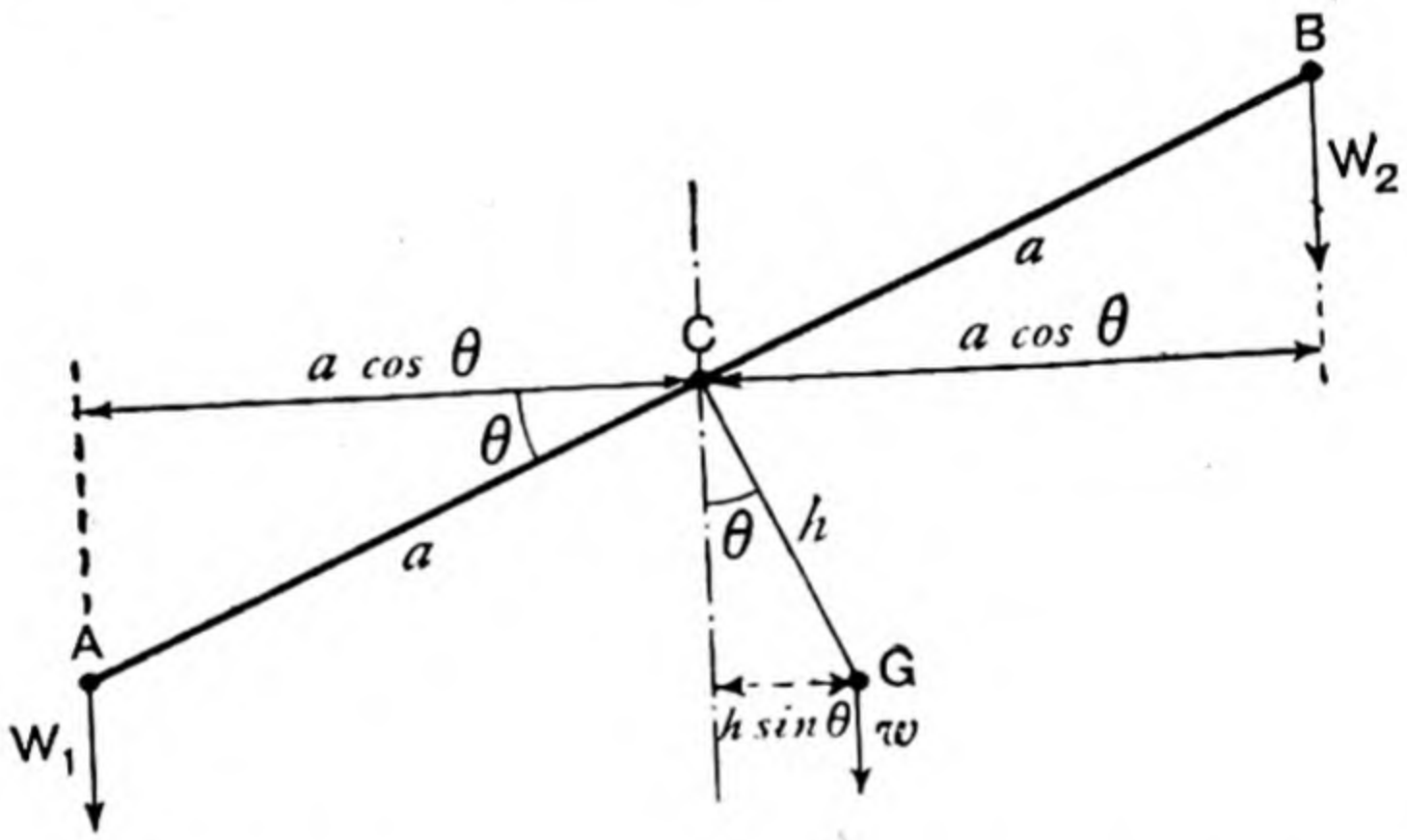
or, when

$$X = M.$$

Exact equality of X and M is of course unattainable. How closely can we hope to match them? One balance may match them to

within 0·01 gm. ; another to within 0·0001 gm. The latter is more *sensitive*, though both if constructed as supposed above, with equal arms, are *true*. What makes one balance more sensitive than another?

All instruments using a rotating pointer make the quantity under observation apply a moment (p. 42) tending to deflect the pointer, and oppose this by a controlling or restoring moment which increases as the deflexion increases, a steady reading occurring when these two moments are equal. Great sensitiveness is obtained when the design gives a large initial deflecting moment and a small initial restoring moment. In the balance, the restoring moment is provided by the *weight* of the beam acting from its centre of gravity, which is below the pivot.



In the diagram, let W_1 and W_2 be the *weights* (that is, forces in dynes) acting at A and B, a the length of each arm, w the weight of the beam acting from G, where $CG = h$. W_1 is greater than W_2, and the beam sets in equilibrium deflected through an angle θ from the horizontal.

Take moments about C :

$$W_1 \,.\, a \cos\theta = W_2 \,.\, a \cos\theta + w \,.\, h \sin\theta,$$

so,

$$(W_1 - W_2)\, a \cos\theta = wh \sin\theta,$$

the left-hand side gives the deflecting moment, and the right-hand side the restoring moment.

Then,

$$\tan\theta = \frac{a(W_1 - W_2)}{wh}.$$

For a given small difference $(W_1 - W_2)$, the term $\tan\theta$ will be large if a is large, w small, and h small. A sensitive balance thus has long arms, a light beam, and the knife-edge C close to the centre of gravity of the beam. Here it is supposed that A, B, and C are collinear; if this is not so, the sensitiveness varies with the load.

In use, the beam swings to and fro through the rest position; the requirements for sensitiveness are also those for a long period of swing, which is inconvenient. Air resistance on the pans themselves damps the swings satisfactorily with most ordinary balances, but many sensitive balances have damping devices.

Standard masses smaller than 0·01 gm. are difficult to handle, so a wire rider, usually weighing 0·01 gm., is used; it is moved along the balance arm, which is graduated for the purpose. The moment about C of the weight of a rider of mass r at D in the diagram of p. 15 is the same as that of the weight of a mass $r \cdot DC/BC$ placed in the pan suspended at B.

If the arms are of unequal length, say a and b, the balance will never be true; though the beam may set horizontally when unloaded if the pans are of unequal masses P_1 and P_2 so adjusted that

$$P_1 a = P_2 b.$$

The true mass of X can still be found, by **double weighing.** First, placing X on the left-hand pan and counterpoising it with mass M_1, we have $(X + P_1)ga = (M_1 + P_2)gb$, or remembering $P_1 a = P_2 b$,

$$Xa = M_1 b.$$

Next, placing X in the other pan, and counterpoising with mass M_2, similarly,

$$Xb = M_2 a,$$

so

$$X^2 = M_1 M_2, \quad \text{or} \quad X = \sqrt{M_1 M_2}.$$

This is known as **Gauss' method of double weighing.**

Borda's substitution method, in which X and weights M are used in turn on one side to balance a counterpoise of sand on the other side, is another way of using an untrue balance to weigh accurately.

Measurement of density. As an example of the measurement of a quantity in derived units, consider the measurement of the density

of a solid material, provided in the form of a rectangular block. The volume, itself a measurement in derived units, needs three separate length measurements, the length, the breadth, and the depth, and is of course (*length* × *breadth* × *depth*), expressed in cm.^{3}. The mass is determined by direct weighing; the density is then found by dividing the mass by the volume, and is expressed in gm. per cm.^{3}, or gm. cm.$^{-3}$.

It seems at first a little illogical, if not pedantic, to ascribe derived units to a property of a *material*. One expresses the length of a body in terms of a standard unit of length, the mass of a body in terms of a standard unit of mass; so surely the property of a material might be expressed in terms of the standard unit of that property, or of that property as possessed by a standard material. But, of course, the unit of the property *is* unit density, a density of 1 gm. cm.$^{-3}$.

Ordinary water, at its temperature of maximum density (nearly 4° C.) has a density of one kilogram per litre, or very nearly 1 gm. per cm.^{3}. The ratio $\frac{\text{density of material}}{\text{density of water}}$, or the density of the material relative to water, is called the **relative density** or **specific gravity** of the material. There is usually some ambiguity as to whether the density of water is supposed to be taken at 4° C. or at the temperature of the determination, but the possible difference involved is small for most purposes.

Specific gravity is a ratio between two like quantities, and is thus a dimensionless number, with no units. Its numerical value is the same if both the two densities to be compared are measured in lb. per cubic inch, gm. per litre, or any other possible units. It can be measured without working out either density directly. For,

$$\frac{\text{density of material}}{\text{density of water}} = \frac{\text{mass of unit volume of material}}{\text{mass of unit volume of water}}$$

$$= \frac{\text{mass of } n \text{ units of volume of material}}{\text{mass of } n \text{ units of volume of water}},$$

and can thus be found by weighing equal volumes of the two materials, without making any volume measurement at all. Further, in the case of liquids, the specific gravity can be found by taking the ratio of the heights of two vertical columns, without doing any

weighing at all; or by comparing the volumes of equal masses of the material and of water, as with an ordinary hydrometer (p. 186).

Graphs

Introduction. In many experiments, matters are so arranged that a quantity P is observed or measured under different conditions varied by the experimenter. The value of P may depend on several factors; for example, the volume of a gas depends on the mass of gas taken, the pressure, and the temperature; the anode current of a triode depends on the filament temperature, the grid voltage, and the anode voltage. In exploring such relationships, it is usual to vary one factor, Q, at a time and to find how P depends on Q, all other conditions being unchanged.

The results of such an experiment are conveniently expressed on a graph, in which values of Q (or some power or function of Q) as the *independent variable* are plotted along the x-axis, and values of P (or some power or function of P) as the *dependent variable* are plotted up the y-axis.

The chief advantages of plotting a graph are: (*a*) If P is plotted against Q, the relationship between them is shown in pictorial form; and corresponding values of P and Q other than those actually observed can be read off from the graph.

(*b*) It is possible to verify a known or expected relationship between P and Q, and to determine the numerical values of constants occurring in it.

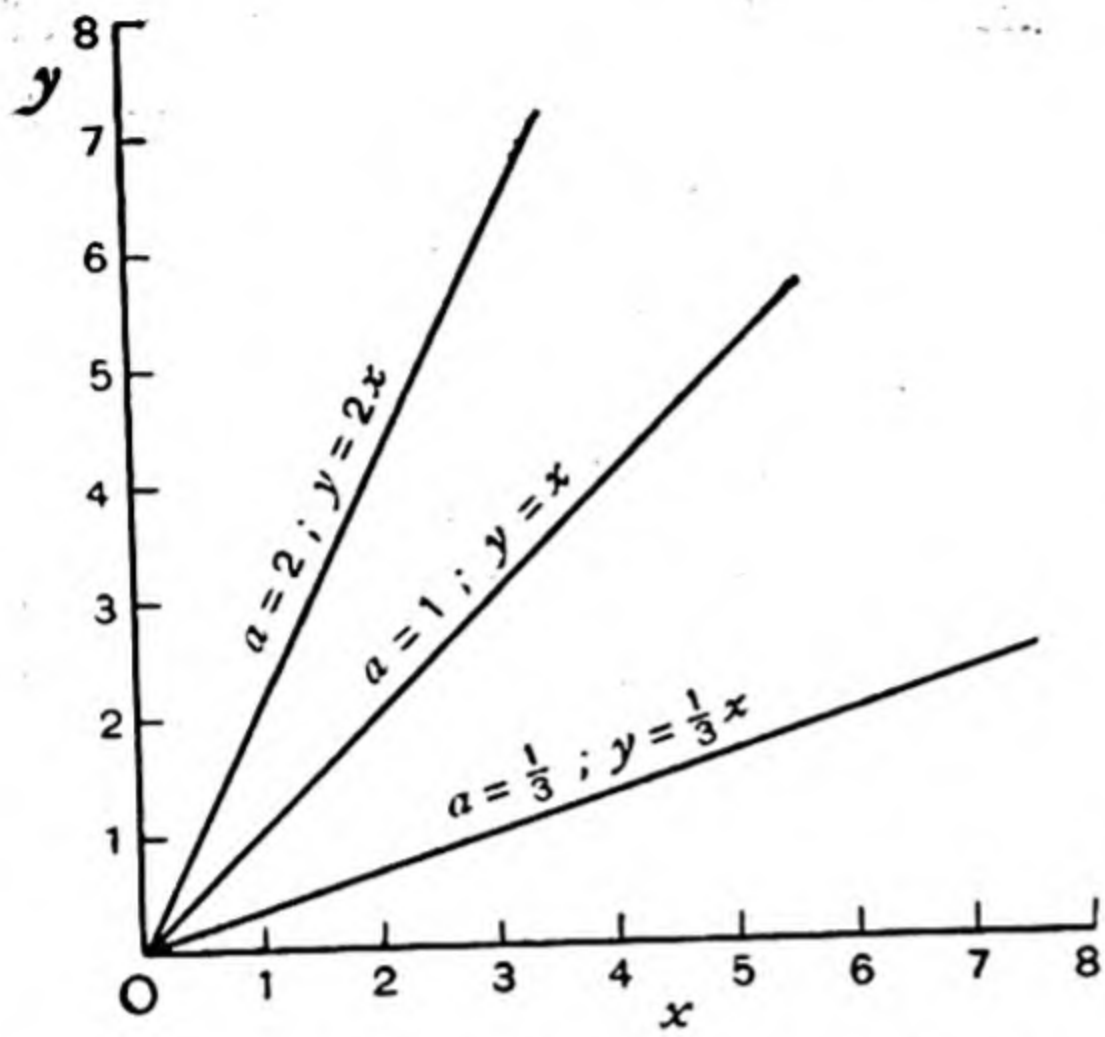
(*c*) In many cases, if the relationship between P and Q is of simple form, and is not known or suspected beforehand, a graph enables us to discover the form of this relationship.

The linear graph. Consider the equation $y=ax$, where a is a fixed number.

When $x=0$, $y=0$; when $x=1$, $y=a$;
when $x=2$, $y=2a$; when $x=3$, $y=3a$, and so on.

The graph of y against x is a straight line, going through the origin (see graph, p 20).

Whatever the values of x and y, $y/x=a$; or, y is directly proportional to x.



The gradient of the line, namely $\frac{\text{increment of } y}{\text{increment of } x}$, is also equal to a.

Thus, a straight line through the origin shows that y is directly proportional to x; and the **constant of proportionality**, a, is given by the gradient of the line.

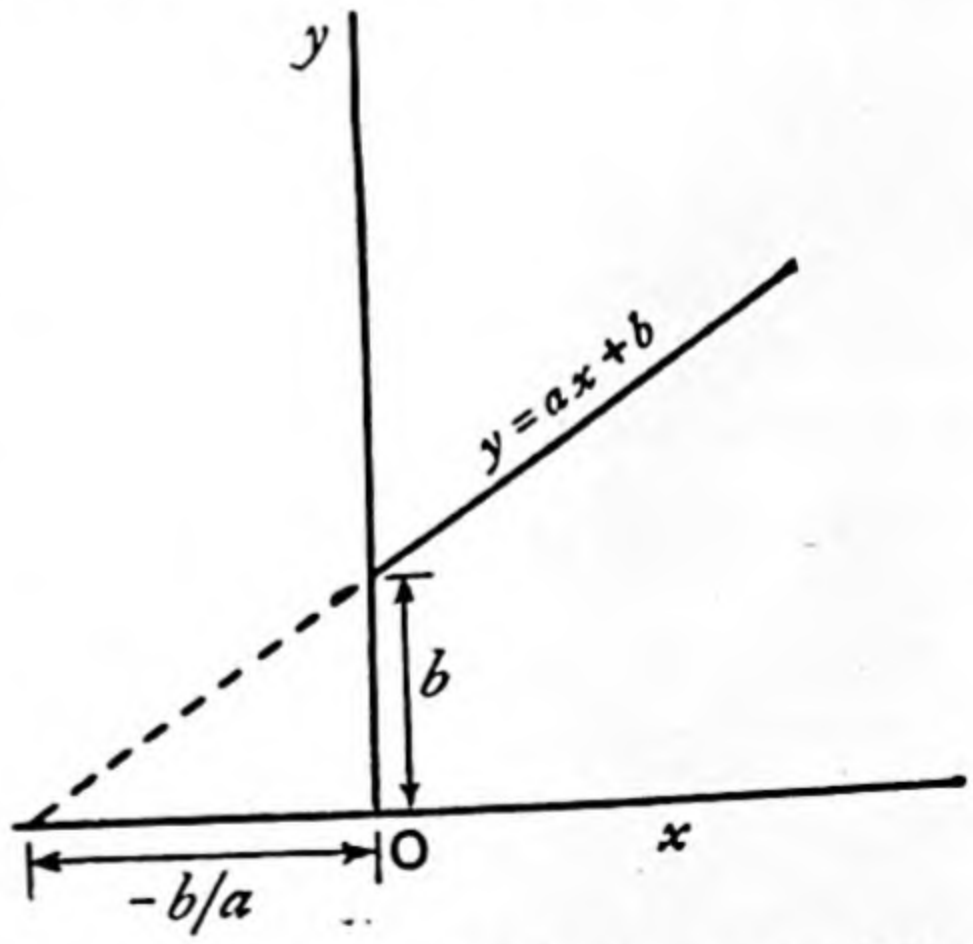
Consider the equation $y = ax + b$, where a and b are fixed numbers.

When $x = 0$, $y = b$; when $x = 1$, $y = a + b$;

when $x = 2$, $y = 2a + b$; when $x = 3$, $y = 3a + b$, and so on.

The graph is a straight line *not* going through the origin (see graph below), for when $x = 0$, $y = b$, and when $y = 0$, $x = -b/a$; these two numbers are called the **intercepts** on the y-axis and x-axis respectively.

The relation between y and x is said to be linear; note that this is *not* the same thing as direct proportion, since y/x is *not* constant in value.



The gradient of the line, $\frac{\text{increment of } y}{\text{increment of } x}$ is equal to a, for increasing x by one unit increases y by a units. If the line slopes downwards, so that y decreases as x increases, the value of a is negative.

To sum up:

1. Whenever the graph of y against x is a straight line, this shows the relation between the two quantities is linear, expressible by the equation $y = ax + b$.

2. The gradient, $\frac{\text{increment of } y}{\text{increment of } x}$ gives the value of a ; if the graph slopes upwards, a is positive, if downwards, a is negative.

3. The intercepts on the y-axis and x-axis respectively are b and $-b/a$.

4. In the particular case when the graph passes through the origin, and in this case only, y and x are directly proportional to one another.

The straight-line graph is extremely important, for it is the only graph that can be identified with certainty at sight ; a curve may " look like a parabola ", or " look like part of a hyperbola "—but the application of a straight-edge will show *for certain* whether or not a graph is a straight line, once it has been drawn.

EXAMPLE. The following values of load and effort were obtained with a certain machine :

Load W :	10 lb. wt.	20 lb. wt.	30 lb. wt.	40 lb. wt.
Effort P :	5 lb. wt.	7 lb. wt.	9 lb. wt.	11 lb. wt.

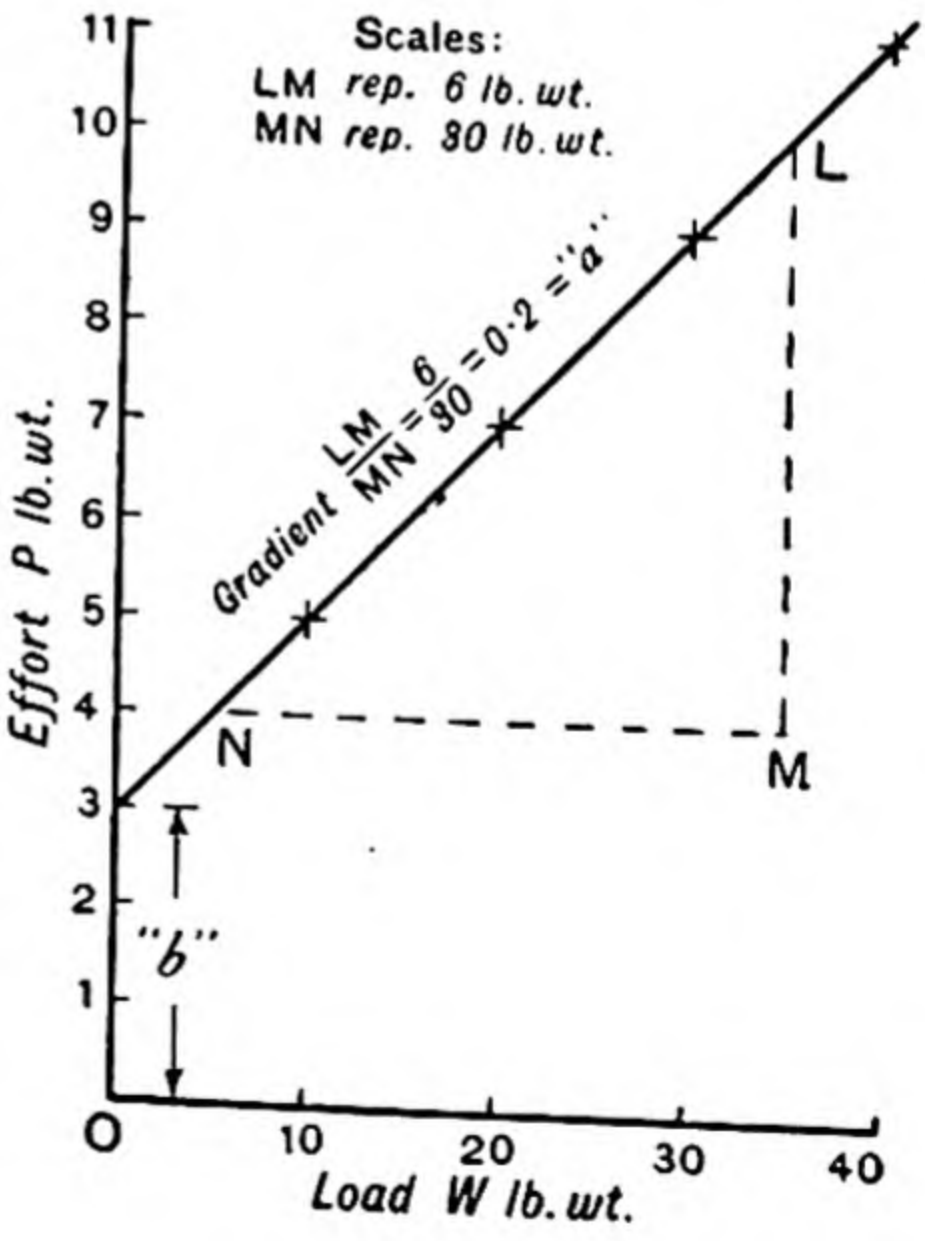
Show that the relation between W and P is of the form $P = aW + b$, and find a and b.

Plotting the graph, the gradient $LM/MN = 0{\cdot}2$, and the intercept on the P-axis is 3 units.

Thus the linear graph shows that the relationship is that stated, and the equation is

$$P = 0{\cdot}2W + 3.$$

It may be objected that the values of a and b could have been obtained from two readings only, by solving two simultaneous equations ; but we were asked to show that the linear relation does hold, and in an actual experiment the points will probably be found to lie close to, but not actually on, the line which best fits them. By drawing the graph we use *all* the readings in helping to find the line, instead of only two, and thus in effect average the experimental readings.



Reduction to the straight line form, when the equation is known. The following examples illustrate typical conversions to obtain a linear plot :

1. The period T and the length l of a simple pendulum are connected by the formula $T = 2\pi\sqrt{\frac{l}{g}}$; on squaring both sides of this equation,

$$T^2 = \frac{4\pi^2}{g} \cdot l;$$

whence if T^2 is plotted as y, and l as x, the graph is a straight line through the origin, of gradient $4\pi^2/g$ (Fig. a).

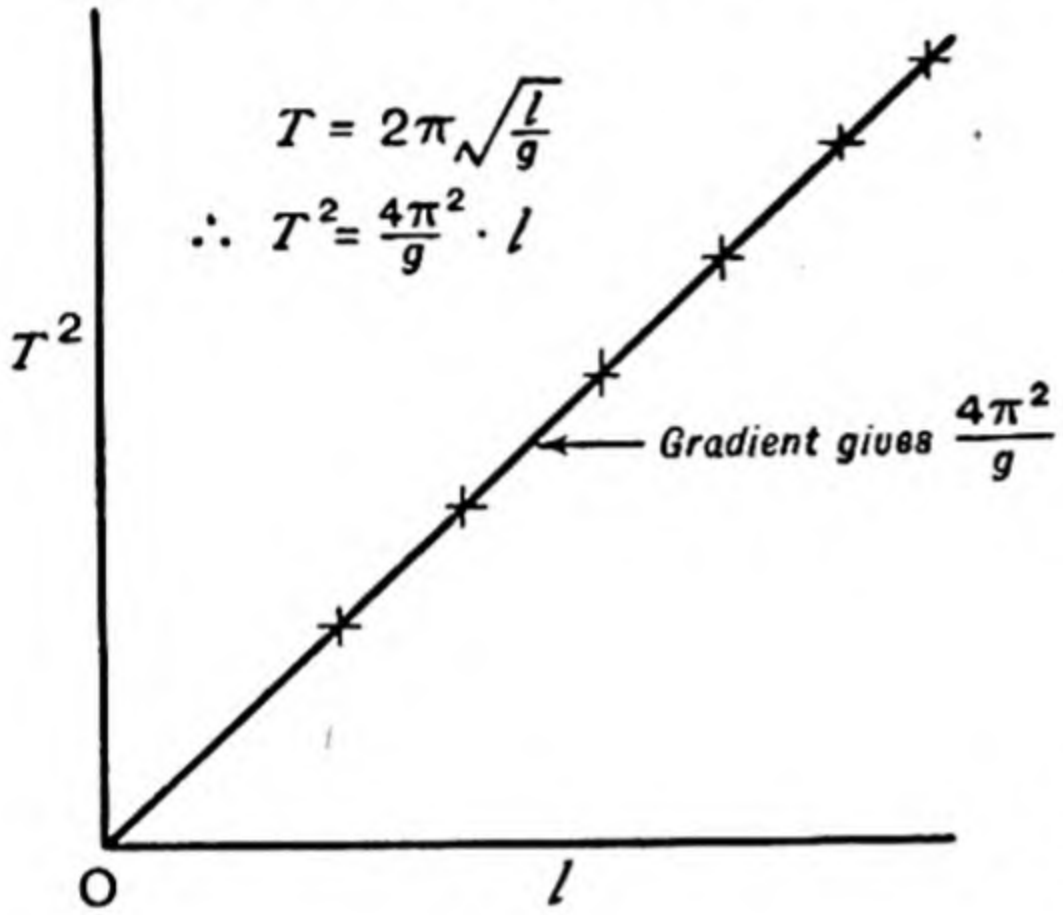


Fig. a

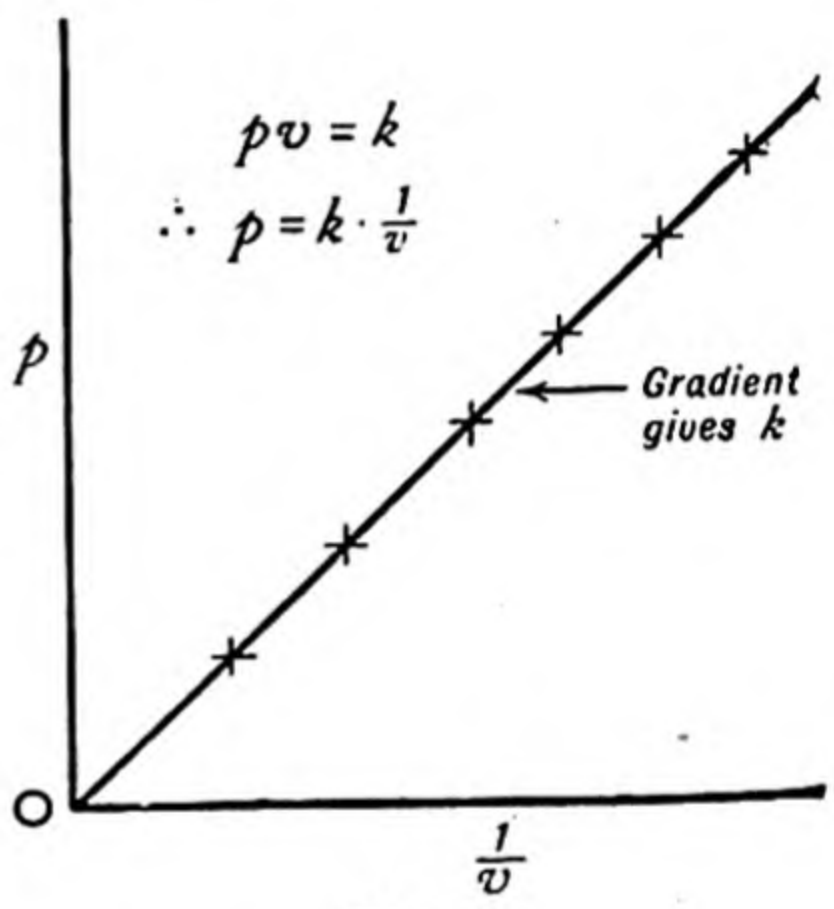


Fig. b

2. Boyle's law states that the pressure p and volume v of a fixed mass of gas are connected by the equation $pv = k$, where k is a constant ; rearranging,

$$p = k \cdot \frac{1}{v},$$

whence the graph of p (plotted as y) against $1/v$ (plotted as x) is a straight line through the origin, the gradient giving k (Fig. b).

3. The period T of oscillation of a spiral spring loaded with a mass M is given by the formula

$$T = 2\pi\sqrt{\frac{M+S}{k}},$$

where k is a constant for the spring, and S its " effective mass ".

Squaring, $T^2 = \frac{4\pi^2}{k}(M+S)$; so the graph of T^2 against M is linear, the intercept on the M axis being $-S$, and the gradient being $4\pi^2/k$ (Fig. c).

4. The period T of oscillation of a compound pendulum about an axis distant h from its centre of gravity is given by

$$T = 2\pi\sqrt{\frac{k^2+h^2}{hg}},$$

where k is a constant for the pendulum called its radius of gyration.

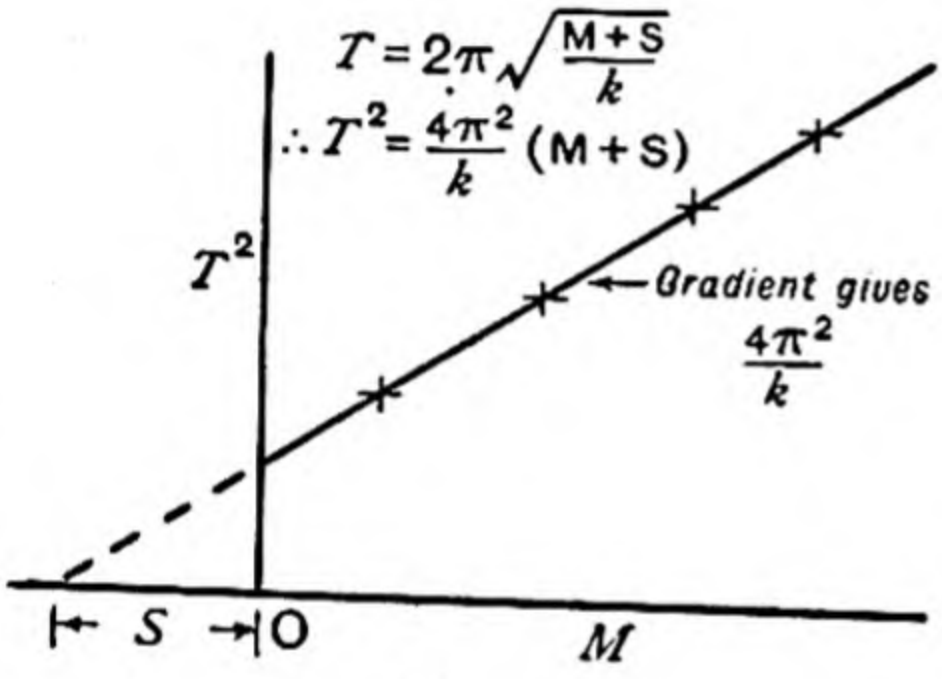


Fig. c

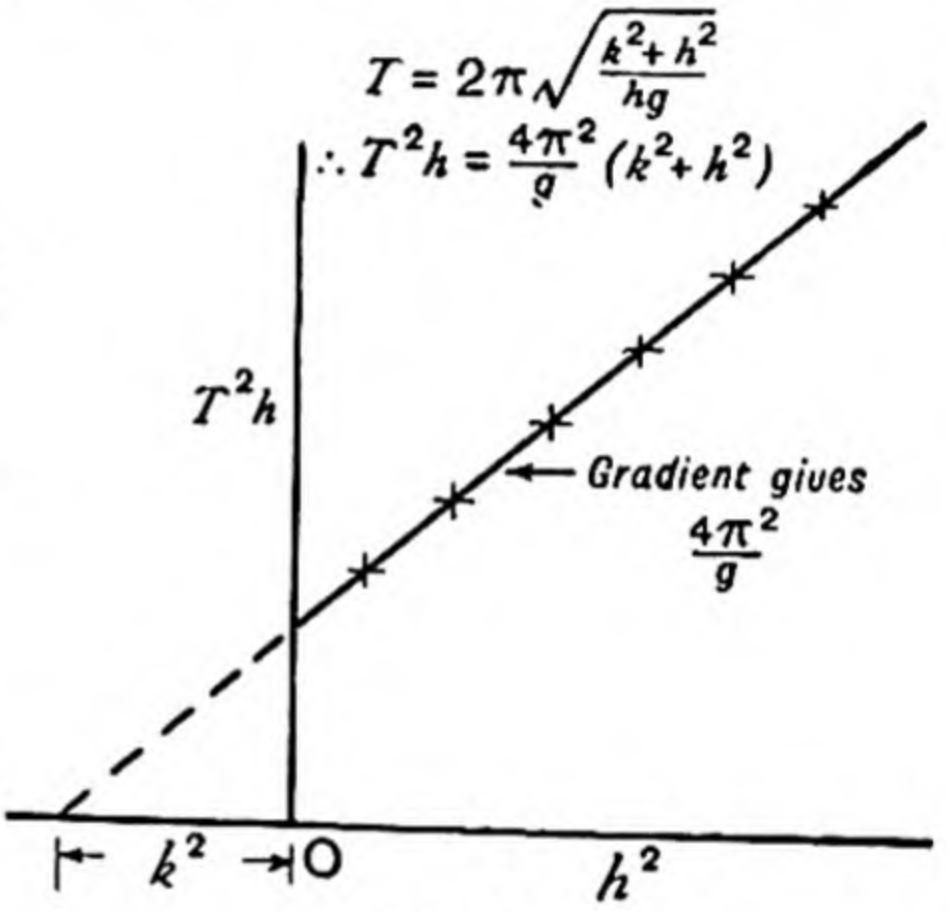


Fig. d

Squaring $T^2 = \frac{4\pi^2}{g} \cdot \frac{k^2+h^2}{h}$, and rearranging, $T^2h = \frac{4\pi^2}{g}(k^2+h^2)$; whence a plot of T^2h against h^2 is linear, the intercept on the h^2 axis being $-k^2$, and the gradient $4\pi^2/g$ (Fig. d).

Ingenuity, and some experience in arranging formulae to give the required straight-line graph, are both required. The above is not an exhaustive list of "types" but rather an indication of the way in which the problem should be approached.

Reduction to the linear form when the relation is not known.

Suppose that P and Q are related by the equation $P = kQ^n$, where k and n are numbers, both of which are unknown.

Then, taking logarithms of both sides, $\log P = n \log Q + \log k$. Plotting $\log P$ as y, and $\log Q$ as x, a straight line results, the gradient being n and the intercept on the $\log P$ axis giving $\log k$.

Care has to be taken in plotting, since the true logarithms, not the conventionally tabulated logarithms, of numbers smaller than unity are required. For example, the logarithm of 0·5, read as $\bar{1}$·6990, is really $(-1+0{\cdot}6990)$, or $-0{\cdot}3010$; an intercept may be $-2{\cdot}73$, which is $(-3+0{\cdot}27)$ or $\bar{3}{\cdot}2700$ when four-figure antilogarithm tables are used to find k from it.

(1)

The following examples, with made-up readings, illustrate the method. In each, the first two columns are meant to stand for actual observations P and Q, supposed related by the formula $P = kQ^n$.

Q	P	log Q	log P
1·0	3·0	0·0	0·477
2·0	12·0	0·301	1·078
3·0	27·0	0·477	1·430
4·0	48·0	0·602	1·680

The graph is linear, so the relation is of the supposed form. The gradient is 2·0, and the intercept on the log P axis is 0·477, so k is the antilogarithm of 0·477, or 3·0. Thus the relation is $P = 3Q^2$ (Fig. *a*).

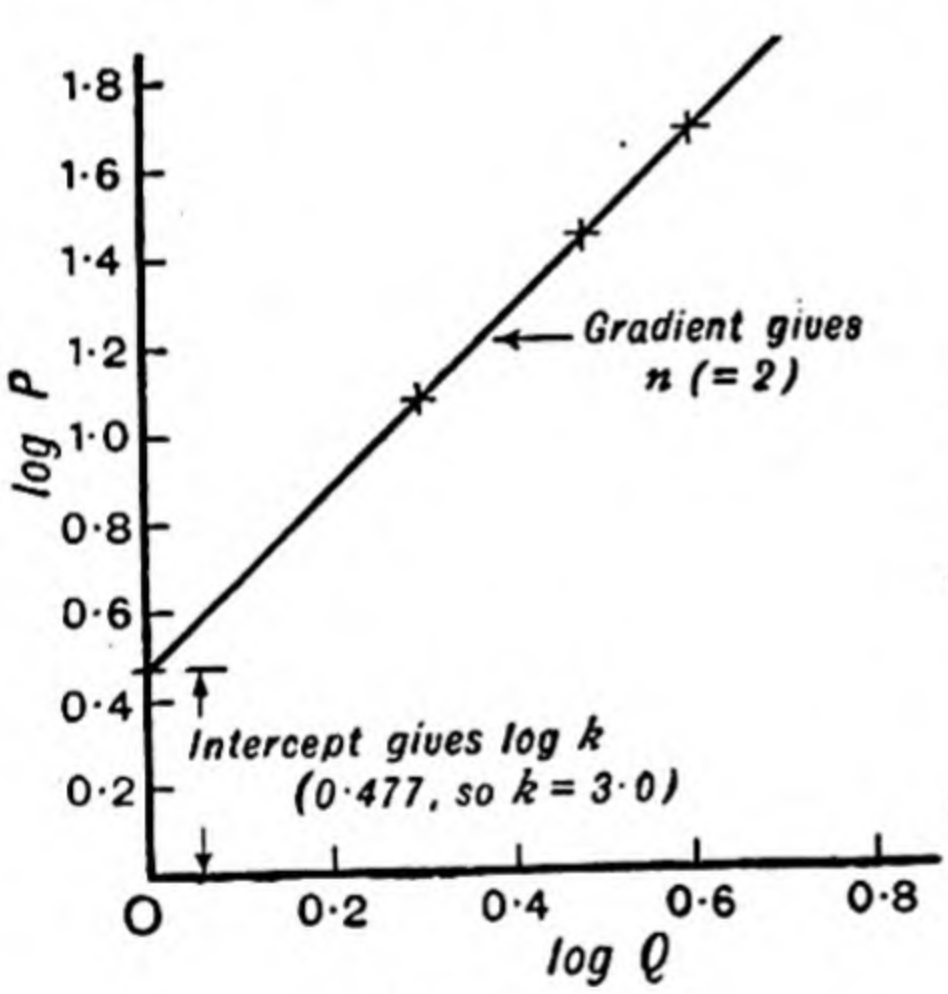


Fig. *a*

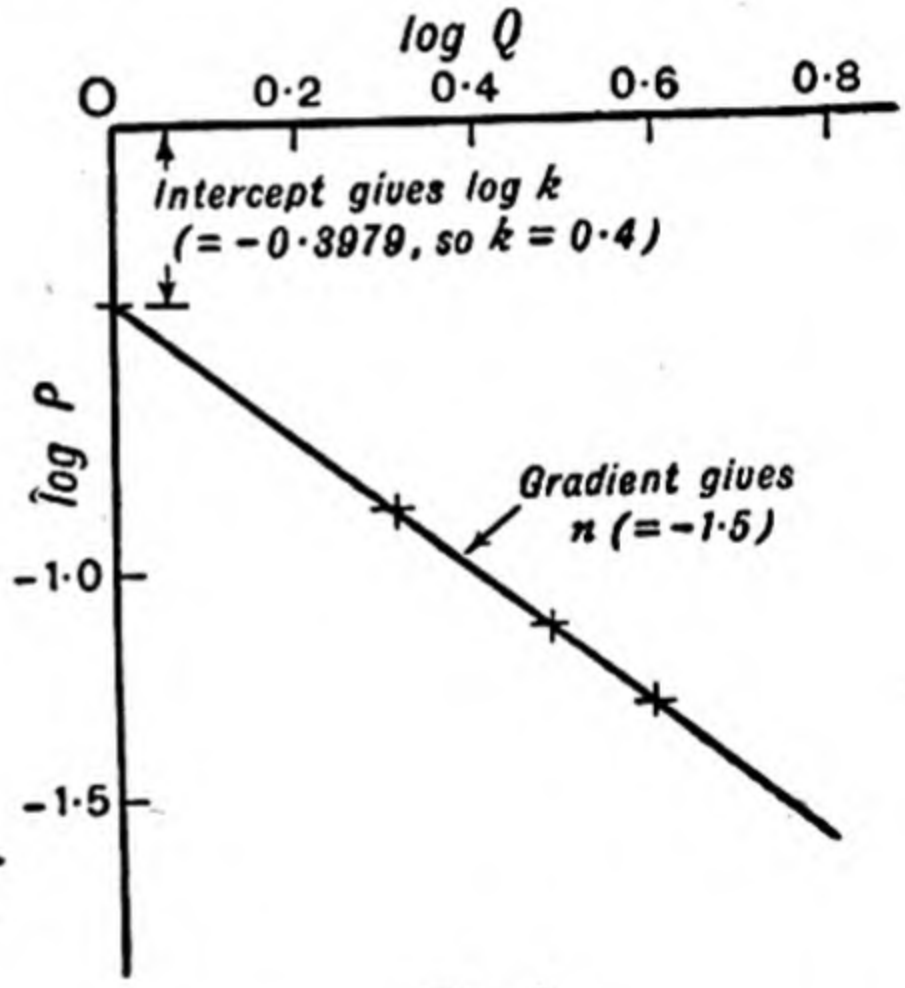


Fig. *b*

(2)

Q	P	log Q	log P
1·0	0·400	0·0	$\bar{1}$·6021 = −0·3979
2·0	0·140	0·301	$\bar{1}$·1461 = −0·8539
3·0	0·077	0·477	$\bar{2}$·8865 = −1·1135
4·0	0·032	0·602	$\bar{2}$·6990 = −1·3010

Again the graph is linear, so the supposed relation holds. The gradient is $-1{\cdot}5$, so n is $-1{\cdot}5$; and the intercept on the log P axis is $-0{\cdot}3979$, which for tabular purposes is $\bar{1}{\cdot}6021$; and k is the antilogarithm of this, which is 0·4. Thus the relation is

$$P = 0{\cdot}4Q^{-1{\cdot}5} \quad \text{(Fig. } b\text{)}.$$

One variable as an index. In the equation,

$$P = a \times 10^{bQ},$$

taking logarithms of both sides,

$$\log P = \log a + bQ,$$

so a straight line results if log P is plotted against Q (not against log Q).

Many important physical formulae are of the type

$$P = ae^{bQ},$$

where $e = 2{\cdot}718$ is the base of Napierian or natural logarithms. Taking logarithms to the base e is the obvious step, but it is actually simpler as a rule to express e as a power of 10 and use common logarithm tables. As $e = 10^{0{\cdot}4343}$, the above expression is

$$P = a \times 10^{0{\cdot}4343bQ},$$

giving

$$\log P = \log a + 0{\cdot}4343bQ.$$

The gradient of the straight line obtained by plotting log P against Q is $0{\cdot}4343b$, and the intercept on the log P axis is log a.

Formulae such as $P = a\sqrt{Q}e^{-bQ^2}$ yield to similar treatment. For, taking logarithms,

$$\log P = \log a + \tfrac{1}{2}\log Q - 0{\cdot}4343bQ^2,$$

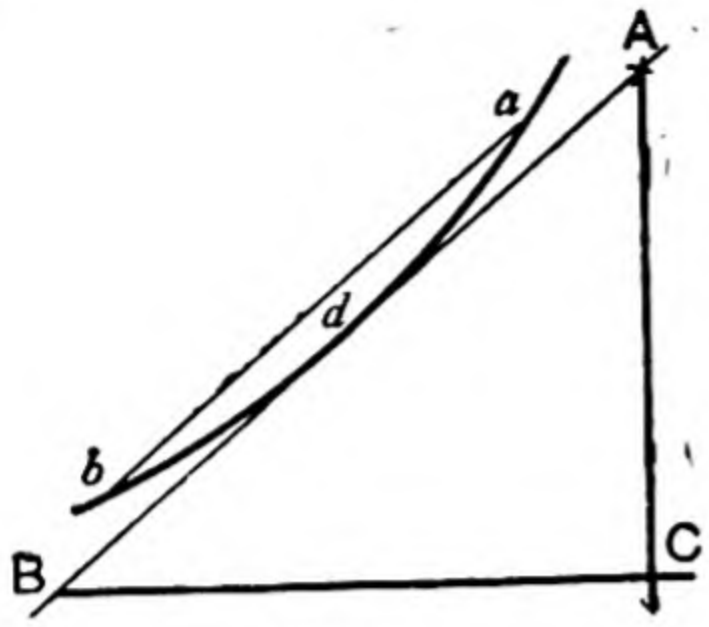
or

$$\log P - \tfrac{1}{2}\log Q = \log a - 0{\cdot}4343bQ^2,$$

and a straight line is obtained by plotting (log $P - \frac{1}{2}$ log Q) against Q^2.

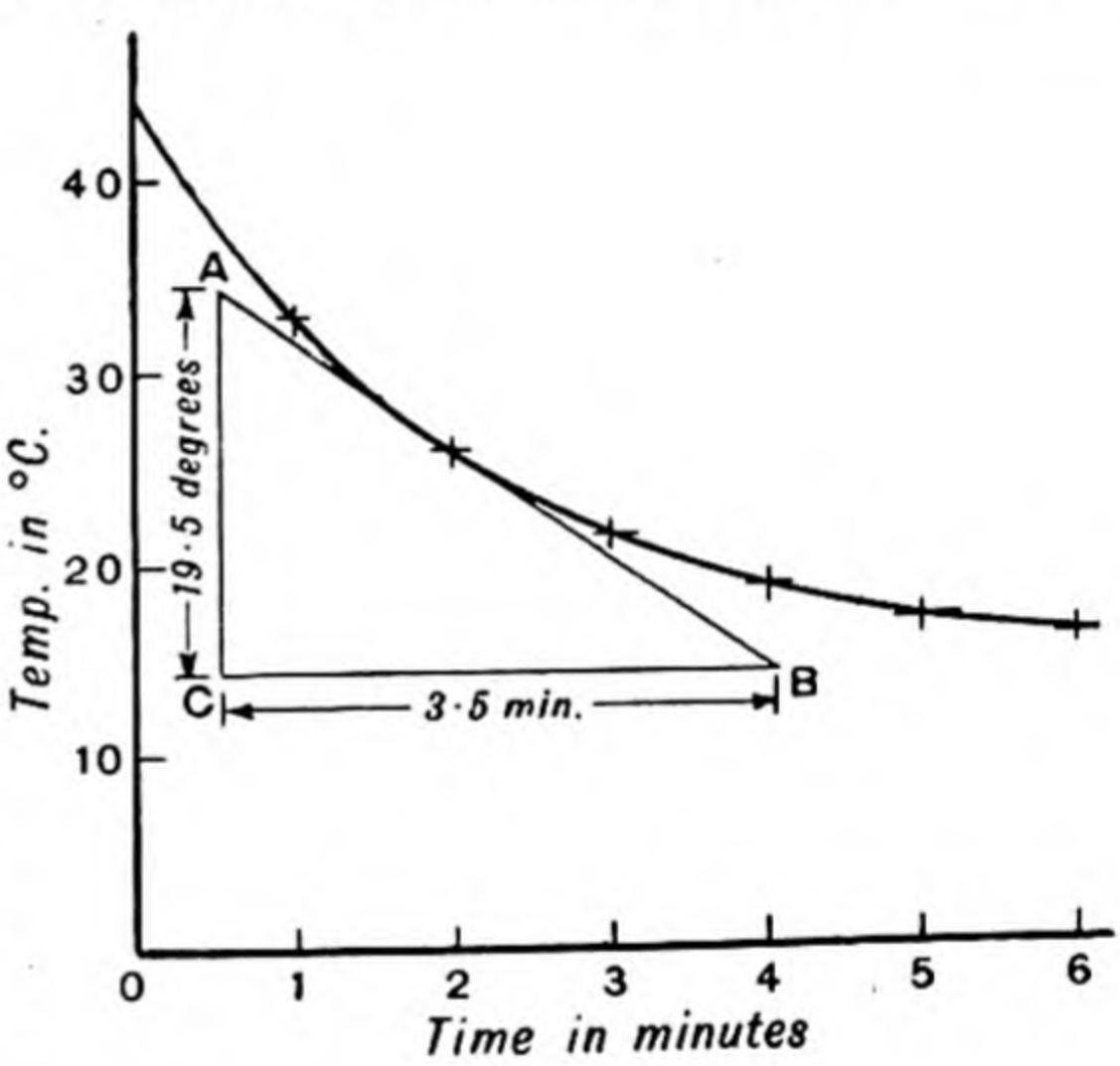
Logarithmic and semi-logarithmic graph paper. Reference to tables can be saved by the use of graph paper in which the rulings are spaced logarithmically, like the graduations on a slide-rule. For plots of the type log P against log Q, both axes are so divided, and the numbers P and Q are plotted straight out on to the paper. Semi-logarithmic paper with one axis divided logarithmically and the other divided uniformly is used for graphs of the type log P against Q.

Gradient of a non-linear graph at any point. If the relation between P and Q is not linear, the gradient varies from point to point. The gradient at any point d (see diagram) differs very little from that of the line ab joining two points a and b close to d and on either side



of it ; this difference vanishes when a and b are very close to d, and the line ab then becomes the tangent at d. In order to measure the gradient of the tangent, it is of course necessary to mark off on it a considerable length AB, and make this the hypotenuse of a right-angled triangle ABC with sides parallel to the axes. Then the gradient is AC/BC, interpreted to scale.

A most important type of graph is that in which the independent variable is *time*. In this case, the gradient measures the *rate of change with time* of the dependent variable. Displacement-time and velocity-time graphs are discussed later in this chapter ; the following example shows the general method of attack in all "rate of change" problems.



EXAMPLE. The figures represent the temperature of a cooling calorimeter at different times ; find the rate of fall of temperature at time 2 min.

Time in minutes :	0	1	2	3	4	5	6
Temp. in ° C. :	45·0	33·0	26·1	21·7	19·1	17·4	16·5

The tangent is drawn at the point for $t = 2$ min. : the gradient of this is AC/BC, representing 19·5° C. per 3·5 minutes, or 5·6° C. per minute.

Area between a graph and one axis. Often the product $P \times Q$ has a definite physical meaning. For example, if P is force and Q distance moved in the direction of the force, the product $P \times Q$ represents work done ; if P is velocity and Q is time, the product $P \times Q$ represents distance travelled.

In this case, the area of each square of the graph paper represents, to scale, a definite number of units of the product PQ. If one unit represents, say, a force of 5 lb. wt., and one unit a distance of 10 ft., then the area of one square represents $5 \times 10 = 50$ ft. lb. of work. If one unit represents 10 ft./sec. up the velocity axis, and one unit represents 4 sec. along the time axis, the area of one square represents a distance of $10 \times 4 = 40$ ft.

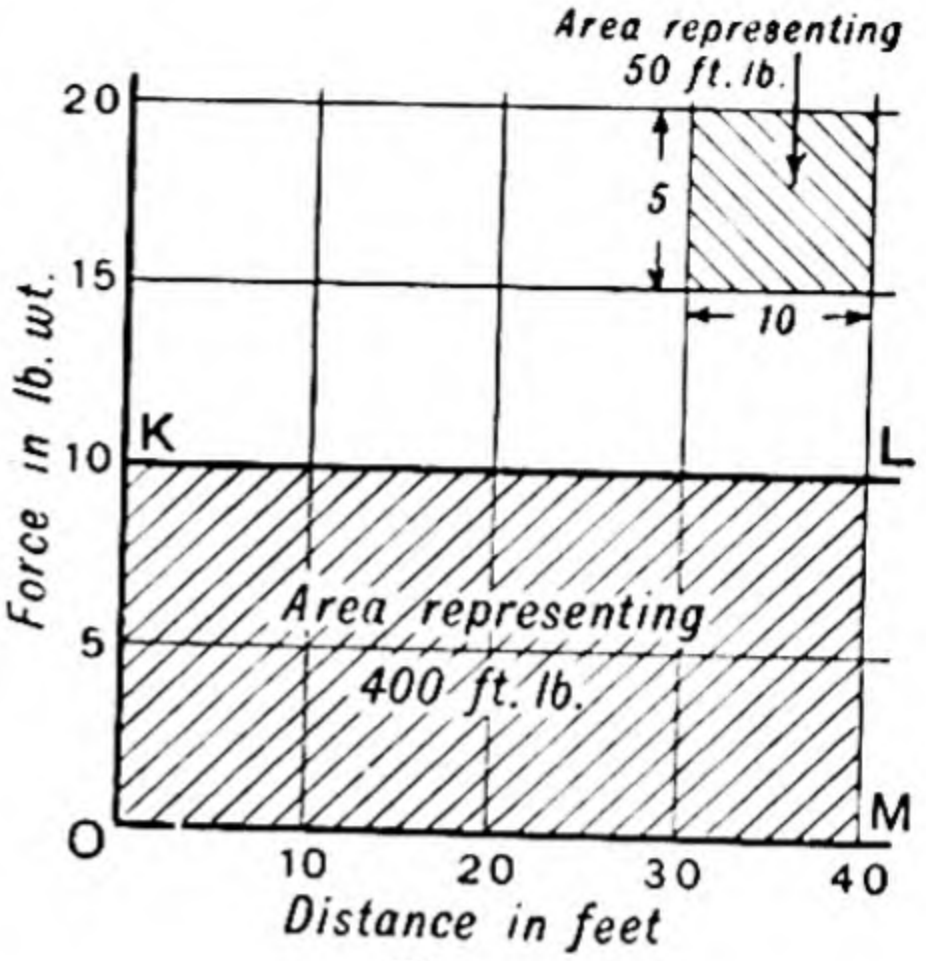


Fig. *a*

In Fig. *a*, representing a constant force of 10 lb. wt. moving its point of application, and drawn to the above scale, the area $OKLM$, containing 8 unit squares, represents 400 ft. lb., which is the work done by the force of 10 lb. wt. moving through 40 ft.

If the force is not constant, the same rule will hold. For the curve $OJKL$ of Fig. *b* can be imagined to be traced in a series of very small steps, for each of which the area beneath the step represents the work done. In this figure, drawn to the same scale, the area $OJLM$ is nearly 7·5 squares, representing 375 ft. lb. of work ; and the area $KLMN$, 3·5 squares, represents the work done in the interval between N and M, which is 175 ft. lb.

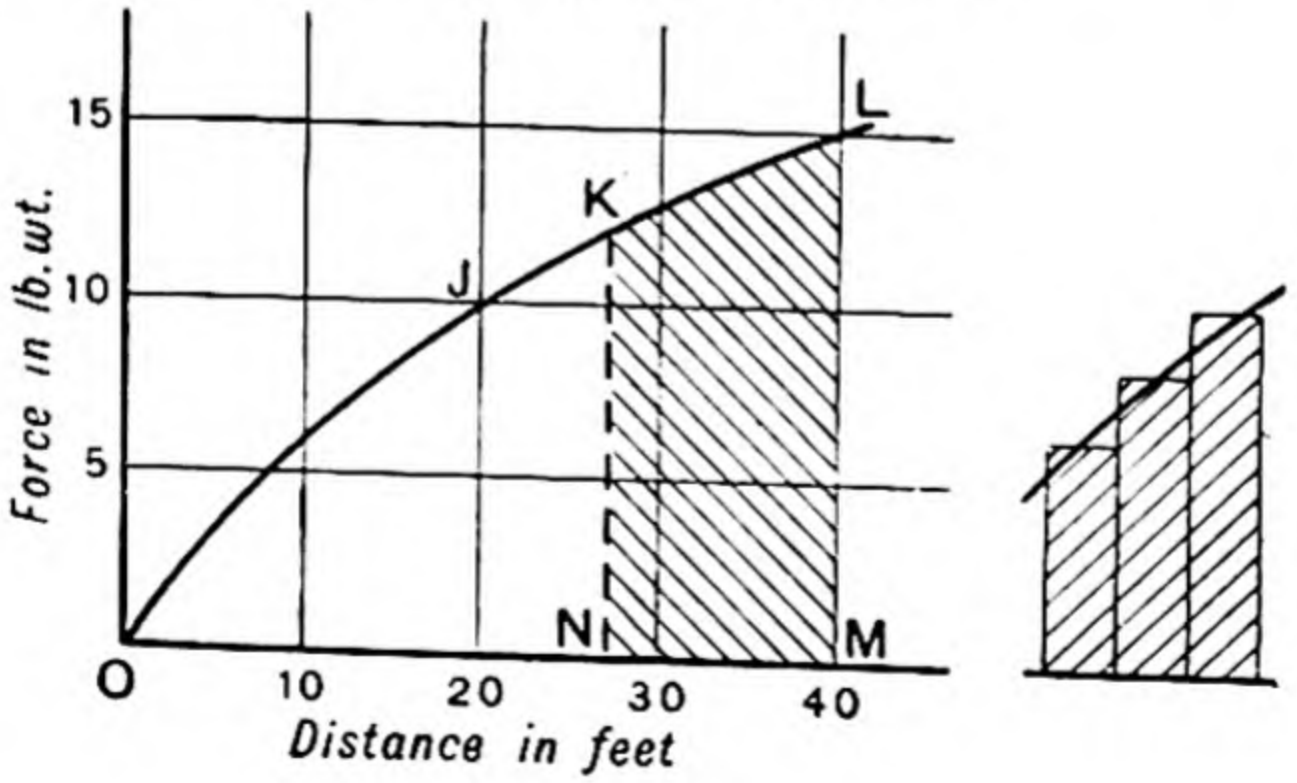


Fig. *b*

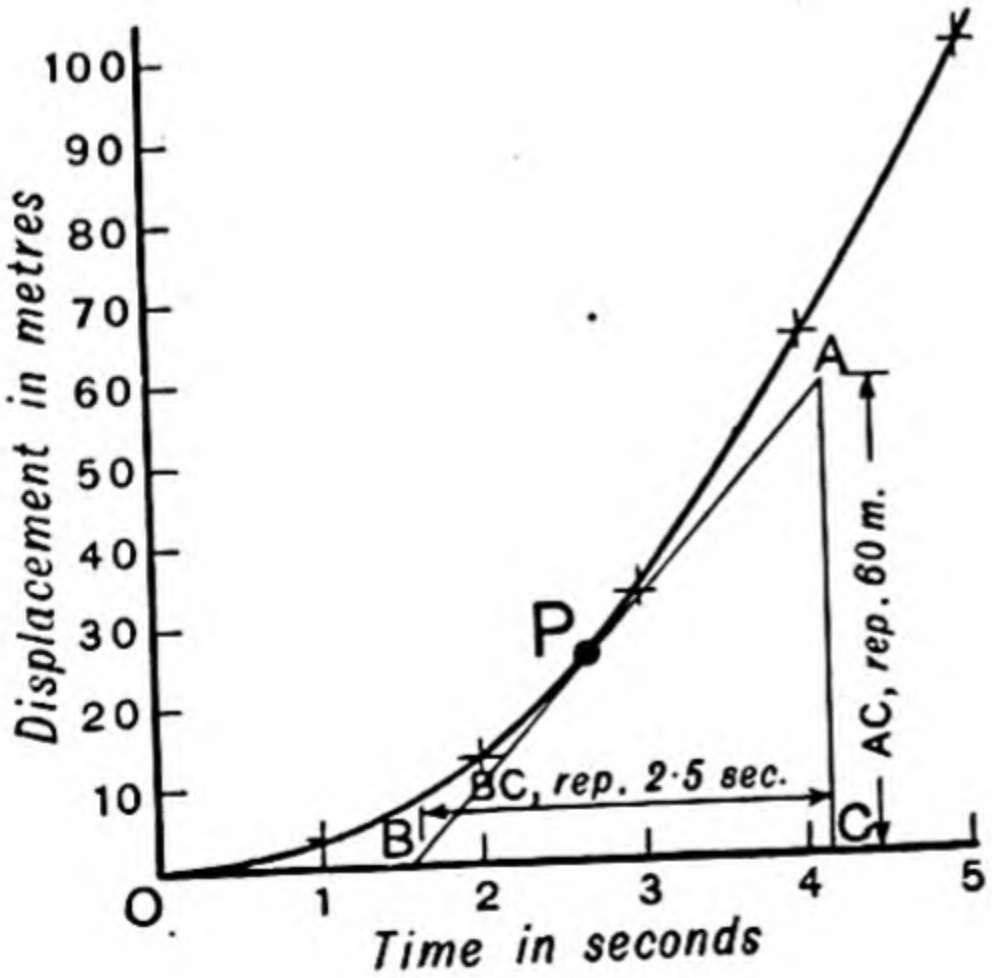
Two important precautions are necessary in area computations:

(1) *The intersection of the axes must be at the origin* (0, 0), *and this must be shown on the paper.*

(2) *The area taken must be that between the curve and the correct axis.*

It is difficult, without invoking the calculus, to state a definite rule about this; but a moment's thought should enable the right choice to be made. In the examples quoted, an increase in the force without any increase in distance leads to *no performance of work*; this force increase would be represented by a vertical line, and there is *no area between this line and the distance-axis.* Thus it must be the *other* area, that between the line and the distance-axis, that we require. Reasoning along these lines, once it is realised that a decision has to be made and that any area will not do, there should be no difficulty about choosing the correct one.

These two points have been emphasised, as many students conscientiously choose scales which utilise the full extent of the graph paper for the range of readings to be plotted, and if (0, 0) does not come within this range the origin is not on the paper. This is an excellent practice in many cases, but often causes mistakes when an area has to be computed. Many also seem to believe that, provided the readings are accurately plotted and the graph properly drawn, it does not matter which way the axes are chosen; examples of this type should convince them that the matter is often important, and deserves some really careful thought.



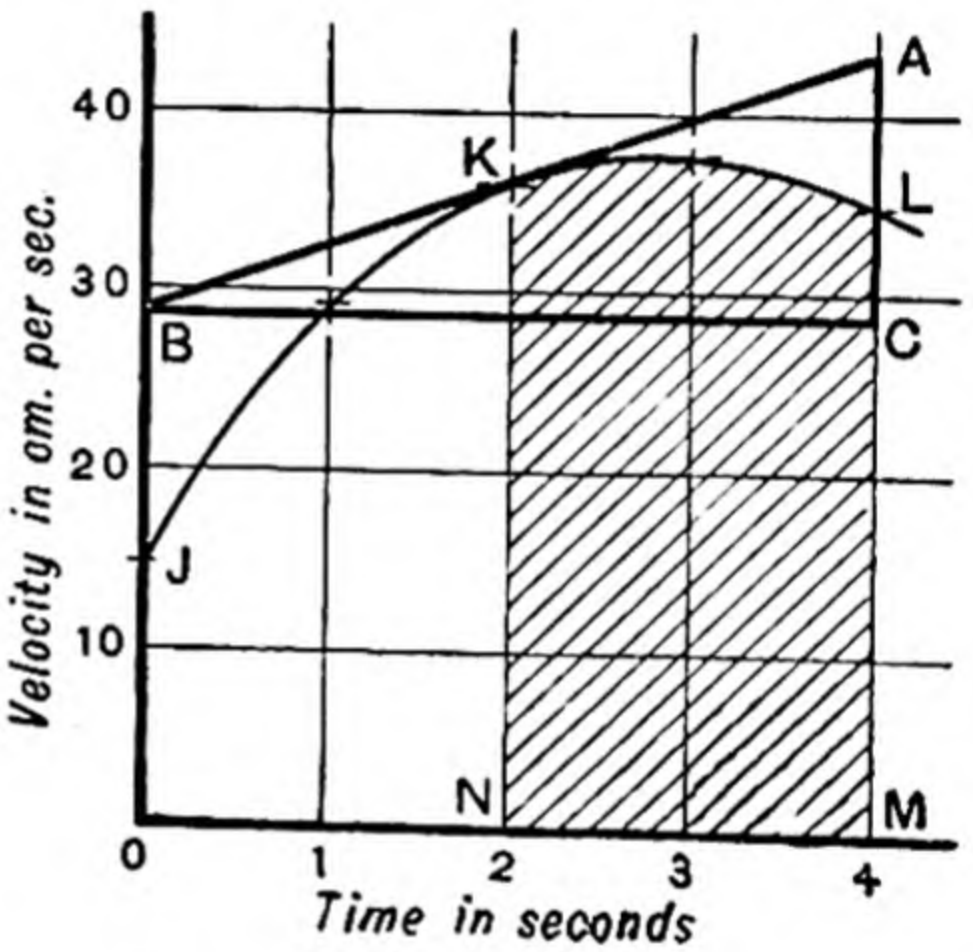
Displacement-time and velocity-time graphs. From readings of the displacement of a body which moves in a straight line from a fixed point on the line, at different instants, a **displacement-time (or distance-time) graph** may be plotted. The gradient of the graph at any instant gives the *velocity* of the body at that instant.

EXAMPLE. Plot the displacement-time graph for the following motion, and find the velocity after 3 seconds:

Time in seconds:	0	1	2	3	4	5
Displacement in metres:	0	4·0	12·0	36·0	64·0	100·0

Drawing the tangent at the point, P, for $t=3$ sec., the gradient AC/BC interpreted to scale is $60/2{\cdot}5=24$ metres per second.

Similarly, from readings of the velocity at different instants, a velocity-time graph is plotted. The gradient of the graph at any instant gives the *acceleration* (or rate of change of velocity) at that instant, and the area between the graph and the time-axis for any interval gives the distance traversed during that interval.



EXAMPLE. Plot the velocity-time graph for the following motion, and find (*a*) the acceleration at $t=2$ sec., (*b*) the distance travelled in the last two seconds:

Time in seconds:	0	1	2	3	4
Velocity in cm./sec.:	15	29·5	36	38	35

Drawing the tangent at the point $t=2$ sec., the gradient AC/BC interpreted to scale is $14/4=3{\cdot}5$ cm. sec.$^{-2}$. One square on the paper represents $10\times1=10$ cm., and the area $KLMN$ is nearly 7·5 squares, so the distance covered in the last two seconds is $10\times7{\cdot}5=75$ cm.

EXAMPLES ON CHAPTER 1

1. Describe with diagrams how you would construct and use (*a*) a vernier to read to 0·05 mm., (*b*) a vernier to read to 0·002 in., (*c*) an angular vernier, with the main scale divided into degrees and 20-minute intervals, to read to 0·5 minutes, (*d*) an angular vernier, with the main scale divided into degrees and 15-minute intervals, to read to the nearest minute.

2. A beam balance has arms of length a, and is pivoted a short distance k above the line joining the pan knife-edges, while the centre of gravity is a distance h below this line. If the weight of the beam is w, show that the deflection θ of the beam when the weights in the pans are P and Q is given by

$$\tan\theta = \frac{(P-Q)\,a}{w(h+k)+(P+Q)\,k}.$$

In a certain balance, $k=0$, $a=12$ cm., $w=250$ gm., and the length of the pointer is 15 cm. It is found that a difference in weight of 0·01 gm. causes the end of the pointer to move through a distance of 5 mm. What is the value of h?

3. Describe a sensitive form of beam balance, giving diagrams and pointing out the purpose of its knife-edges and of the method of hanging its pans. On what does the sensitiveness and the accuracy of such a balance depend?

With a balance of which the arms were 10 cm. long, it was found that 0·010 gm. extra load on one pan deflected the beam of mass 20 gm. through 1° and that this deflexion was independent of the loads placed on the pans. What can you deduce from these measurements? (O.)

4. Describe how you would measure the specific gravity of a solid accurately, and give the theory of the method you use.

The densities of ordinary water and of heavy water at 12° C. are respectively 0·999 and 1·106 gm. cm.$^{-3}$. Find the proportion by weight of heavy water in a sample of water which is found to have a density of 1·035 gm. cm.$^{-3}$ at this temperature. (O.)

5. Define *density* and *specific gravity*.

Describe an experiment to find the density of air at room temperature and pressure.

The density of a certain liquid is 0·800 gm./c.c. at 0° C. and 0·762 gm./c.c. at 50° C. What mass will be expelled from a 50 c.c. flask, which is exactly full at 0° C., when it is heated up to 50° C.? Assume the volume of the flask does not change. (O.)

6. A volume of 50 c.c. of water is mixed with an equal volume of alcohol of specific gravity 0·8. The specific gravity of the mixture is found to be 0·93. Find the volume contraction which occurs on mixing. (O.)

7. What are the fundamental units of the C.G.S. system, and how are they defined?

Show how the units of the following may be derived from the fundamental units: acceleration, momentum, force, pressure, work, power. (O. & C.)

8. Describe the construction of a good physical balance, stating what features determine the accuracy and sensitiveness respectively. Explain how you would measure the sensitiveness of such a balance, and how you would find the mass of a body even if the balance were not true. (O. & C.)

9. Discuss the advantages to be gained by a graphical representation of a set of physical measurements.

How, after making the necessary experiments, would you plot a set of observations (*a*) on the length of an air column and the frequency of the note to which it resounds, (*b*) on the cooling of a hot body in a constant temperature enclosure, so as to bring out as clearly as possible the relation between the quantities involved? (O. & C.)

10. The following values were obtained for the period of vertical oscillation for a spiral spring carrying different loads :

Load M	25	50	75	100	125 gm.
Period T	0·96	1·14	1·28	1·41	1·54 sec.

Supposing the formula relating T and M to be

$$T = 2\pi\sqrt{\frac{M+S}{k}},$$

where S is the effective mass of the spring and k a constant for the spring, find the values of S and k.

11. A metre stick was used as a rigid pendulum, and swung on pivots at different distances h from the centre of gravity : the formula for the period T for small swings is

$$T = 2\pi\sqrt{\frac{k^2+h^2}{hg}},$$

where k is a constant for the stick, and g the acceleration due to gravity. The following values were obtained :

h	5	10	20	30	40	50 cm.
T	2·76	2·03	1·63	1·57	1·60	1·66 sec.

By means of a suitable linear graph, determine the values of k and g.

12. In an experiment with a cooling calorimeter, the following results were obtained :

Rate of fall of temperature, r	5·6	3·5	2·3	1·7	1·3	0·8 C. deg. per min.
Temperature excess over air temp., θ	79·4	56·2	39·8	31·6	25·0	18·0 C. deg.

Assuming that the law of cooling is given by the formula $r = a\theta^n$, find the values of a and n.

13. A thin wooden beam is supported horizontally upon two knife-edges equidistant from its mid-point, at which a fixed load is suspended. The separation x of the knife-edges is varied, and the corresponding depression y of the mid-point is observed. The following values are obtained :

Separation x	40·0	50·0	60·0	70·0	80·0 cm.
Depression y	0·32	0·63	0·99	1·72	2·56 cm.

If the relation between y and x is of the form $y = ax^n$, find a and n.

14. Determinations are made of the resistance R and the power consumption in watts W of a small metal-filament lamp at different temperatures, with the following results :

Resistance R	1·6	2·0	2·4	2·9	3·4 ohm
Wattage W	2·2	6·3	20·0	31·6	63·0 watts

Show that the relation between W and R is of the form $W = aR^n$, and find a and n.

15. The following values (taken from de Broglie) of the frequency ν of the K_α X-ray lines in the spectra of elements from calcium to neodymium are tabulated against the atomic number N of the element :

Frequency ν	0·9	2·1	3·9	6·0	$9{\cdot}0 \times 10^{18}$ sec.$^{-1}$
Atomic Number N	20	30	40	50	60

Show that the relation between ν and N is of the form $\sqrt{\nu} = AN + B$, and find the values of A and B.

16. Values for the anode current I for different values of the anode voltage V for a diode were obtained as follows :

Anode voltage V	24	36	60	72	100 volts
Anode current I	0·13	0·25	0·52	0·66	1·27 milliamp.

Show that the current varies as the $\frac{3}{2}$ power of the voltage.

17. Given certain simplifying assumptions, the atmospheric pressure p at height h above sea-level may be represented by the formula

$$p = Ae^{-Bh}.$$

Show that the following figures are consistent with this formula, and find the values of A and B :

Pressure p	31·0	11·5	2·87	$0{\cdot}88 \times 10^4$ dyne cm.$^{-2}$
Height h	10	20	30	40 km.

18. Andrade's formula for the variation of the viscosity of a liquid with the absolute temperature T is $\eta = Ae^{b/T}$. Find the values of the constants A and b for water from the following figures :

Temperature T	273	293	313	333	353 deg. abs.
Viscosity η	0·0179	0·0100	0·0065	0·0047	0·0036 poises

19. The following table gives values for the coefficient of viscosity η of air at different absolute temperatures T :

Temperature T	83	194	300	371	673 deg. abs.
Viscosity η	5·7	13·3	18·4	22·2	$30{\cdot}0 \times 10^{-5}$ poises

Show that they are consistent with Sutherland's formula

$$\eta = \frac{A\sqrt{T}}{1 + B/T},$$

and find the values of the constants A and B for air.

20. MacLeod's formula relating the surface tension T_s of a liquid and the difference d between the densities of the liquid and its saturated vapour is $T_s = Cd^r$, where C and r are constants.

Show that the following figures are consistent with this formula, and find the values of C and r :

Surface tension T_s of water	71·2	68·0	64·5	61·0 dyne cm.$^{-1}$
Density difference d	0·9957	0·9880	0·9776	0·9649 gm. cm.$^{-3}$

21. The following figures give the velocity v at different times t of a body moving in a straight line :

Time t	0	1	2	3	4	5 sec.
Velocity v	0	55	75	85	89	91 cm. sec.$^{-1}$

Plot a velocity-time graph ; find the acceleration when $t = 2$ sec., and the total distance travelled during the 5 sec.

22. The distance travelled from rest, s, at different times, t, by a body falling freely are as follow :

Time t	0	1	2	3	4	5 sec.
Displacement s	0	16	64	144	256	400 ft.

Plot a displacement-time graph ; find from it the velocity at the end of each second of the motion, and plot a velocity-time graph from these results. What is the value of the acceleration as obtained from the velocity-time graph?

CHAPTER 2

VECTOR QUANTITIES

VECTORS

MANY physical quantities are represented by *directed* numbers, and the direction concerned, as well as the actual magnitude, must be specified before such a quantity is completely determined. Directed quantities are called **vectors**.

Examples are :

Displacement, which is distance from a chosen reference point *in a given direction*.

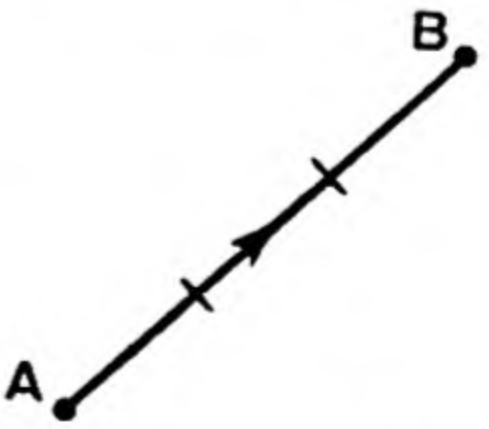
Velocity, which is rate of change of displacement, or speed *in a given direction* ; and **momentum**, which is mass × velocity.

Acceleration, which is rate of change of velocity.

Force, since the effect of a force is directed along a line called its *line of action* ; and **weight**, which is a vertically directed force.

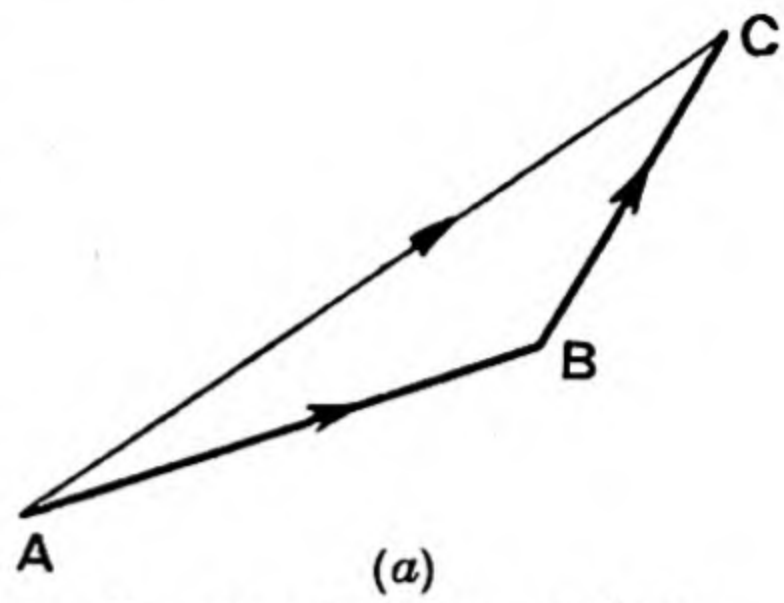
A vector can be completely represented by a straight line, of length representing the magnitude on a chosen scale, drawn in a direction representing that of the vector, with an arrow for complete identification. For example, the diagram represents a displacement of 3 miles in a north-east direction from the point A ; the line AB is 3 units long, is drawn in the appropriate direction, and is arrowed to indicate that the sense is $A \to B$, not $B \to A$.

Vectors are often indicated in print by an arrow over the symbols for the line representing them (as $\overrightarrow{AB}$), or by clarendon (or black-faced) type, as **AB**, or **s**. But at this stage it is best always to draw a diagram to depict the vectors, rather than use special symbols to write them.



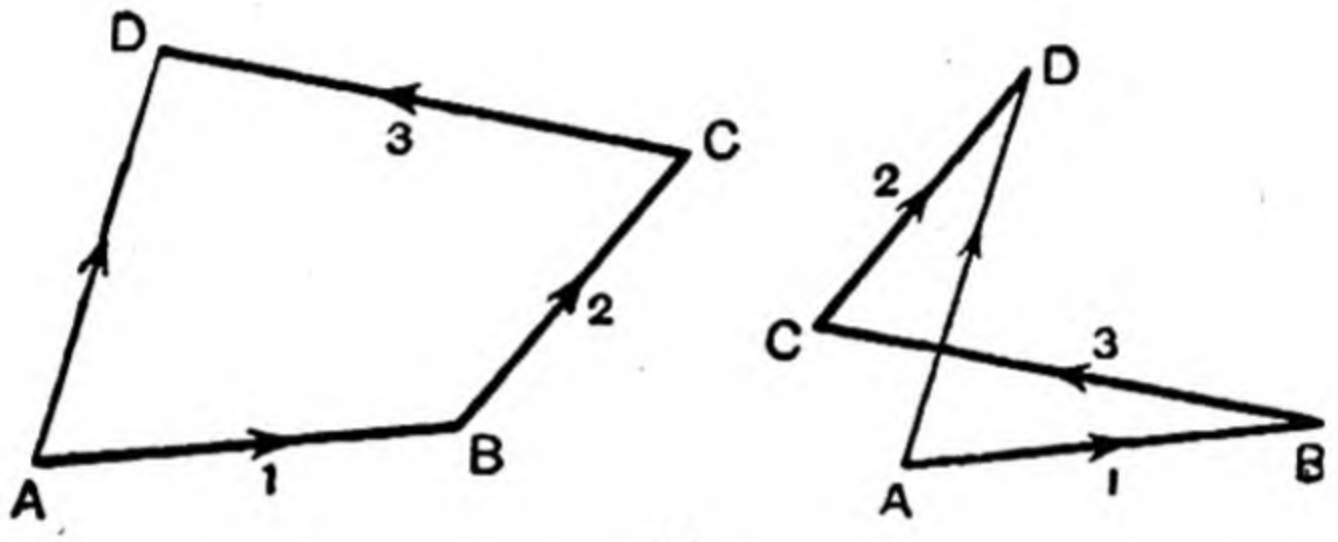
Representation of a vector.

Compounding vectors. The resultant of two or more vectors is the single vector to which they are collectively equivalent. The process of finding the resultant is called compounding, composition, or vector addition. The same rule holds for all vectors, so we shall consider the self-evident case of displacements.

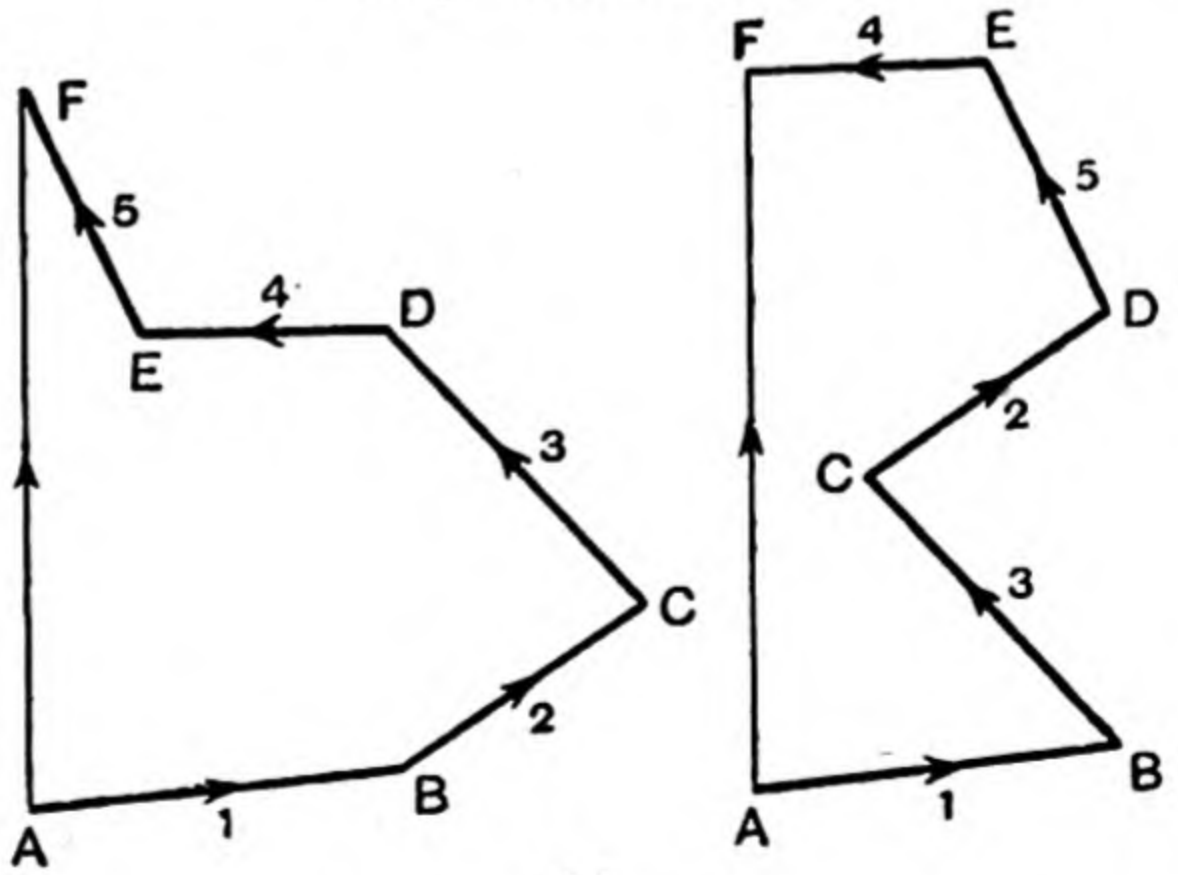


(*a*)
Resultant of two vectors.

The resultant displacement from A of a body undergoing two displacements, first from A to B and then from B to C is AC (Fig. *a*). Consideration of Figs. *b* and *c* shows that for any number of



(*b*)
Displacements in the order 1, 2, 3 or 1, 3, 2 give same resultant displacement AD.



(*c*)
Displacements in the order 1, 2, 3, 4, 5 or 1, 3, 2, 5, 4 give same resultant displacement AF.

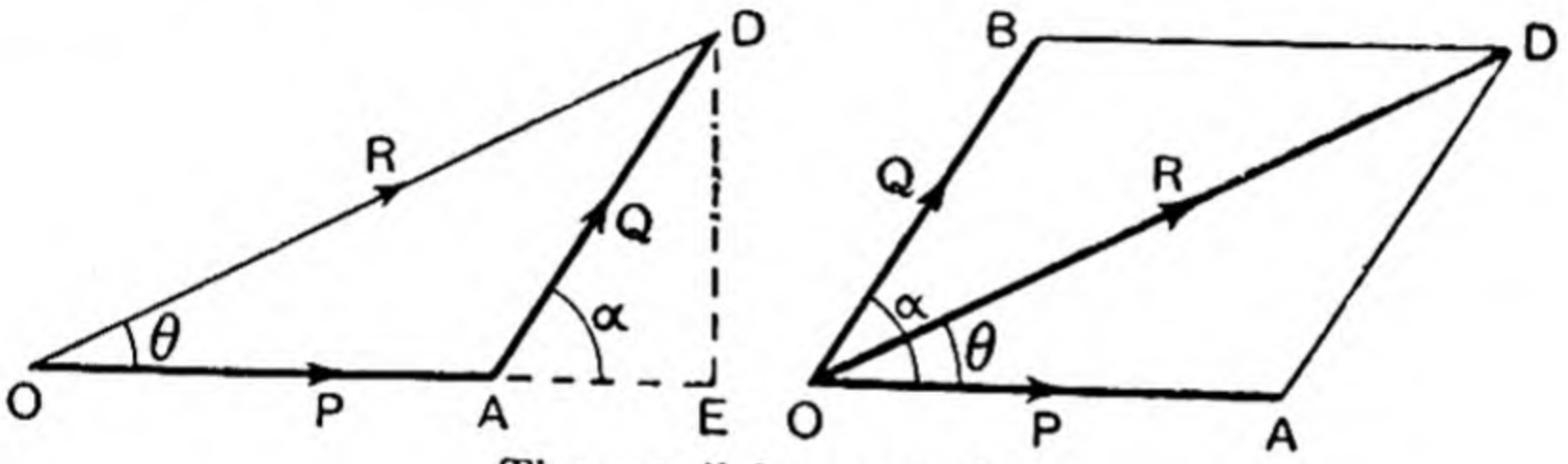
displacements, the resultant displacement is simply that of the finishing point from the starting point ; and it is represented by the line which makes the figure obtained by setting out the individual displacements a closed polygon. The order in which the individual displacements occur is immaterial ; indeed, if they take place simultaneously (as may occur, for example, to the hand of a passenger which he *waves* while walking across a *moving* ship) the result is the same, and they can clearly be drawn in any sequence.

With only two vectors, the closed figure formed by drawing the resultant is of course a triangle.

The general rules for compounding vectors are :

(1) Vectors must be compounded geometrically, not added arithmetically.

(2) If the vectors are set out in sequence to scale, in any order, the resultant is represented by the line completing the diagram as a closed figure.



The parallelogram rule.

Parallelogram rule. If OA and AD (see diagram) represent two vectors P and Q making an angle α with one another, OD, representing the resultant R, can be calculated using the cosine formula.

For, $$OD^2 = OA^2 + AD^2 - 2OA \,.\, OD \cos(180 - \alpha),$$

so $$R^2 = P^2 + Q^2 - 2P.Q \cos(180 - \alpha),$$

or, $$R^2 = P^2 + Q^2 + 2P.Q \cos\alpha.$$

The angle θ between R and P is given by

$$\tan\theta = \frac{DE}{OE} = \frac{Q\sin\alpha}{P + Q\cos\alpha}.$$

Now, the angle between P and Q, here given, appears as an exterior angle in the triangle. For this, and for another reason to appear later, it is often considered best to draw $OB = AD$ to represent Q, so that a completed parallelogram is formed, of which the diagonal through O represents R.

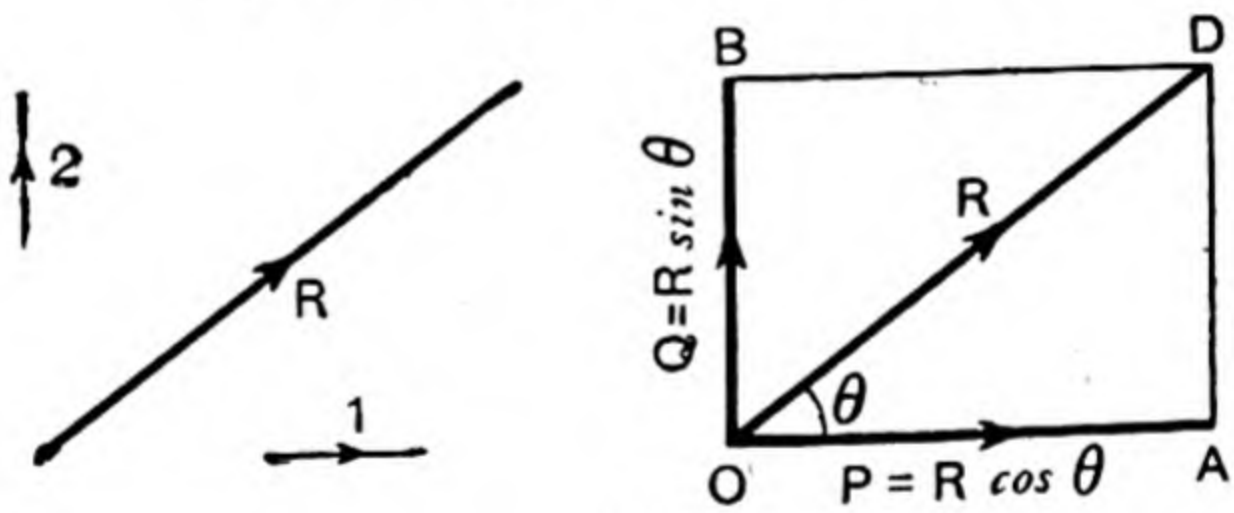
Thus, if OA and OB represent in size and direction two vectors P and Q, their resultant R is represented in size and direction by the diagonal OD of the parallelogram $AODB$.

Resolution of a vector. If P and Q are together equivalent to R, then R is equivalent to P and Q together. Thus we can resolve any given vector into two others in any two given directions, by working the parallelogram rule backwards—starting with the diagonal and drawing the adjacent sides in the chosen directions. This can always be done, whatever the directions. The most important case is when the two directions are at right angles to one another; the two vectors then obtained are called the **components** of R, or the **resolved parts** of R, in the given directions.

Thus, given R, to find the components in two directions 1 and 2 at right angles to one another, OD is drawn to scale to represent R (Fig. a), the rectangle $OADB$ is constructed with sides in the given directions, when OA represents P, the component in the direction 1, while OB represents Q, the component in the direction 2.

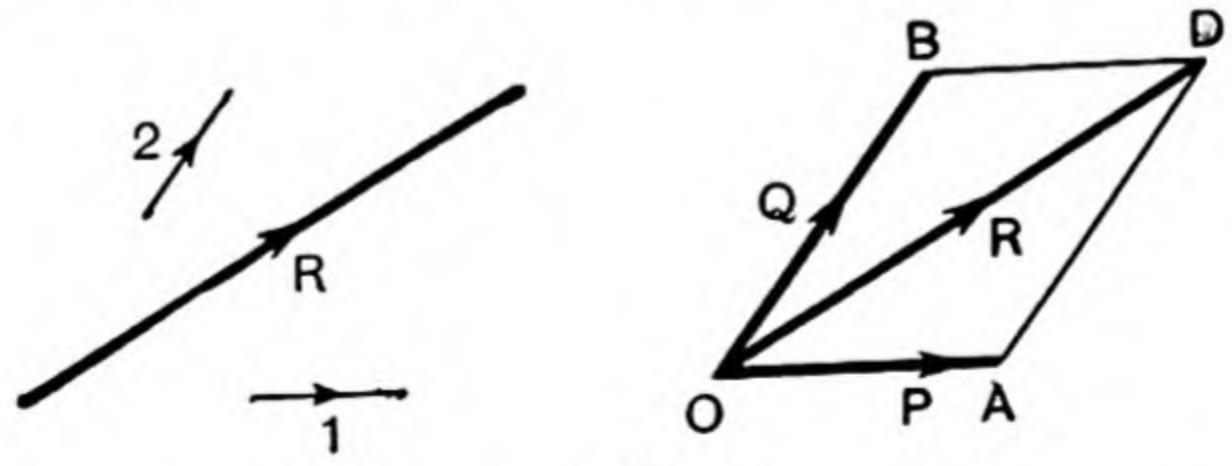
As $OA/OD = \cos\theta$ (referring to the figure),

$$P/R = \cos\theta, \quad \text{or} \quad P = R\cos\theta\,;$$



(a)

Resolution of R in directions 1 and 2 at right angles.



(b)

Resolution of R in directions 1 and 2 not at right angles.

and, as $OB/OD = AD/OD = \sin\theta$,

$$Q/R = \sin\theta, \quad \text{or} \quad Q = R\sin\theta.$$

It will be noted that if the directions are not at right angles to one another, a complete separation into independent vectors is not achieved; for in Fig. *b*, Q can itself be resolved parallel to and at right angles to P, and has itself therefore a component parallel to P.

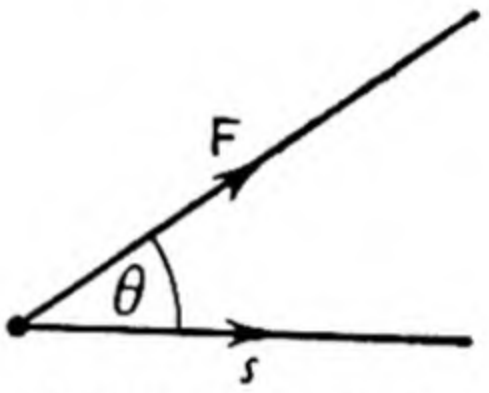
Scalars

Scalars are *non-directed* quantities, such as mass and volume (which are unquestionably undirected), length and area (which are usually considered apart from the direction in which they lie, but may sometimes be directed), and work and energy.

Scalars (of the same kind) may be added together according to the ordinary rules of algebra.

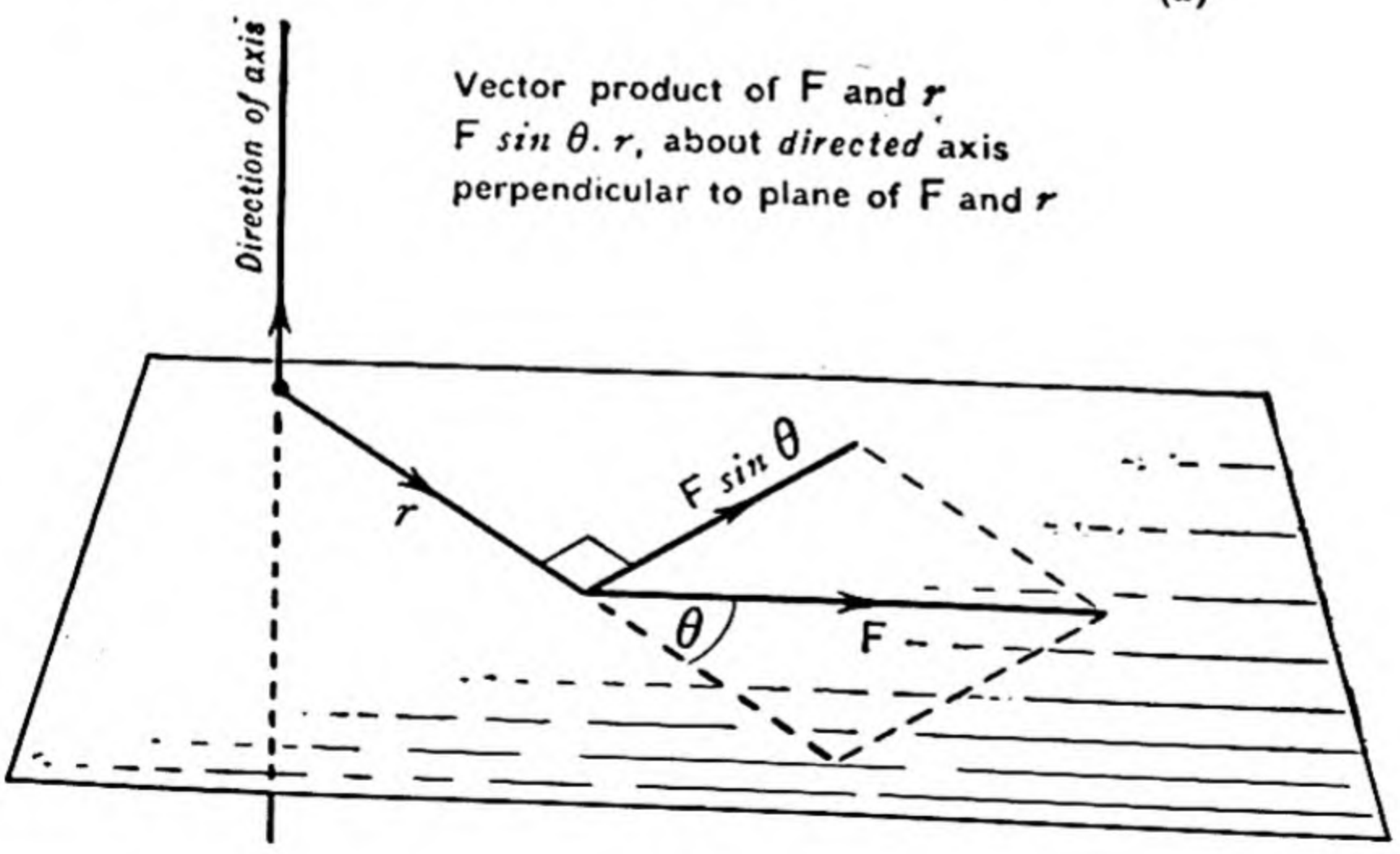
Many derived quantities are expressed as the product of a scalar and a vector; for example, momentum (= mass × velocity). All such derived quantities are vectors.

The product of a vector and a vector



Scalar product of F and *s*.
F *cos* θ. *s*

(*a*)



Vector product of F and *r*
F *sin* θ. *r*, about *directed* axis perpendicular to plane of F and *r*

(*b*)

may give either a vector or a scalar. Work, for example, is measured by the product (force × distance moved in the direction of the force); both are vectors, yet work is a scalar. This is because the direction of the two vectors *with respect to one another*, and not with respect to any independent reference point, is involved. In the figure below, the scalar product of F and s inclined at an angle θ is $(F \cos \theta) \times s$, or $Fs \cos \theta$.

On the other hand, the moment of a force about an axis (Fig. *b*)

(= force × perpendicular distance from axis to line of action of the force)

is a vector, since the perpendicular is directed *from a fixed reference line*. The direction involved in the product is that of the axis, as indicated in Fig. *a*.

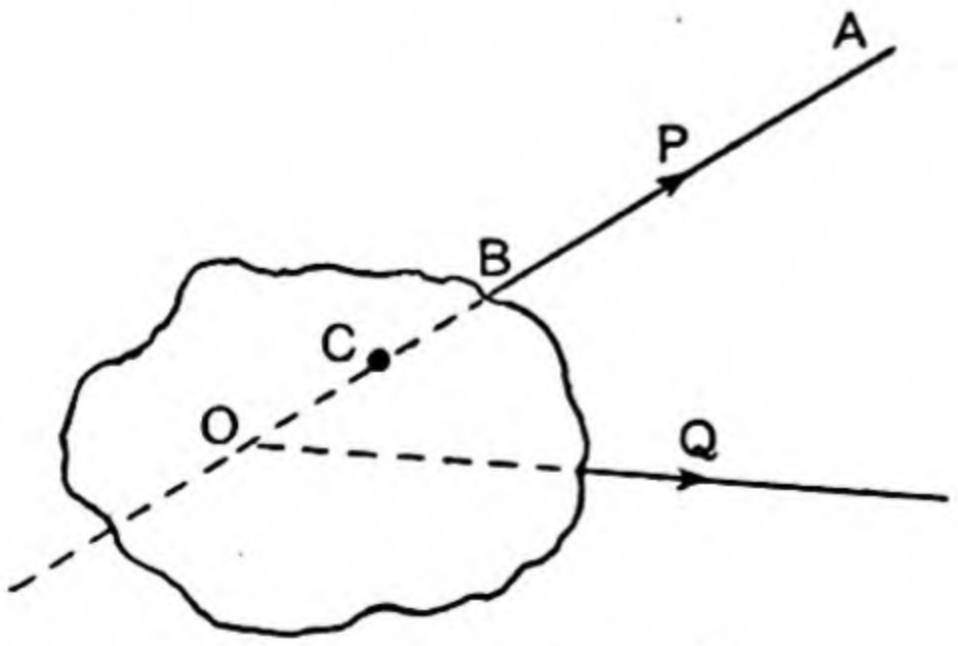
Here, the vector product of F and r is

$$F \sin \theta \,.\, r.$$

Forces

A force applied to a body by means of, for example, a pull P in string AB held in the hand at A and attached to the body at B (see diagram), acts on the whole body; its effect would be the same if the string were attached to any other point C in the line AB produced (the *line of action* of the force). The lines of action of two non-parallel forces P and Q applied to a body must meet in some point O (not necessarily on the body), and the two forces can be said to "act" at this point.

It can now be seen why a parallelogram, rather than a triangle, is preferred when forces are to be compounded; for the figure will represent not only the forces themselves, but also the point at which they act.

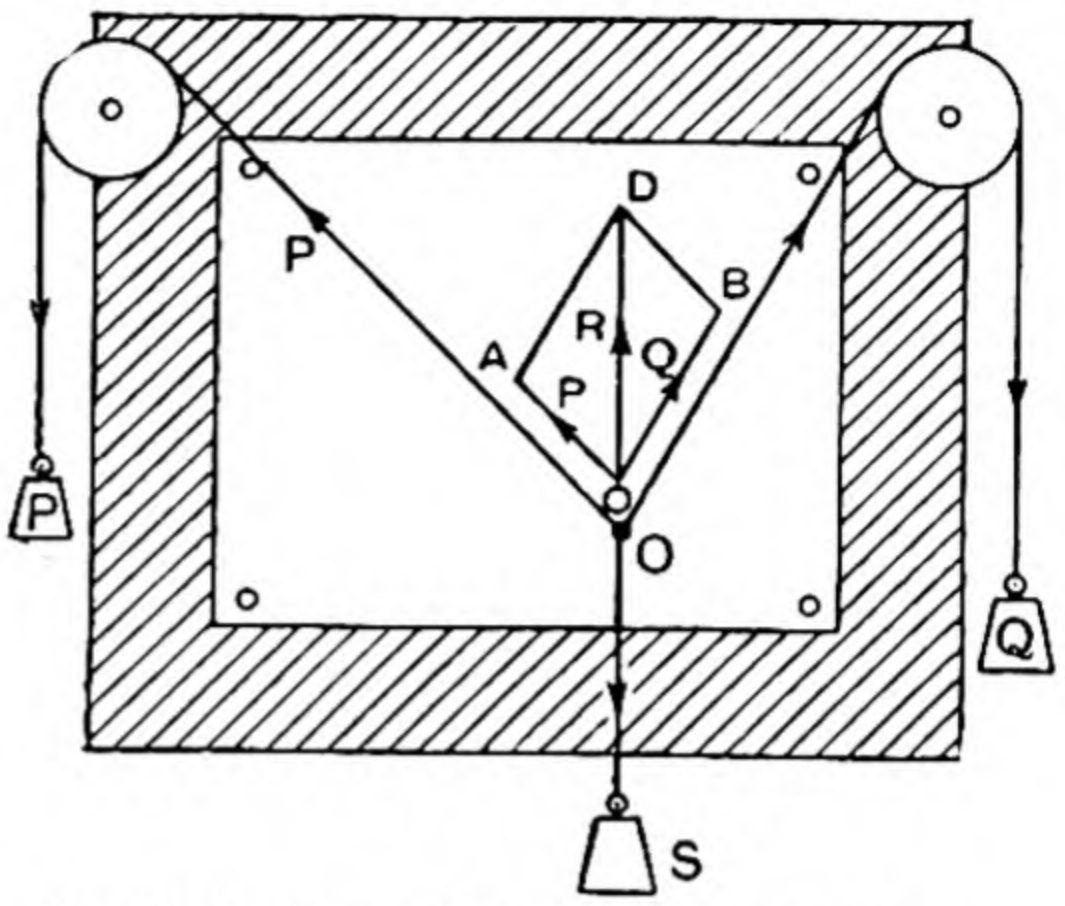


Lines of action of two forces.

Parallelogram of Forces Rule. If OA and OB represent in size and direction two forces acting at a point O, their resultant is represented in size and direction by the diagonal OD of the parallelogram $OADB$.

To verify the rule experi-

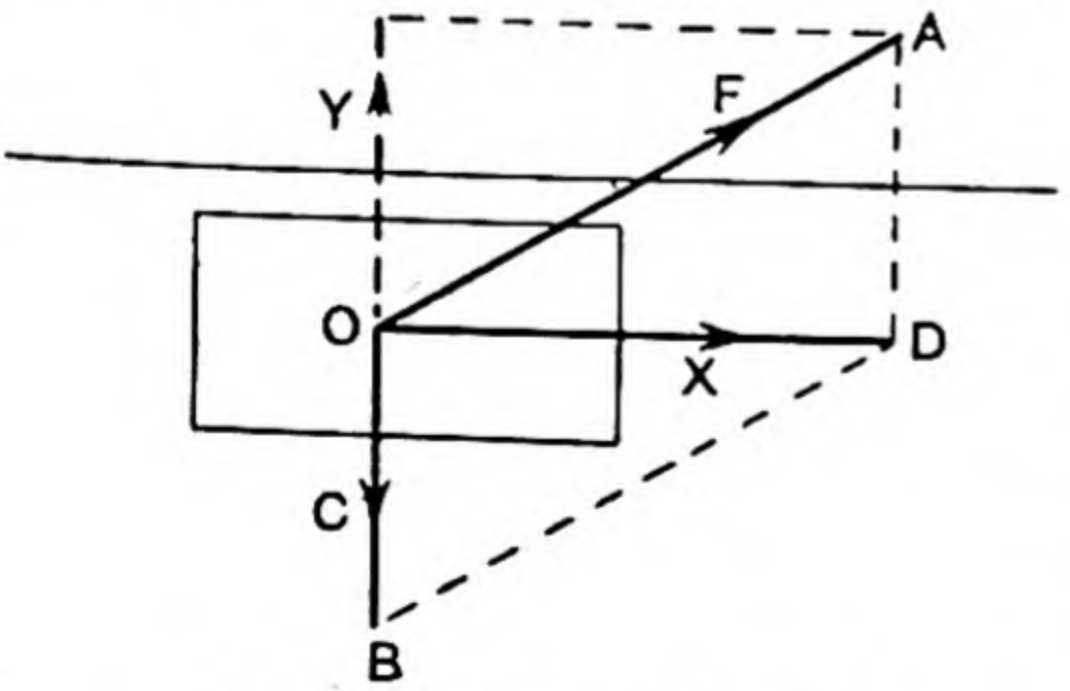
mentally, the apparatus illustrated is used. Forces P and Q, applied at O by strings over pulleys, are balanced by a third force S; S, called the **equilibrant** of the two forces, is equal in magnitude and opposite in direction to the resultant R. When the system is steady, the lines of the strings are traced on paper behind them, OA and OB are drawn to scale to represent P and Q, and the diagonal OD of the completed parallelogram drawn. Thus R is calculated, and found to be equal in magnitude and opposite in direction to the third force S, so that the law is verified.



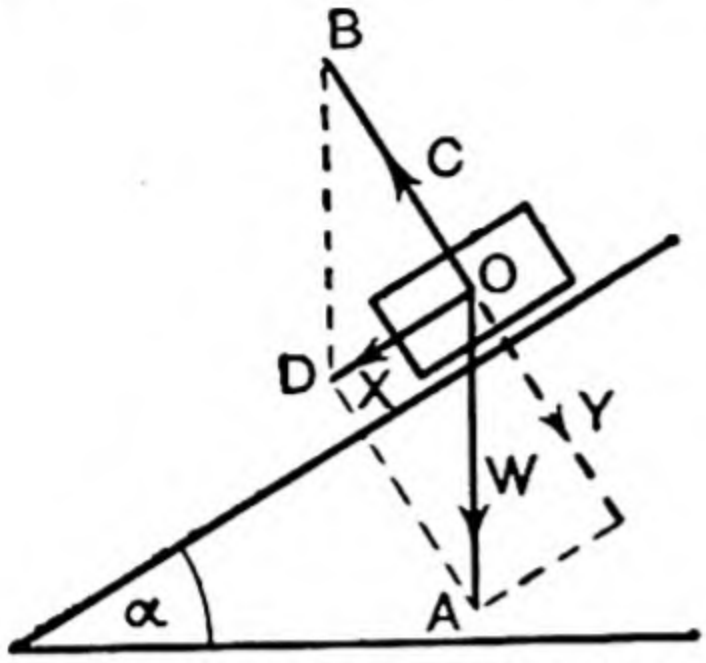
Parallelogram of forces demonstration.

Resolution by constraint. A proper explanation of the resolving action of constraints such as smooth rails, smooth planes, sails, and keels involves Newton's Third Law (p. 67).

Consider a trolley on smooth rails pulled by a horizontal force F at an angle AOD to the rails. As these are smooth, the constraining force C of the rails on the trolley acts at right angles to the rails; thus the force acting on the trolley is the resultant of the two forces F and C, represented by the diagonal OD of the parallelogram $OADB$. But C is equal and opposite to the force Y exerted by the trolley on the rails; and as both X and Y are provided by F, they must be the resolved parts $X = F\cos\theta$, $Y = F\sin\theta$, where θ stands for the angle AOD. The important point is, that though we have of course



Resolution by smooth rails.

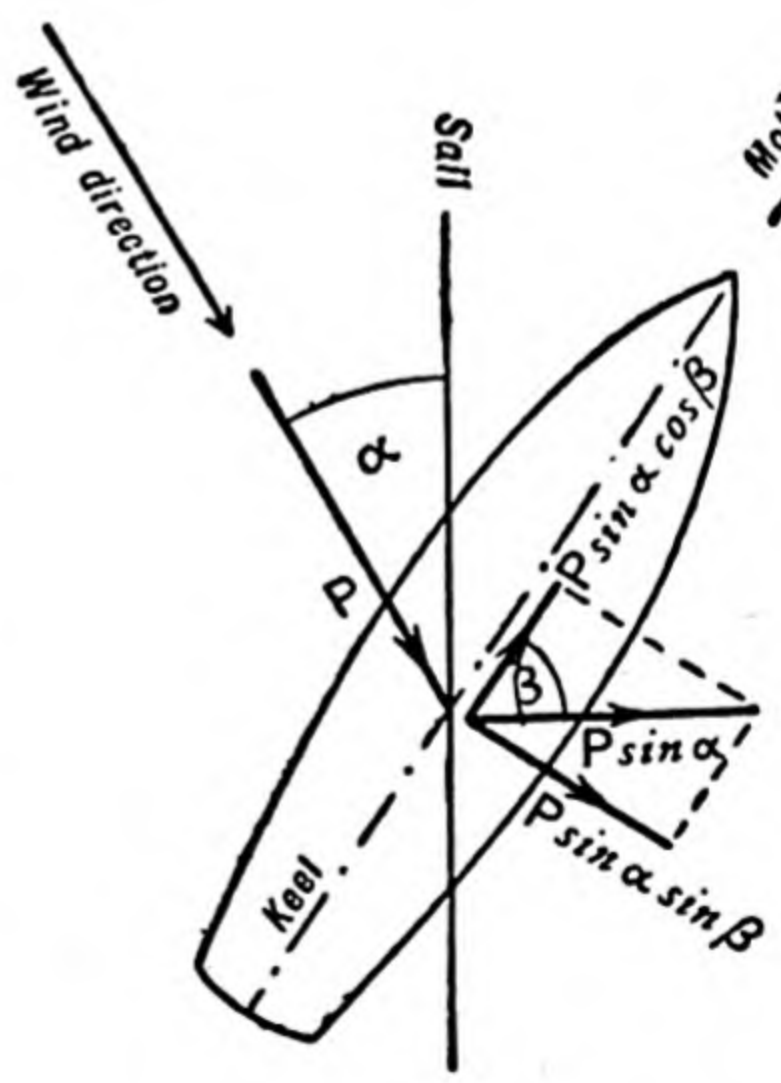


Resolution of weight by inclined plane.

resolved the force F acting on the trolley into *two* components at right angles, the resultant force X acting on the trolley is *one only* of the resolved parts—that parallel to the direction of the constraint.

Similarly, the *resultant* force acting on a body of weight W on a smooth plane inclined at an angle α to the horizontal is X, which is the resolved part of the weight W parallel to the constraining plane, whence $X = W \sin \alpha$.

When a sailing vessel is tacking against the wind, both sail and keel act as resolving constraints. The direction of the sail must lie between those of the wind and the keel. If the thrust of the wind is equivalent to a force P at an angle α to the sail, that normal to the sail is $P \sin \alpha$, and this is the only effective component; the keel now acts as a rail-constraint, and if β is the angle between the normal to the sail and the keel, the resolved part $P \sin \alpha \cos \beta$ in the direction of the keel is the resultant force producing motion. The forces $P \sin \alpha \sin \beta$ corresponding to the C and Y of the inclined plane (above) do not act in the same line, (since Y is at the sail and C at the keel) but give a couple (p. 48) causing the boat to heel over. The larger the keel, the more effective the constraint and counterweight it affords (it will be remembered from Arthur Ransome's stories that *Amazon*, with her centre-board, could sail closer to the wind than *Swallow* with her permanent six-inch keel).



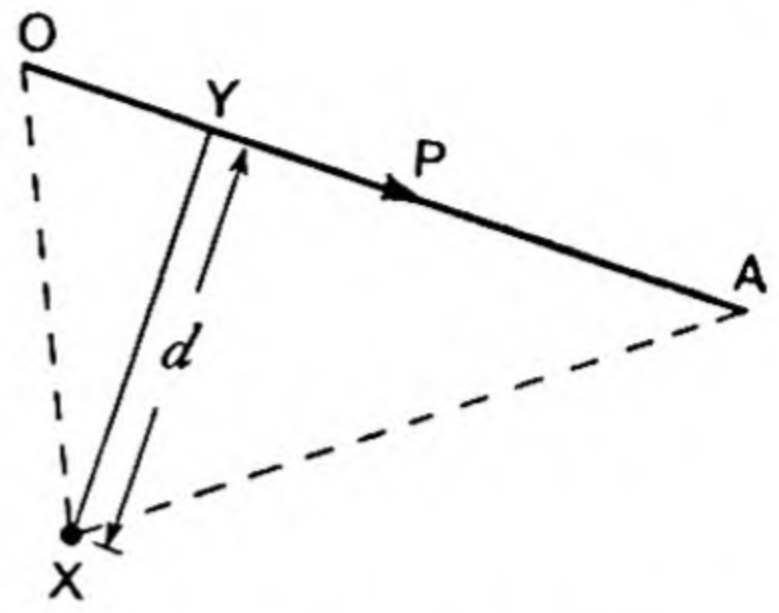
Action of sail and keel.

Moment of a force. The moment of a force about a point

(representing the intersection of a possible axis of rotation with the plane of the paper) is the tendency of the force to turn the body to which it is applied about that point. It is measured by the product

(*force* × *perpendicular distance from the point to the line of action of the force.*)

Moments are clockwise ↷ or anticlockwise ↶.



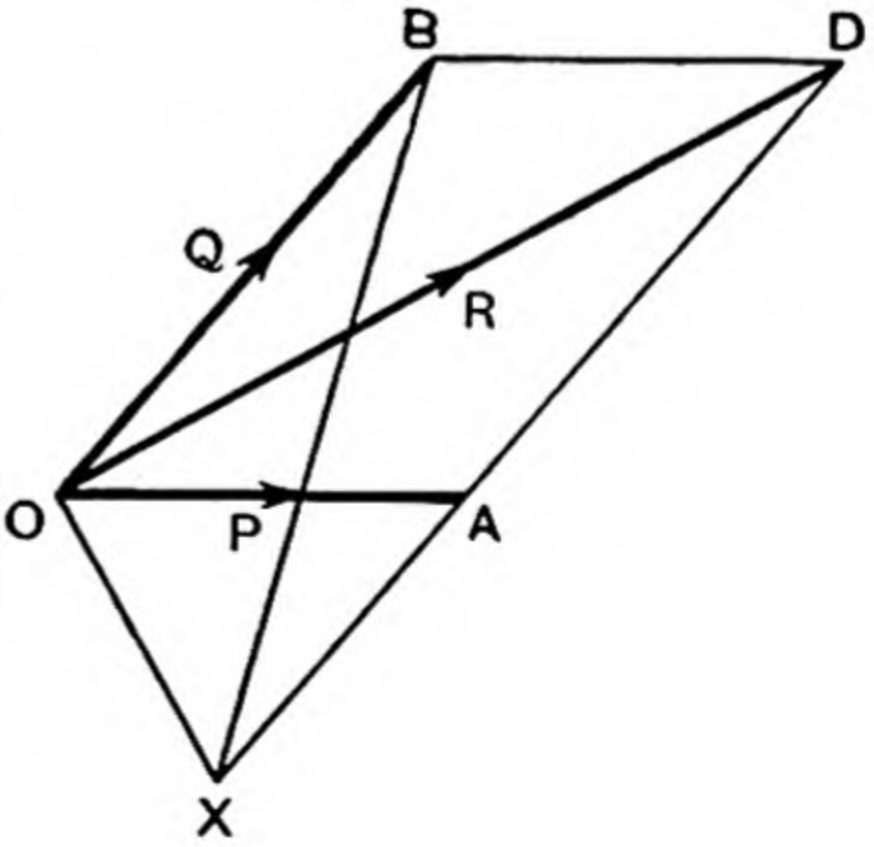
Representation of moment by an area.

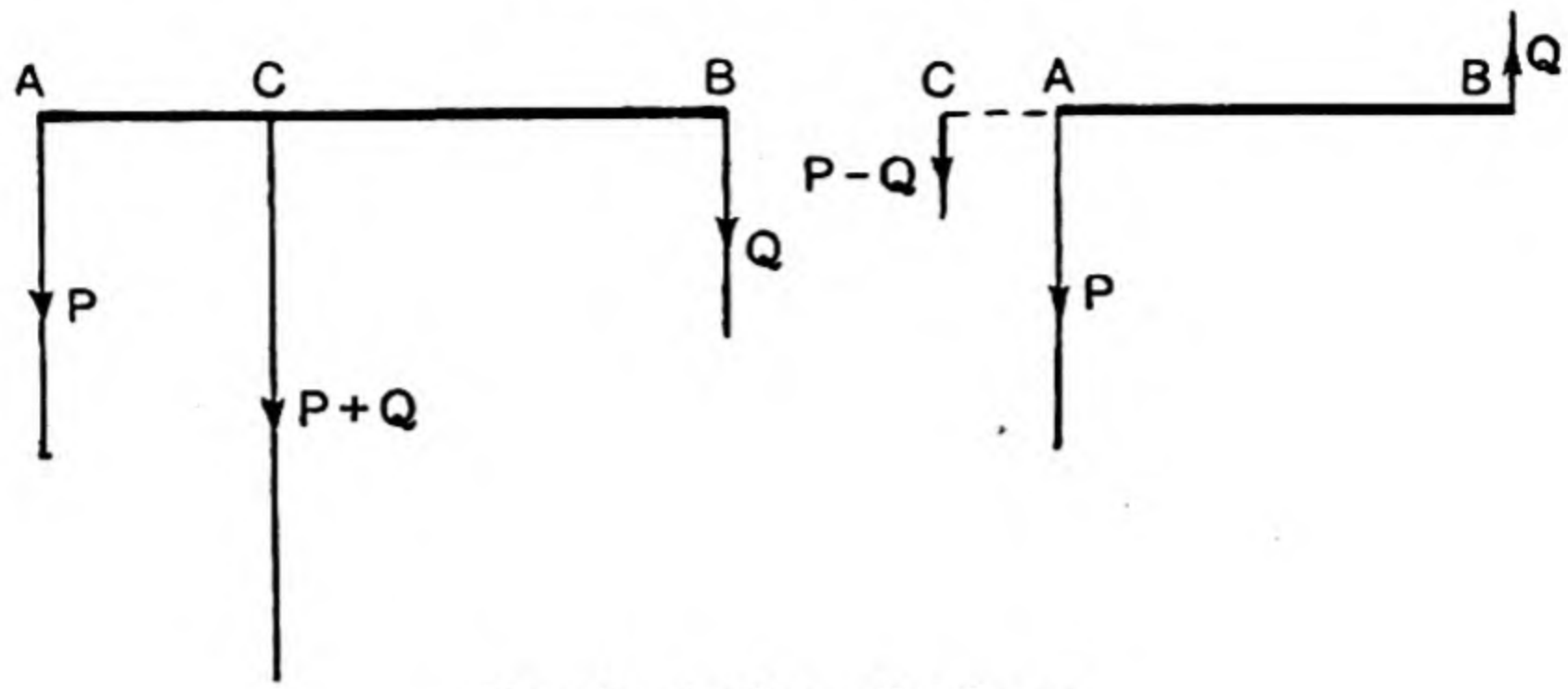
In the diagram, if OA represents a force P, the moment about the point X is $P \times d$, where d is the length of the perpendicular XY from X on to OA. As the area of a triangle is $\frac{1}{2}$(base × perpendicular height), the moment is represented to scale by twice the area of the triangle OXA.

The definition of the term resultant implies that the moment of the resultant of a set of forces about any point must equal the sum of the moments of the individual forces about that point—for otherwise the resultant is not their complete equivalent. This can be proved as a corollary of the parallelogram rule ; for the moments of P, Q, and R (see below) about the point X (chosen on DA produced for convenience) are each twice the area of the appropriate triangles OAX, XOB, XOD ; and it is easily proved that

$$\text{area } XOD = \text{area } XOA + \text{area } XOB.$$

Thus, although moment is a "vector product", moments about a given axis can be compounded by writing down the moments of individual forces, and then taking the sum of the moments. For this purpose clockwise and anticlockwise moments are considered as of opposite algebraic sign (for example, anticlockwise + ; clockwise −) and the *algebraic* sum is taken.





Resultant of parallel forces.

Parallel forces. The magnitude of the resultant of a set of parallel forces acting on a body is found by taking one sense as positive and the other as negative, and finding the algebraic sum. The line of action of the resultant must be parallel to those of the individual forces, and is determined by using the fact that it must be such that the moment of the resultant about any point is equal to the algebraic sum of the moments of the individual forces about that point.

Thus for P and Q acting at A and B in the same sense (see diagram), the resultant is $(P+Q)$ passing through a point C on AB such that (taking moments about A),

$$(P+Q) \times AC = Q \times AB.$$

Similarly, if P and Q are in opposite senses, the resultant is $(P-Q)$ and its line of action passes through C where

$$(P-Q) \times AC = Q \times AB.$$

Centre of gravity. A body can be subdivided into smaller portions almost as finely as we wish, and each of these small portions, taken alone, experiences a force towards the centre of the earth which is the weight of that small portion. The force which we call the weight of the whole body is really the resultant of a very large number of forces, namely, the weights of the individual particles comprising it, which we can regard as parallel.

For each position of the body, this resultant has a definite line of action which can (in theory at least) always be found by applying the principle that the moment of the resultant weight about any point equals the sum of the moments of the individual weights about

that point; and all possible lines of action of the total weight pass through a point called the **centre of gravity** of the body, which can be defined as follows:

The centre of gravity of a body is that point, fixed with respect to the body, through which the line of action of the whole weight always passes, for all positions of the body.

The term **centre of mass** is also used, and perhaps more appropriately; if the line of action of the weight always passes through the point, it is just as if the whole *mass* were concentrated at that point; and this effective concentration is not confined solely to the weight, since *for most dynamical purposes* the body acts as if the whole mass were concentrated at this point.

The position of the centre of gravity (often abbreviated as C.G.) of a completely symmetrical body made of uniform material is determined by symmetry. Thus the centre of gravity of a thin uniform rod is at its mid-point; of a triangular sheet at the point of intersection of the medians; of a disc and a sphere at their centres; of a cylinder half-way along the axis, and so on. If the body is symmetrical only about one axis, the centre of gravity must lie somewhere along this axis. If the body is composed of two portions of which the centres of gravity are at H and K, the centre of gravity G of the whole must lie in the line HK; for if HK is vertical, the moments of the individual weights about any point in the line are zero, as also must be the moment of the total weight.

The position of the centre of gravity of a composite body, or one that can be conveniently divided up into a limited number of simple symmetrical parts, is found as shown in the following examples. A large number of cases in which this subdivision is impracticable can be treated by calculus methods, but belong rather to the province of mathematics, though of course the same physical principles are employed. However intractable from the mathematical point of view the shape of a body may be, and whether it is uniform in composition or not, the centre of gravity can always be located experimentally.

EXAMPLE. Find the position of the centre of gravity of the shape shown on p. 46, cut out of thin uniform plywood.

Let unit area have unit mass, and hence unit weight. Divide the

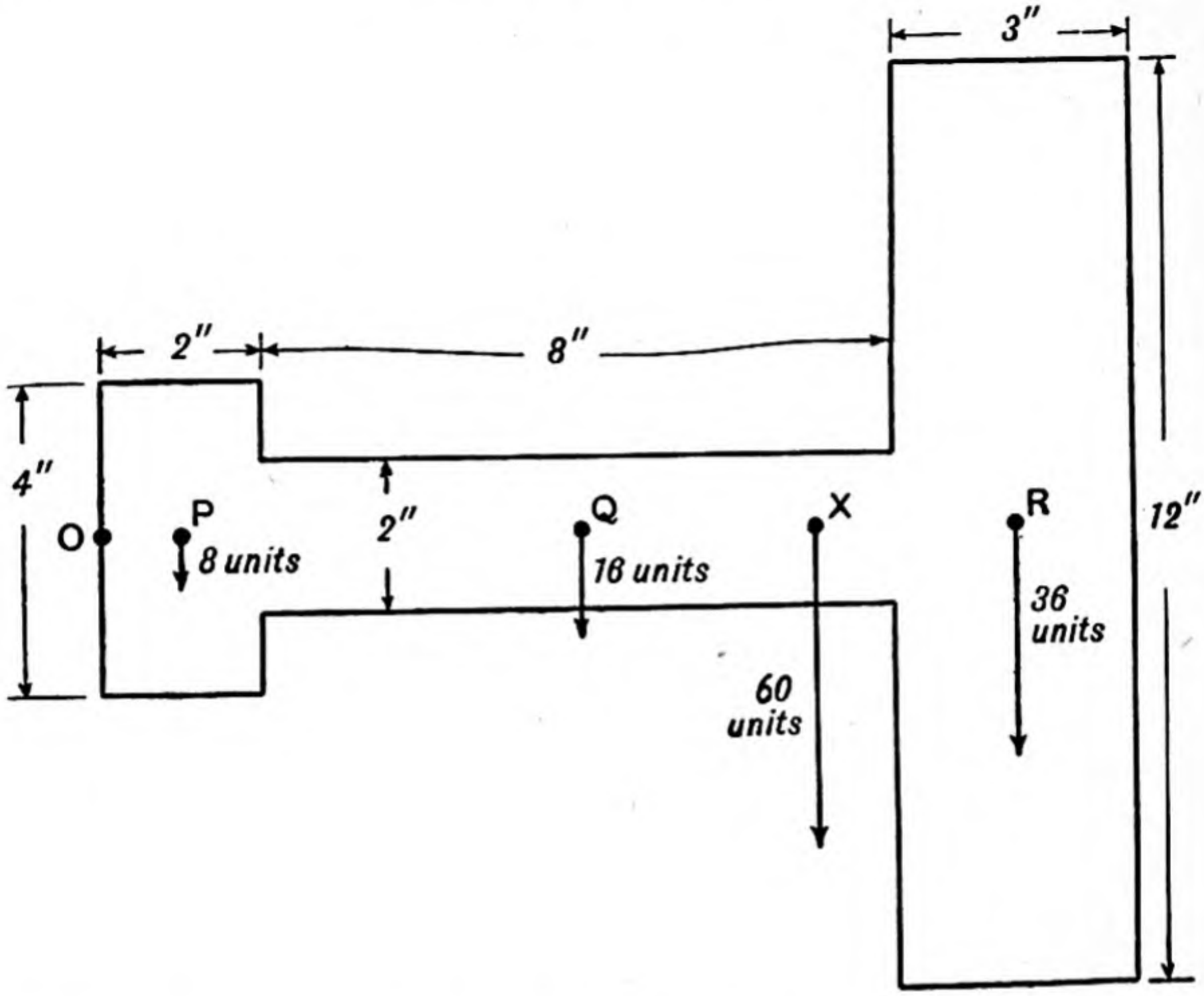


figure into three rectangles, with centres of gravity at P, Q, and R. 8 units of weight act from P, 16 from Q, 36 from R, and the total weight of 60 units acts from the C.G. X, at distance x from the point O, the mid-point of the left-hand edge, which we will take as reference point. The figure is symmetrical about the line $OPQR$, and X must lie on this line.

The distances concerned are : $OP = 1$ in., $OQ = 5$ in., $OR = 11{\cdot}5$ in., $OX = x$ in.

Take moments about O,

$$8 \times OP + 16 \times OQ + 36 \times OR = 60 \times OX,$$

so
$$8 \times 1 + 16 \times 5 + 36 \times 11{\cdot}5 = 60x,$$

$$502 = 60x,$$

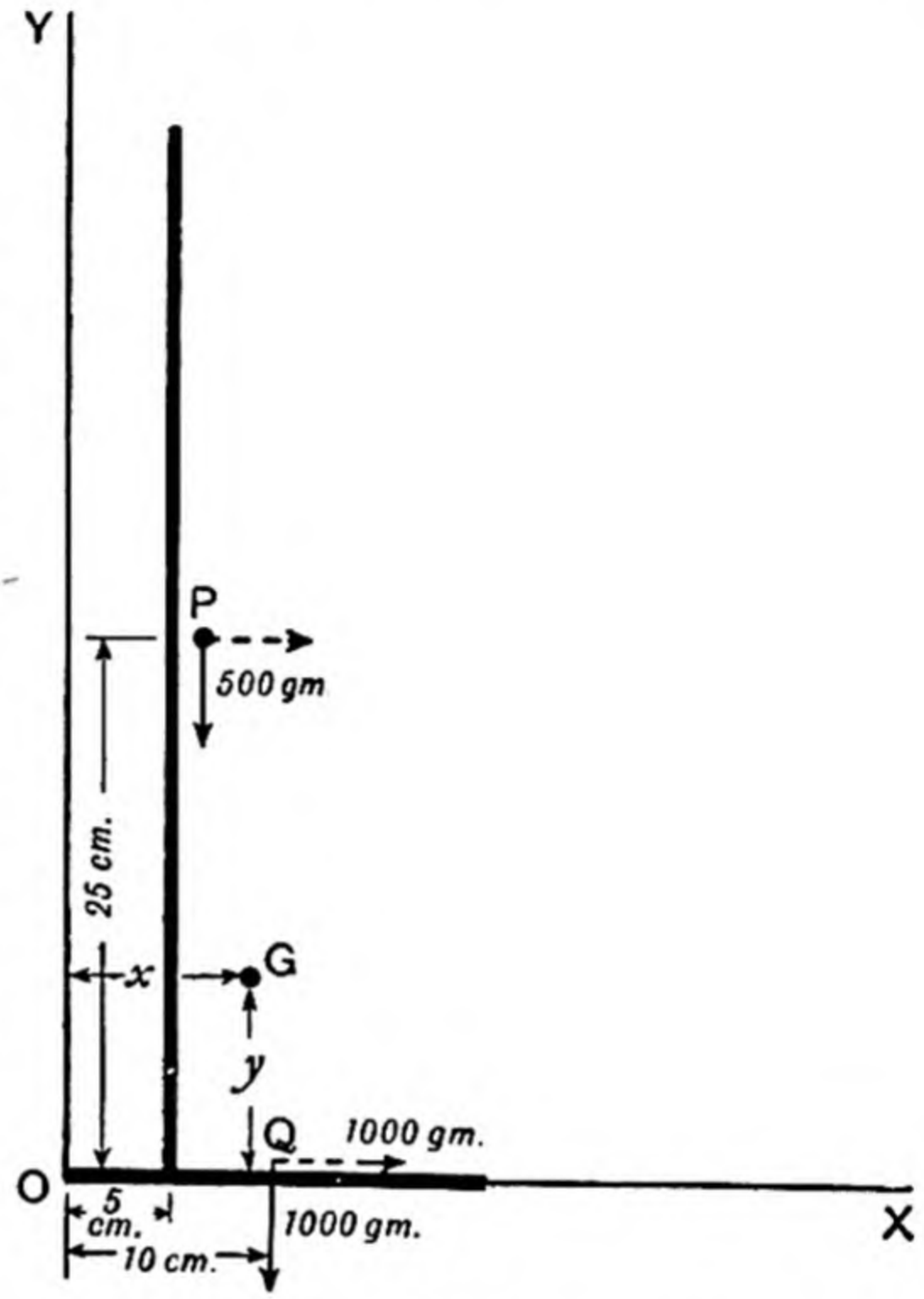
and
$$x = 8{\cdot}37 \text{ in. from } O.$$

EXAMPLE. A retort stand consists of a thin uniform rod of mass 500 gm. and length 50 cm., set in a thin rectangular base of mass 1000 gm. and dimensions 20 cm. × 15 cm., at a distance of 5 cm. from one of the shorter edges. Find the position of the C.G.

Take O (see diagram), the mid-point of one of the short edges, as reference point. The rod is placed on the line of symmetry of the base passing through O, and the figure represents a vertical section through this line; by symmetry, the C.G. lies in the plane of the figure.

Take co-ordinate axes, OX, OY as shown; the problem is to find the co-ordinates of the C.G. with respect to these axes. Let G be the point, with co-ordinates x, y. The total weight, 1500 gm. (not indicated in the figure) acts from G.

With the stand vertical as shown, take moments about O. The centres of gravity of the rod and the base, P and Q, are 5 cm. and 10 cm. respectively from OY, so



$$1500x = 1000 \times 10 + 500 \times 5 = 12{,}500$$

$$\therefore\ x = 8{\cdot}33 \text{ cm.}$$

Now imagine it tipped up so that OX is vertical and OY horizontal; the weights and their resultant all pass through the same points as before, but are now parallel to OX, as indicated by the dotted lines. The distances of P and Q from OX are 25 cm. and 0 cm. respectively.

Taking moments about O,

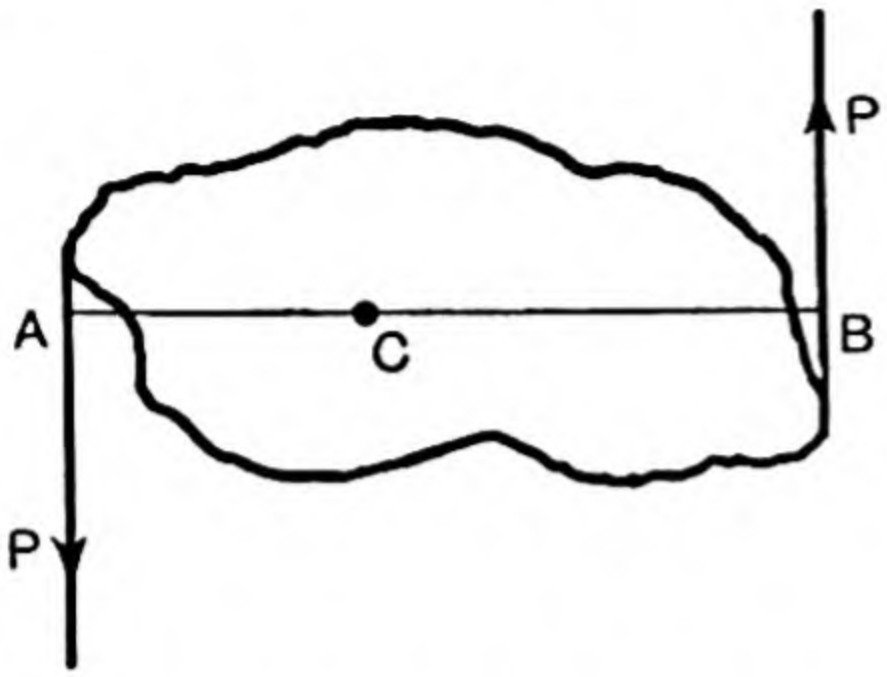
$$1500y = 500 \times 25,$$

so

$$y = 25/3 = 8{\cdot}33 \text{ cm.}$$

Thus the co-ordinates of G are 8·33 cm., and 8·33 cm.

Couples. The resultant of two equal and opposite forces which are not acting in the same line is apparently zero. This does not mean that they have no mechanical effect, of course, but simply



that *no single force can be found to replace them*. Clearly, the forces indicated in the diagram must affect the body to which they are applied, for both tend to rotate it in an anticlockwise direction.

A pair of equal parallel forces, acting in opposite senses, and not in the same line, is called a **couple**.

Take moments about any point C in AB (see diagram) ; the moment of P at A about C is $P \times AC$ ↺ ; the moment of P at B about C is $P \times BC$ ↻ ; the sum of the moments about C is

$$P \times (AC + BC) = P \times AB \circlearrowleft .$$

The same result is obtained for the sum of the moments of the two forces wherever the point C is taken, whether on AB, or anywhere else in the plane of the figure. Always it is $P \times AB$ ↺ , or

(*one force* × *perpendicular distance between the two forces*).

This is called the **moment of the couple**. AB is called the **arm** of the couple.

If the point C is kept fixed, and the lines of application of the forces are moved about in the plane in any way that preserves the same value of AB, the moment of the couple about C (and hence about any point in the plane) is unchanged.

A couple has several other important properties, the chief of which are summarised below :

1. A couple tends to produce pure rotation ; and whenever a body rotates, or tends to rotate, a couple is acting, or can be considered to act.

2. The moment of a couple is the same about any point in the plane of the lines of action of the forces, and is measured by the product :

(one force × arm of couple).

3. A couple can never be replaced by a single force (but can be replaced by any other couple of the same moment).

4. The resultant of a set of two or more couples is a couple, of moment equal to the sum of the moments of the individual couples.

5. A couple can never be balanced by a single force (but can be balanced by any other couple of equal moment and opposite sign).

These facts, which are usually encountered at a more advanced stage in Mechanics, should be grasped and understood now, otherwise the student will never grasp the real working of even the simplest rotating device.

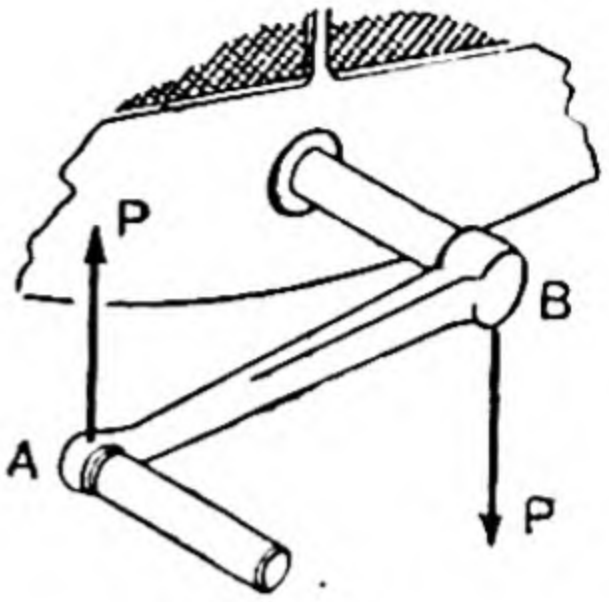


Fig. (*a*)

Experience suggests that the difficulty lies, not in understanding these statements, but in *believing* them, for in so many everyday examples we find rotation apparently resulting from the application of a single force—and it is unfortunately possible to solve the majority of elementary problems from this superficial point of view.

Take, for example, the starting handle of a motor-car. It is grasped and pulled, with a force as P at A in Fig. *a*. But when the force P is applied at A an equal and opposite reaction P at the bearing B is called into play, and this reaction, together with the force at A, comprises the couple that rotates the crankshaft.

Again, the steering-wheel of a motor-car is designed on the assumption that a couple will be applied to it by simultaneously pulling with one hand and pushing with the other (Fig. *b*); but if one hand only is used, a couple is likewise employed—the other force being the reaction at the bearing.

But, it may be asked, one can kick a football, or strike a golf ball, and set them spinning in spite of an earnest desire to the contrary—and no reaction can be invoked here. How is this? In such cases,

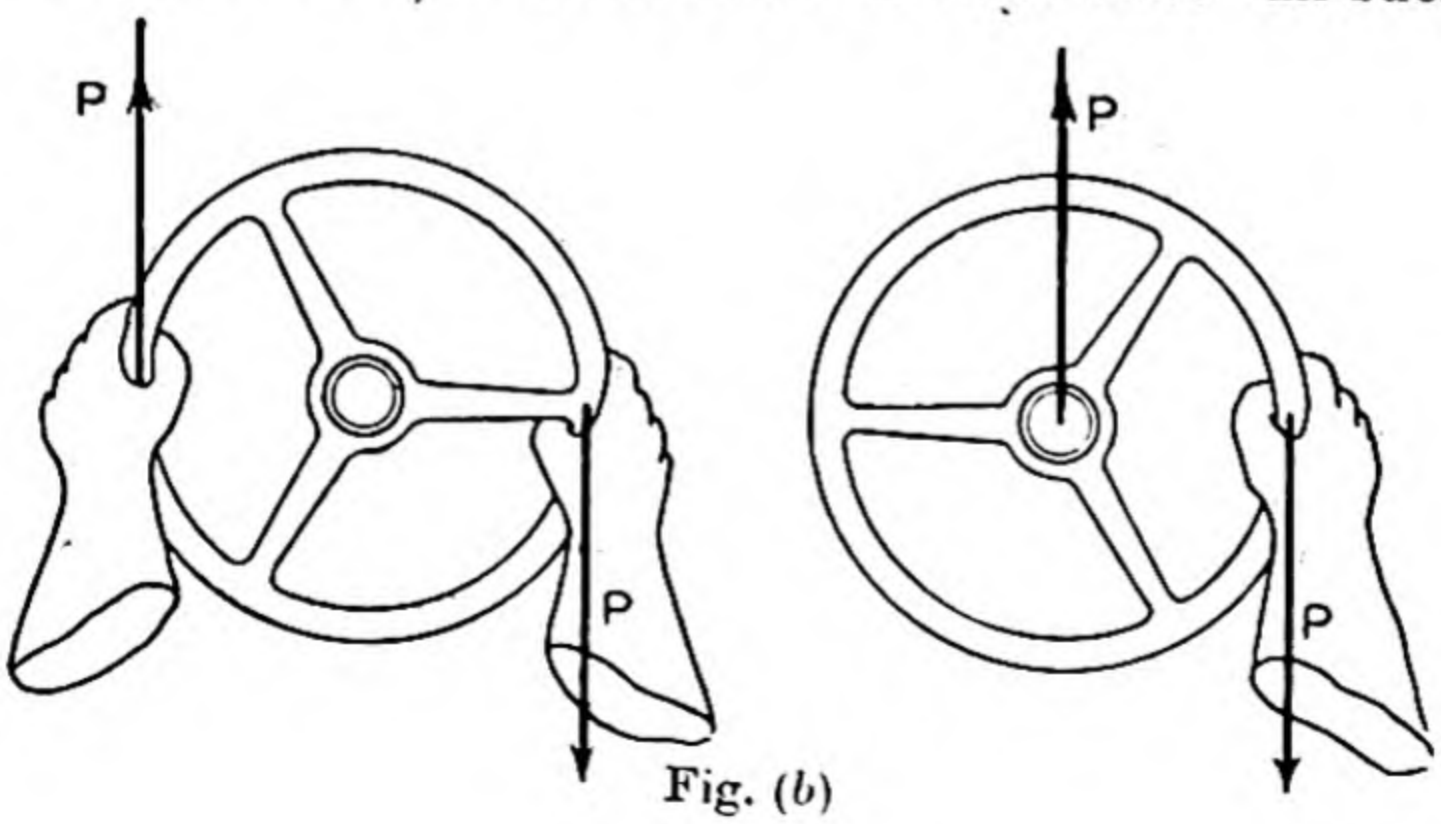


Fig. (*b*)

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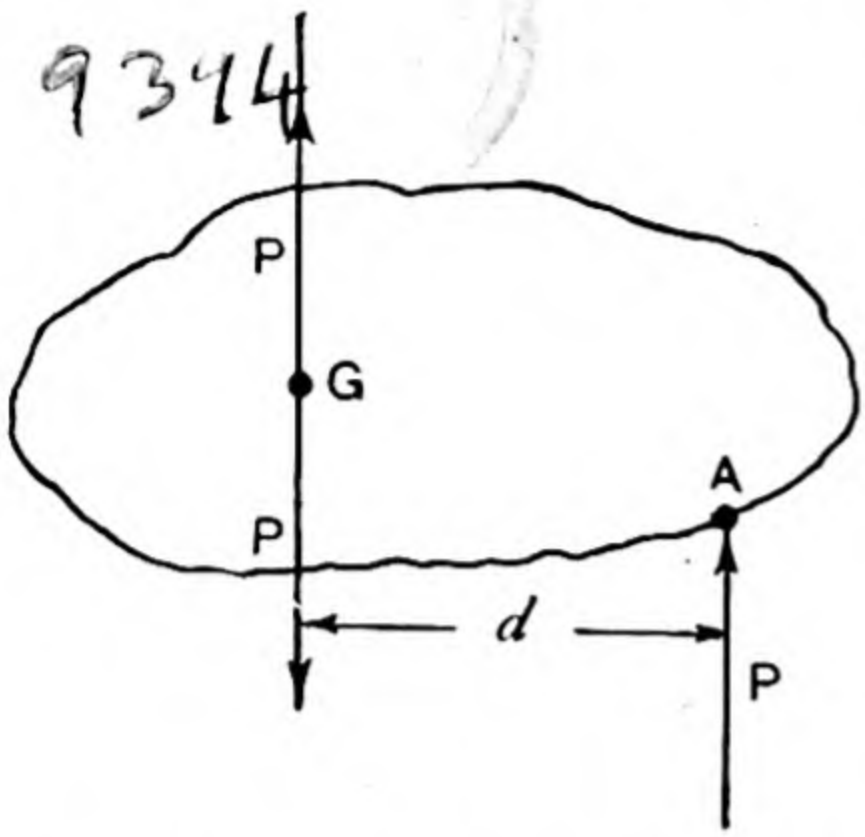


when the body as a whole is accelerated, a force really intended to act through the centre of gravity is unskilfully aimed ; yet the centre of gravity is accelerated, and so we can suppose that the misdirected force is after all effectively applied there.

In the diagram, if P is the applied force acting at A, we can put in two equal and opposite forces P at G, the centre of gravity. Since these obviously cancel one another out, we have not added or taken anything away from, the original conditions. But we now see that there is the equivalent of a force P acting through the centre of gravity G, and a couple of moment $P \times d$, where d is the perpendicular distance between G and the line of action of the applied force.

Thus, a single force applied to a rigid body along a line *not passing through the centre of gravity* of the body is equivalent to an equal and parallel force passing through the centre of gravity, together with a couple of moment equal to the product of the force and the perpendicular distance of the centre of gravity from its actual line of action. The line of action of the force may similarly be supposed transferred to pass through any point, accompanied by the appropriate couple.

Velocity as a Vector

Resultant velocity. At first, the suggestion that a body may possess several independent velocities sounds confusing, for we never see a body moving in several directions at once. But what we observe is a resultant velocity ; and if we consider, for example, an aircraft travelling at a certain velocity relative to the air (its " airspeed ") while the air itself moves as a wind with a certain velocity, then the velocity of the aircraft relative to the ground (its " groundspeed ") is the resultant of these two velocities.

Example. The airspeed of an aircraft headed due north is 200 m.p.h., in a south-west wind blowing at 40 m.p.h. Find the

magnitude and direction of the velocity of the aircraft relative to the ground.

In the diagram (in which the top of the page represents north), if OA represents the airspeed, and OB the wind velocity, the resultant R is represented by OD. Using the cosine formula, adapted as on p. 37,

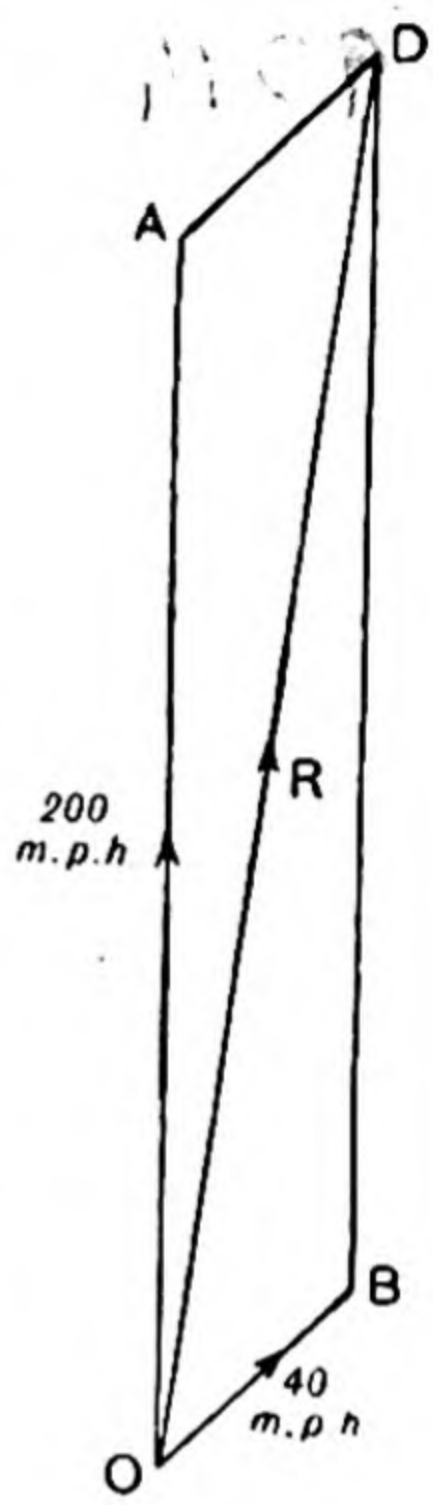
$$R^2 = 200^2 + 40^2 + 2 \,.\, 40 \,.\, 200 \,.\, \cos 45°,$$

$$\therefore \; R = 230 \text{ m.p.h.}$$

The tangent of the angle AOD between R and due north is

$$40 \sin 45° / (200 + 40 \cos 45°) = 0{\cdot}124,$$

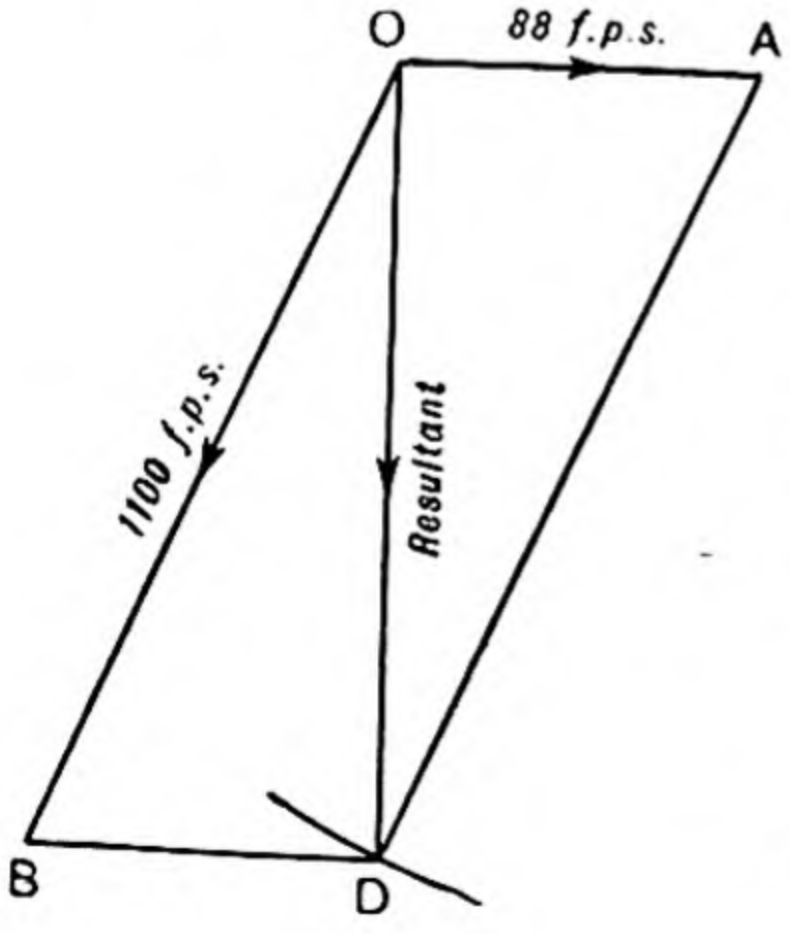
whence R is directed 7° 6′ east of north.



A similar type of problem occurs with sound. The velocity of sound is its " airspeed ", and the actual velocity in a wind between source and observer is the " groundspeed ".

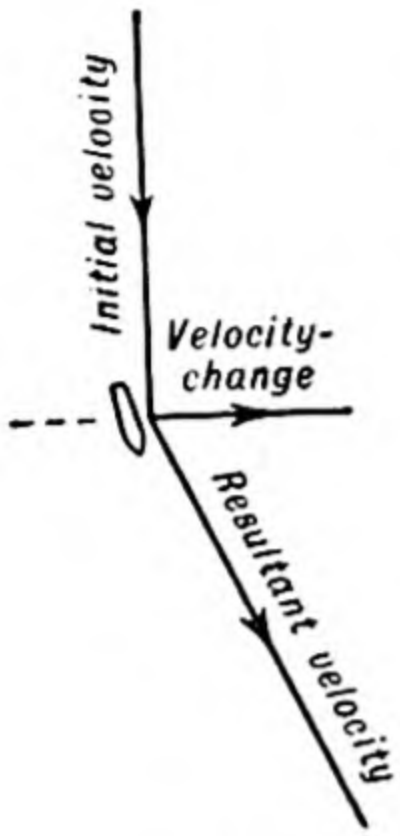
EXAMPLE. An observer is 5 miles due south of a siren ; the wind blows from west to east at 60 m.p.h. The velocity of sound is 1100 ft. per sec. Find the time taken for the sound to reach the observer.

By construction (see diagram), the velocity is the diagonal OD of the parallelogram $OADB$, of which adjacent sides are represented by OA (60 m.p.h. = 88 ft. per sec.) and OB (1100 ft. per sec.). The direction of OB is not given, so the figure is constructed by first laying off OA, and then with centre A and radius representing 1100 ft. per sec. describing an arc to cut the north-south line through O at D. The scales are rather ill-matched, and it is probably best to calculate ; if v be the velocity between source and observer,

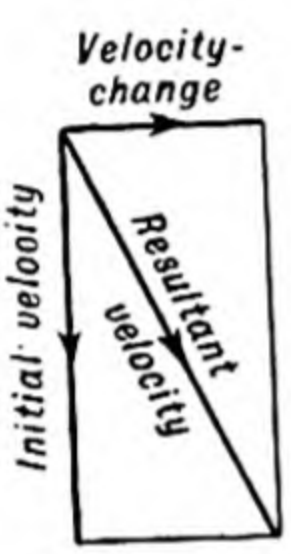


$$v = \sqrt{1100^2 - 88^2} = 1096 \text{ ft. per sec.},$$

and the time taken is $5 \times 5280/1096$
$= 24{\cdot}09$ sec.



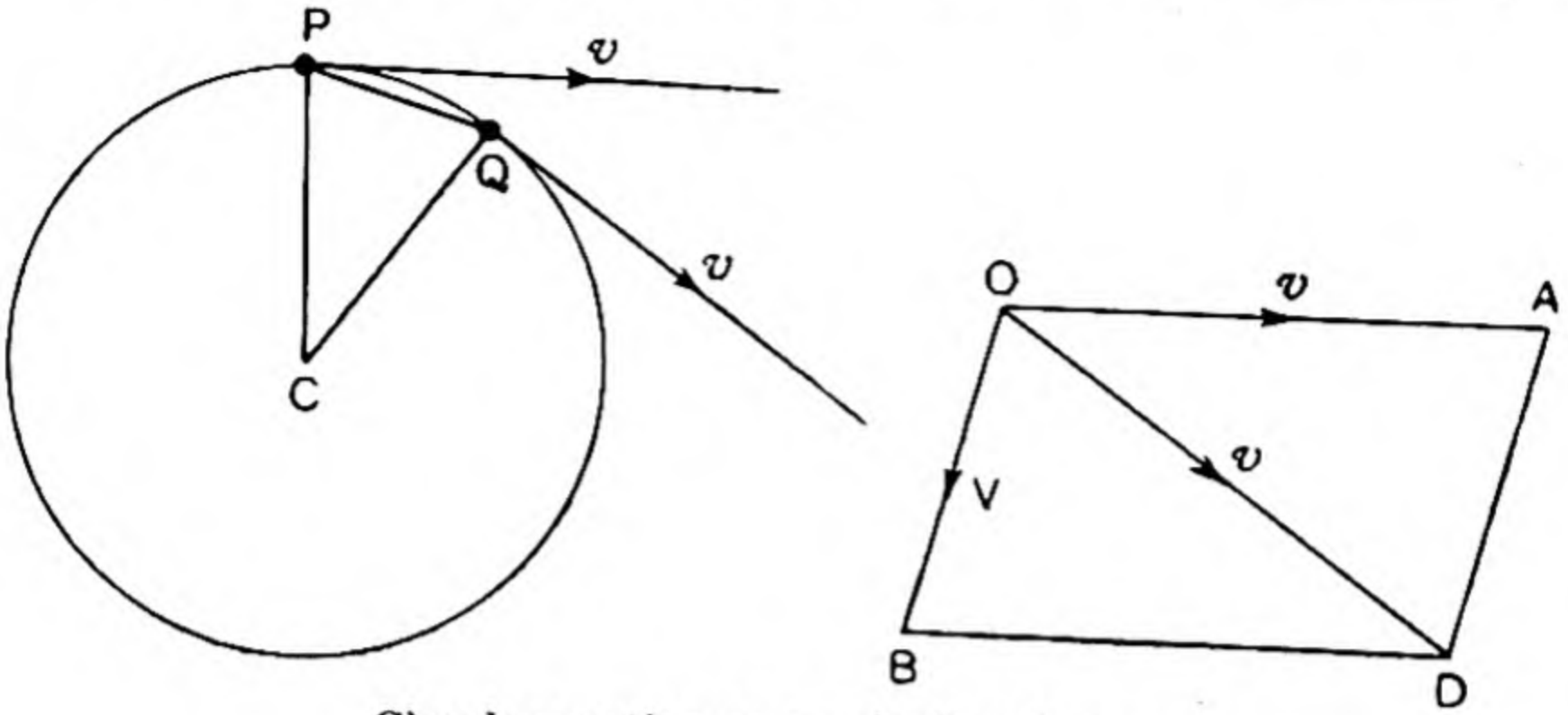
Velocity-change problems. A body moving in a given direction may acquire, either suddenly or gradually, a velocity in another direction due to the action of a force in the second direction. For example, a cricket ball struck in a direction at right angles to its path (say towards "point") suffers a change in velocity in the direction of the blow. Its subsequent velocity, in the direction of "third man", is the resultant of the original velocity and the velocity-change applied at the moment of impact (see diagrams).



Or, a body projected horizontally with a certain velocity u cm. per sec. suffers a continuously acting vertical acceleration g cm. per sec. per sec., so that after t sec. it has acquired a vertical velocity gt cm. per sec., without any alteration of the original horizontal velocity u. The actual velocity of the body at time t sec. is then the resultant of the independent horizontal and vertical velocities; that is, $\sqrt{u^2+g^2t^2}$, at an angle to the horizontal the tangent of which is u/gt.

The case of a body describing a circular path at uniform speed combines the features of both the previous examples; for it experiences a velocity-change at right angles to its motion, just as the cricket ball did; and this velocity-change is operating continuously. Before considering this in detail, it is well to call to mind some familiar examples of bodies describing circular paths, and reflect that this circular motion is only possible if there is some means of applying a force steadily *towards the centre of the circle*—it may be a string, or road friction, or rails and flanged wheels, or (as in the solar system where the planets describe orbits which are in most cases very nearly circular) gravitational attraction. That is, although the *speed* is constant, in every case the *velocity* is changing, and there is a continuously operating velocity-change directed at right angles to the path and *towards the centre*; this is due to an externally applied force acting towards the centre of the path.

Consider a body moving with uniform speed v cm. per sec. in a circular path of radius r cm. Let us find the velocity-change occurring in a short time interval τ seconds.



Circular motion as a velocity change.

Let P and Q (see diagram) be the positions of the body at the beginning and end of the interval and let the lines labelled v, v represent in length and direction the velocities at P and Q. What velocity-change V applied to v at P gives as resultant v at Q?

Construct a parallelogram of velocities, with OA representing the velocity at P, and OD as diagonal the resultant velocity, that at Q. The side OB then represents that velocity which, compounded with the velocity at P, gives the velocity at Q; it is the required velocity change, V.

Now OA and OD are parallel to the tangents at P and Q; thus the angle AOD is equal to the angle between these tangents, which also equals PCQ. Hence the angle BDO is equal to the angle PCQ, and the triangles BDO and PCQ are similar.

So
$$\frac{OB}{OD}=\frac{PQ}{QC},$$
$$\therefore\ \frac{V}{v}=\frac{PQ}{r},$$
or,
$$V=\frac{v\,.\,PQ}{r}.$$

Now, if τ is a very short time interval, the straight line PQ is very nearly equal to the distance PQ along the arc, which is $v\tau$.

Thus
$$V=\frac{v\,.\,v\tau}{r}=\frac{v^2}{r}\,.\,\tau.$$

This is the change in velocity in time τ, directed towards the centre.

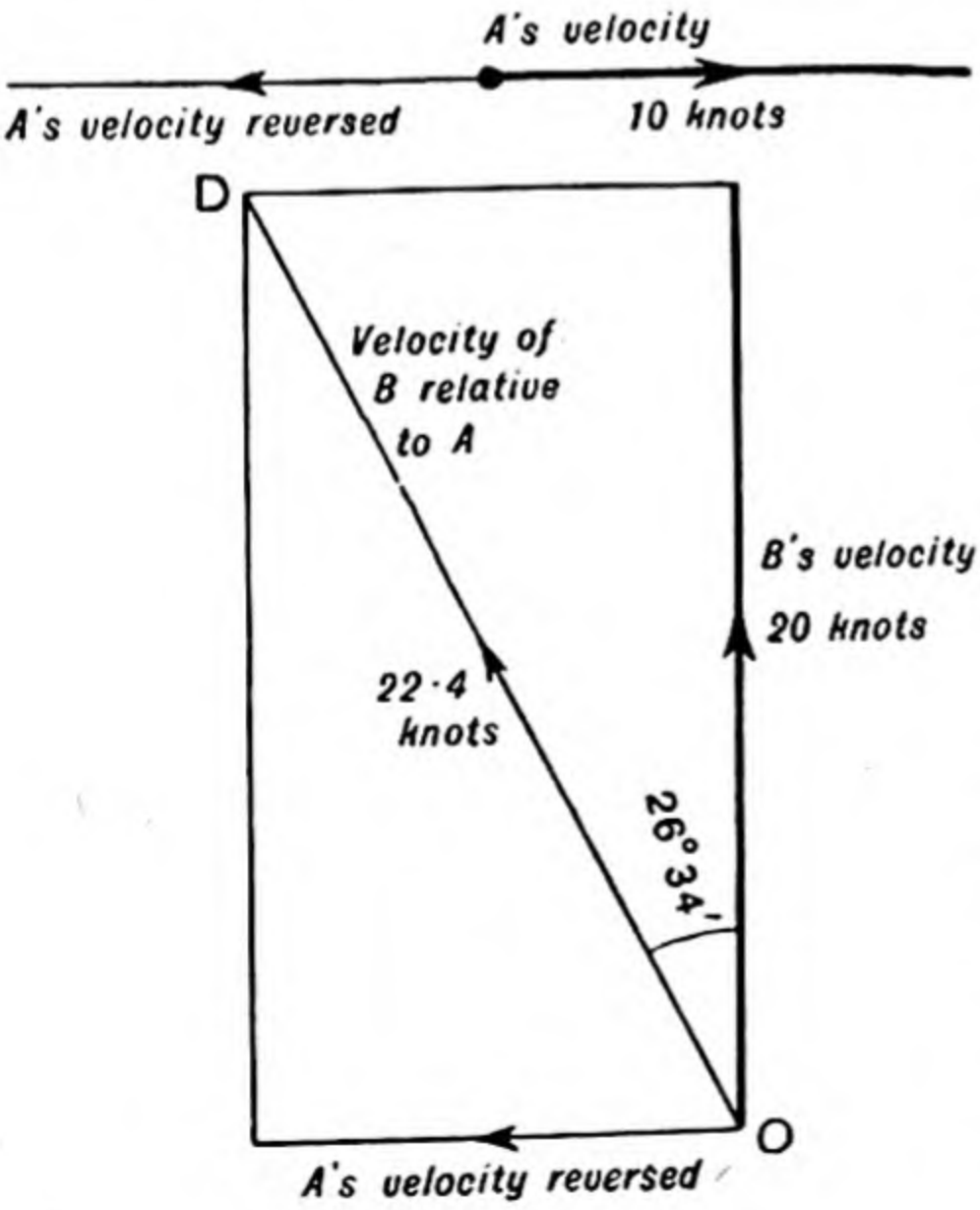
The rate of change of velocity is V/τ, or v^2/r, directed towards the centre.

Hence, a body describing a circle with uniform speed experiences an acceleration towards the centre of the circle of magnitude $\frac{v^2}{r}$, where v is the speed, and r the radius of the circle.

Relative velocity problems. The essential thing to realise in dealing with problems on relative velocity—that is to say, the velocity of one moving body as it appears to an observer who is also moving—is the observer's point of view. *He supposes himself to be a fixed reference point, that is, at rest.* We can achieve this for him, on paper, by giving him a velocity equal and opposite to his actual velocity. The relative velocity is then preserved unchanged if we also give the same velocity (the observer's velocity reversed) to the observed. The observed body now has two velocities, and the resultant of these is the velocity of the observed relative to the observer.

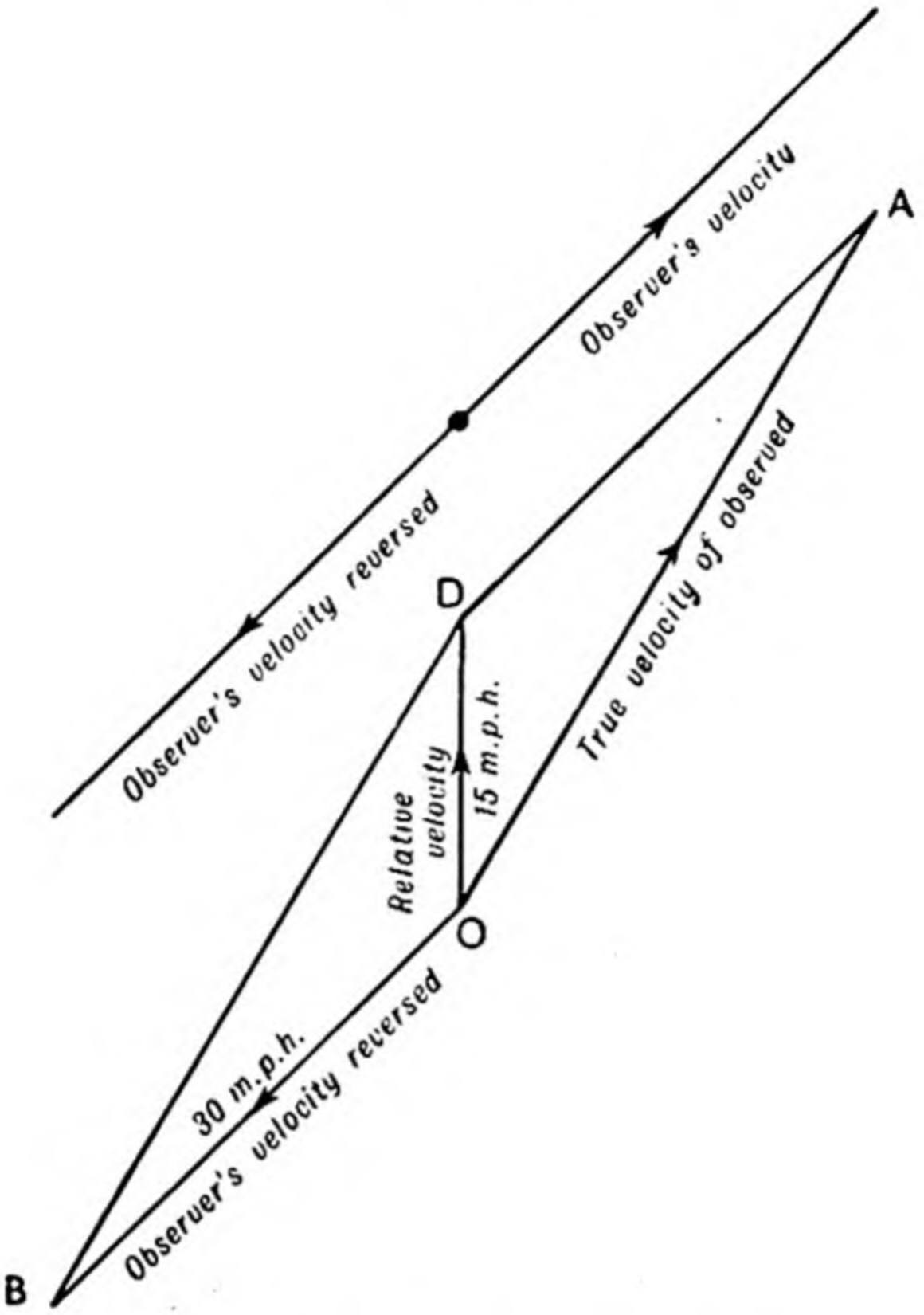
EXAMPLE. A ship A steaming at 10 knots due east sights another vessel B sailing at 20 knots due north. Find the velocity of B relative to A.

Bring A to rest, by applying a velocity of 10 knots due west. Now to keep the relative velocity unchanged, apply the same velocity to B; find the resultant of the two velocities of B, namely, its own 20 knots due north, and the 10 knots due west we have applied. This is represented by OD in the figure, and is calculated to be 22·4 knots, in a direction 26° 34′ west of north.



EXAMPLE. To an observer travelling at 30 m.p.h. in a north-easterly direction, a ship appears to be travelling at 15 m.p.h. due north. Find the true speed and course of the ship.

Here (p. 55) we first apply to both observer and observed



the observer's velocity reversed. Drawing the parallelogram for the observed, the 15 m.p.h. (which is a resultant) is drawn in as the diagonal OD, the side OA giving the true velocity. The answer, by calculation, is 41·97 m.p.h., in a direction 30° 22′ east of north.

EXAMPLES ON CHAPTER 2

1. Explain the rule for the compounding of vectors.

A man walks 600 yards on a compass bearing of 47°, then 500 yards bearing 128°, then 300 yards bearing 209°, then 400 yards bearing 283°. Find his resultant displacement.

2. Forces P, Q, R, S, all in the same plane, act at a point O ; the angle between P and Q is 75°, that between Q and R is 100°, and that between R and S is 90°. Find the resultant, if each force equals 5 kgm. wt.

3. Explain what is meant by resolving a force into its components in two given directions.

A roller weighing 500 gm. is placed on a smooth slope inclined at 30° to the horizontal. Find the effective force acting down the slope, and at right angles to it.

4. What do you understand by the moment of a force about a point, and the moment of a couple?

A cord, nailed to a point P on the edge of a pulley of radius 20 cm., is pulled with a force of 500 gm. wt. in a direction making 40° with the tangent at P. Find the moment of this force about the axis of the pulley, the moment of the couple acting on the pulley, and the magnitude of the single force which, applied tangentially at the rim, would give rise to a couple of the same moment.

5. Find the resultant of each of the following sets of parallel forces:

(a) 7 lb. wt. and 15 lb. wt., in the same sense, their lines of action being 11 ft. apart.

(b) 12 kgm. wt. and 8 kgm. wt., in opposite senses, their lines of action being 1 metre apart.

(c) 5 lb. wt. at A, 7 lb. wt. at B, and 6 lb. wt. at C, all in the same sense, and 9 lb. wt. at D in the opposite sense, where $AB = 2$ ft., $BC = 3$ ft., $CD = 2$ ft.

6. Show that a single force applied at a point X can be replaced by an equal force acting at another point Y, together with a couple.

At a certain instant, a uniform beam AB which is 2 metres long and weighs 6 kgm. is at an angle of 30° to the horizontal and experiences parallel forces of 3 kgm. wt. and 4 kgm. wt. acting vertically upwards at A and B. Find the resultant force acting on the beam, and, if this is regarded as equivalent to a force acting through the centre of gravity of the beam together with a couple, find the magnitude of the couple.

7. A boat travelling at 3 metres per second sets course across a stream flowing at 5 metres per second. Find the resultant velocity of the boat.

8. A car travelling at 20 m.p.h. due north turns a corner and continues at 20 m.p.h. due west. Find the change in velocity. If the time taken is 3 sec., find the average acceleration.

9. A cricket ball bowled at 40 ft. per sec. is struck and travels on at 60 ft. per sec., at 120° to the original path. Find the change in velocity.

10. Show that a body describing a circular path of radius r with speed v experiences an acceleration v^2/r towards the centre of the circle.

If the speed of the moon in its orbit is 3340 ft. per sec., and the radius of this orbit, supposed circular, is 240,000 miles, find the acceleration of the moon towards the centre of the orbit.

11. A beam of electrons, traversing a cathode-ray tube at 10^{10} cm. per sec., is caused by the application of a magnetic field to describe a circular arc of radius 40 cm.

Calculate the acceleration of each particle towards the centre of the circle.

12. If masses of 1, 2, 3 and 4 lb. are placed at the corners of a square of 10 in. side, taken in order, calculate the position of the centre of gravity. (O. & C.)

13. From a thin square plate $ABCD$ of uniform density, and with centre E and sides of 8 in., the triangle AED is cut out. Determine the position of the centre of gravity of the figure which is thus obtained. (O. & C.)

14. A plank of uniform material and cross-section weighs 100 lb. and is 10 ft. long. It is supported at one end and at a distance of 3 ft. from the other end. Find the greatest load that can be placed at the end which overhangs without upsetting the plank. (O. & C.)

15. A telescope consists of four concentric tubes, each 10 in. in length, their masses being 4, 3, 2 and 1 lb. Find, by calculation, the position of the centre of gravity. (O. & C.)

16. $ABCD$ is a square of thin uniform cardboard, of 10 cm. side, from which a square $AXOY$ of 5 cm. side is cut away. Find, by a graphical construction or otherwise, the position of the centre of gravity of the remainder. (O.)

17. Explain the terms *moment of a couple, centre of gravity.*

Explain carefully why a body, pivoted so that it can swing freely, comes to rest with the centre of gravity vertically below the point of support.

Four forces, $P = 10$ lb. wt., $Q = 7$ lb. wt., $R = 12$ lb. wt., and $S = 8$ lb. wt., act in cyclic order in the same sense along the edges of a wooden square of 6-inch side; find the moment of the couple required to keep the body from rotating.

18. Two forces, one of 60 lb. wt. and the other of 80 lb. wt., are applied at the same point, the angle between their lines of action being 48°. Find the size of their resultant and the angle its line of action makes with the smaller of the two forces. Explain your method fully. (O.)

19. State the principle of the *parallelogram of forces.* How would you verify it by experiment?

Explain why, although two given forces can have only one resultant, a given " resultant " can be " resolved " into many different pairs of components. (O.)

20. Explain what is meant by *resolving a force* in a specified direction.

Describe and explain how you could determine experimentally the components of a force in two directions perpendicular to each other.

Find, graphically or otherwise, the resultant of forces of 100 gm. wt. acting due east, 141·4 gm. wt. acting north-east, and 141·4 gm. wt. acting north-west, the forces all acting at one point. (O. & C.)

21. A ship at sea sends out simultaneously a wireless signal above the water and a sound signal through the water, the temperature of the water being 4° C. These signals are received by two stations, A and B,

25 miles apart, the intervals between the arrival of the two signals being $16\frac{1}{2}$ sec. at A and 22 sec. at B. Find the bearing from A of the ship relative to AB. The velocity of sound in water at $t°$ C. $= 4{,}756 + 11t$ ft. per sec. (N.)

22. Define relative velocity.

A boatman, whose speed of rowing is u, crosses a river of width b to a point at a distance c upstream and returns. Find the time of the double journey if the speed of the stream is V. (C. S.)

23. Two bodies, A and B, are moving with constant velocities in different directions. Show how to find the velocity of B relative to A.

The upper part of the wind-screen of a car is 12 inches wide from top to bottom and overlaps the lower part by 1 inch. If the car is travelling horizontally at 30 m.p.h., and rain is falling vertically at 10 m.p.h., what is the maximum angle to the vertical of the upper section of the wind-screen so that no rain enters the car? Assume that the normal position of the screen is vertical, and that the upper section is pivoted about its top edge. (O. & C.)

24. Two bodies, A and B, are moving in different directions. How may the velocity of B relative to A at a given instant be determined?

To a man in a car moving due eastward with a velocity of 20 miles per hour the wind appears to be coming from a direction 60° east of south. When the car moves with the same speed due westward the wind appears to come from a direction 30° west of south. Find the velocity and true direction of the wind. (O. & C.)

CHAPTER 3

THE LAWS OF MOTION

Motion in a Straight Line

We shall, during this chapter, use the term " body ", and describe experiments on a body of very substantial size. But we shall not have to consider its dimensions, and shall really be considering the motion of a certain point—the centre of gravity (or centre of mass) of the body. Vector considerations also will be in the background only ; for we are restricting the present inquiry chiefly to cases of motion in one fixed straight line.

Uniform velocity. A body moves with *uniform velocity* when it undergoes equal changes in displacement in equal successive intervals of time, however short these intervals may be.

If v is the uniform velocity (in cm. per sec.), then the displacement s (in cm.) after time t sec. is given by

$$s = vt.$$

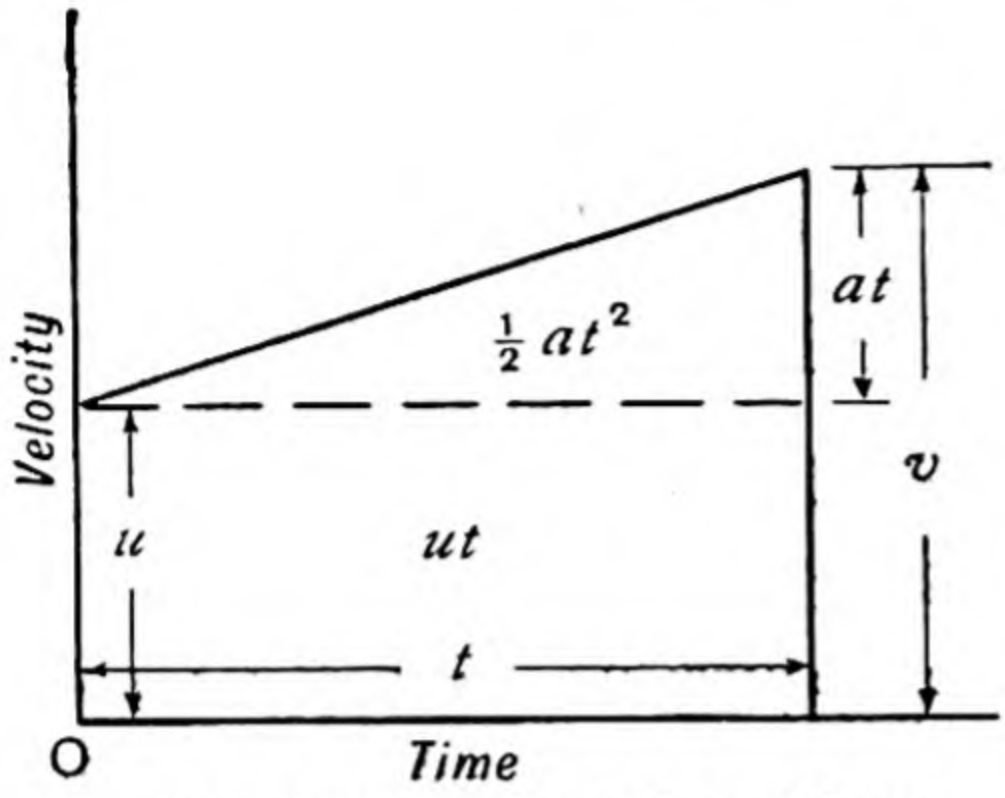
Uniform acceleration. A body moves with *uniform acceleration* when it undergoes equal changes in velocity in equal successive intervals of time, however short these intervals may be.

If a (cm. per sec. per sec.) is the uniform acceleration, u (cm. per sec.) the initial velocity, v (cm. per sec.) the final velocity after time t sec., then from the definition of acceleration, the gain in velocity in t sec. being at cm. per sec.,

$$\mathbf{v = u + at.} \quad \text{.....(1)}$$

The velocity-time graph (see p. 60) is thus a straight line, sloping upwards as shown if a is positive, and sloping downwards if a is a negative acceleration or retardation.

Remembering that the displacement is given by the area beneath the graph for the appropriate interval, we can see, regarding the area as a trapezium,



Velocity-time graph for uniform acceleration.

$$s = \tfrac{1}{2}(u + v)t. \quad \ldots\ldots(2)$$

That is, displacement is (*average velocity* × *time*), where the average velocity is the mean of the initial and final velocities.

Or, regarding it as a rectangle and a triangle,

$$s = ut + \tfrac{1}{2}at^2. \quad \ldots\ldots(3)$$

These equations, embodying all the information that the graph can give in this connexion, clearly contain all that is necessary for the solution of any problems. But supposing we are asked to calculate the velocity of the body after it has attained a given displacement s. Calculation of t from (2) or (3), and subsequent substitution in (1) will solve the problem. But it is convenient to perform this elimination of t once for all, and to remember the equation giving v in terms of a and s.

Rearranging (1), $\quad v - u = at$;

rearranging (2), $\quad v + u = 2s/t$;

multiplying the two left-hand sides and the two right-hand sides,

$$(v - u)(v + u) = 2as,$$

or,

$$(v^2 - u^2) = 2as,$$

whence

$$v^2 = u^2 + 2as. \quad \ldots\ldots(4)$$

It should be pointed out again that the equations 1–4 are simply the algebraic equivalents of the velocity-time graph for uniform acceleration. It is useful to commit them to memory ; and it is most essential to remember also that they are *not* applicable to all types of motion, but hold only for *uniform acceleration*, and when the motion is *rectilinear*.

These equations of course apply to the vertical motion of a body subjected only to the action of its own weight, when the appropriate value for g is the uniform acceleration a of the formulae.

Laws of motion ; Galileo. Galileo (1581-1642) investigated the motion of falling bodies and projectiles, and concluded :

(1) that a body falling freely, under the action of its weight alone, falls with uniform acceleration, and that all freely falling bodies have the same uniform acceleration ;

(2) that a body set moving on a smooth horizontal plane which supported its weight would continue moving with uniform speed in a straight line ;

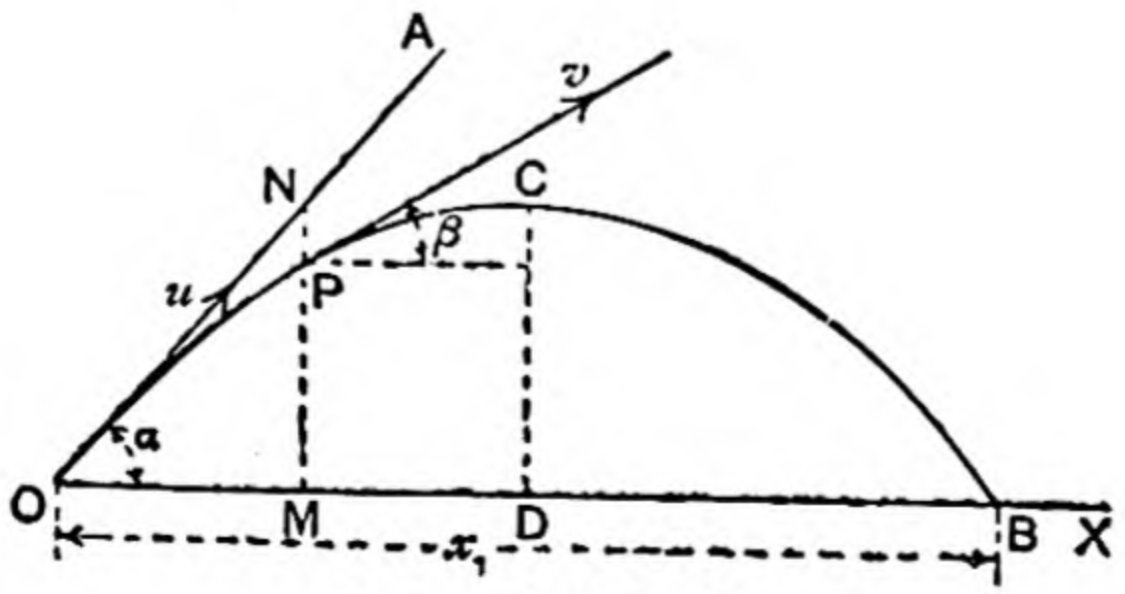
(3) that if a body is projected in any direction, the action of the weight is unaffected by the initial speed, and the motion is that obtained by compounding the two independent motions.

These conclusions embody the first two laws of motion (p. 67) ; for in (2) and (3) we find the first law (the " principle of inertia "), that a body does not of itself accelerate, and that acceleration is produced by the action of weight ; while the fact that all freely-falling bodies have the same acceleration, so that weight is proportional to mass, is consistent with the second law.

Projectiles. The horizontal and vertical motions of a body projected in any direction and at the same time able to fall freely under gravity are independent. The projectile, though air resistance does in practice prevent the realisation of the ideal conditions supposed, offers a good demonstration of the first law of motion, and this led Galileo to the conception of inertia. Neglecting air resistance, and reckoning the upward direction to be positive for displacements, velocities, and acceleration,

$$\textit{Horizontal acceleration} = 0.$$
$$\textit{Vertical acceleration} = -g,$$

If the initial velocity is u at angle α to the horizontal, then resolving u horizontally and vertically,



Path of projectile.

Horizontal velocity at time $t = u \cos \alpha$,
Vertical velocity at time $t = u \sin \alpha - gt$.

The vertical velocity becomes zero when $t = u \sin \alpha/g$, which thus gives the time to the greatest height. We can use equation (1) of p. 59 to show that it will take as long to fall from the greatest height as it took to get there ; hence the total time of flight is twice this, or $2u \sin \alpha/g$.

The actual velocity at time t is as represented at P in the figure, namely v at an angle β to the horizontal, where the horizontal and vertical components are compounded to give

$$v = \sqrt{u^2 \cos^2 \alpha + (u \sin \alpha - gt)^2}$$

and $$\tan \beta = (u \sin \alpha - gt)/u \cos \alpha.$$

For the displacements at time t,

Horizontal displacement, $x = ut \cos \alpha$,
Vertical displacement, $y = ut \sin \alpha - \frac{1}{2}gt^2$.

When $y = 0$, $t = 0$ or $2u \sin \alpha/g$, which is the total time of flight, as already indicated. The range on a horizontal plane is found by substituting this value of t in the expression for x, when

$$x = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{g}.$$

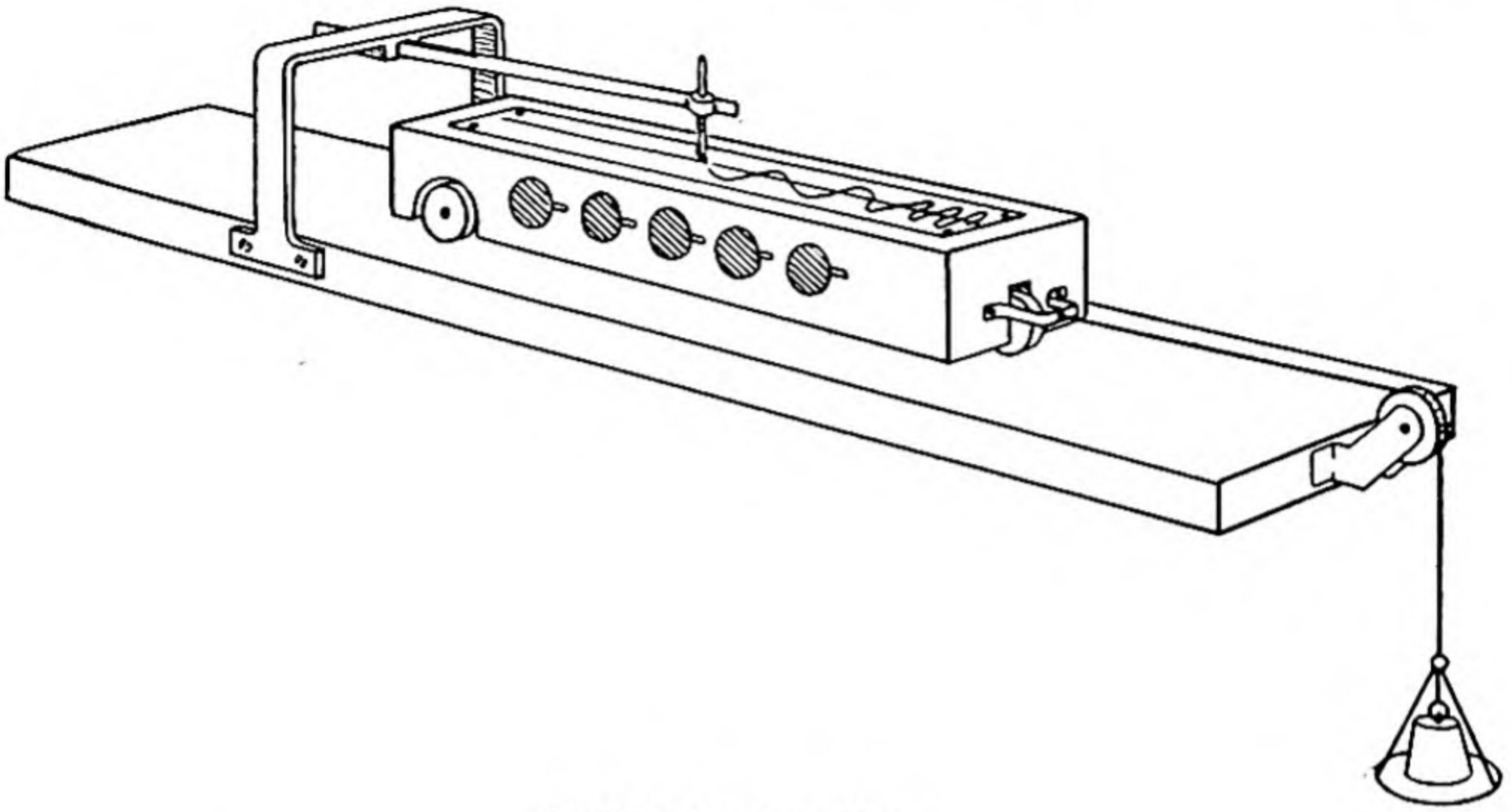
The maximum value of the range for a given initial velocity u is found when $\sin 2\alpha = 1$, so $\alpha = 45°$ and then $x = u^2/g$.

The equation of the path is found by eliminating t between the expressions for x and y, giving

$$y = x \tan \alpha - \frac{g}{2u^2 \cos^2 \alpha} x^2,$$

which is the equation for a parabola.

Motion of a body acted on by a constant force. The apparatus illustrated, known as Fletcher's Trolley, is a simple and convenient device for examining the laws of rectilinear motion. The trolley itself, a large rectangular block mounted on three light wheels, bears several removable metal cylinders, so that its mass can be varied. On its upper surface a strip of paper is pinned. It travels on a track beneath a vibrating arm carrying an inked brush. The period of the vibrator is constant if its length is fixed, and waves traced out on the



(*a*) Fletcher's trolley.

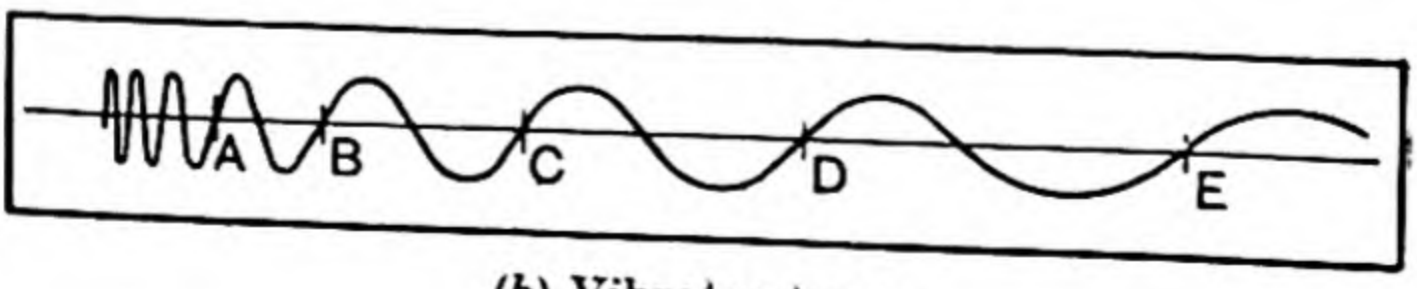
paper indicate the distances travelled in one vibrator-period ; thus the average velocity over a given vibrator-period is found by dividing the length of the chosen wave by the time of this period, τ sec.

The trolley is pulled along by a string attached to a mass hanging over a light pulley. Without going into further details at the moment, it is reasonable to suppose that the trolley when released experiences a resultant force in the direction of its motion, and that this force will not vary as the trolley pursues its path.

The trace made on the paper under these conditions appears as Fig. *b*. Measure the distances AB, BC, CD, DE, etc., of successive waves, taking the starting point A at some convenient place after the trace has become sharp enough.

The average velocity over successive " waves " is then

$$AB/\tau, \quad BC/\tau, \quad CD/\tau, \ldots \text{ etc.}$$



(*b*) Vibrator trace.

The gain in velocity between adjacent "waves" is

$$(BC - AB)/\tau, \quad (CD - BC)/\tau, \ldots \text{etc.};$$

this is the acceleration in cm. per sec. gained per τ sec., and the acceleration in cm. per sec. gained per sec. is

$$\frac{(BC - AB)}{\tau^2}, \quad \frac{(CD - BC)}{\tau^2}, \ldots \text{etc.}$$

These figures are found to be the same, within the limits of experimental error, so the acceleration is uniform—a fact that can be ascertained without actually working out the acceleration since the differences between successive "waves" are all the same. Hence, **a body moving under the action of a constant force experiences uniform acceleration.**

The following example illustrates the calculation of the acceleration.

Period of Vibrator: 1/5 sec.

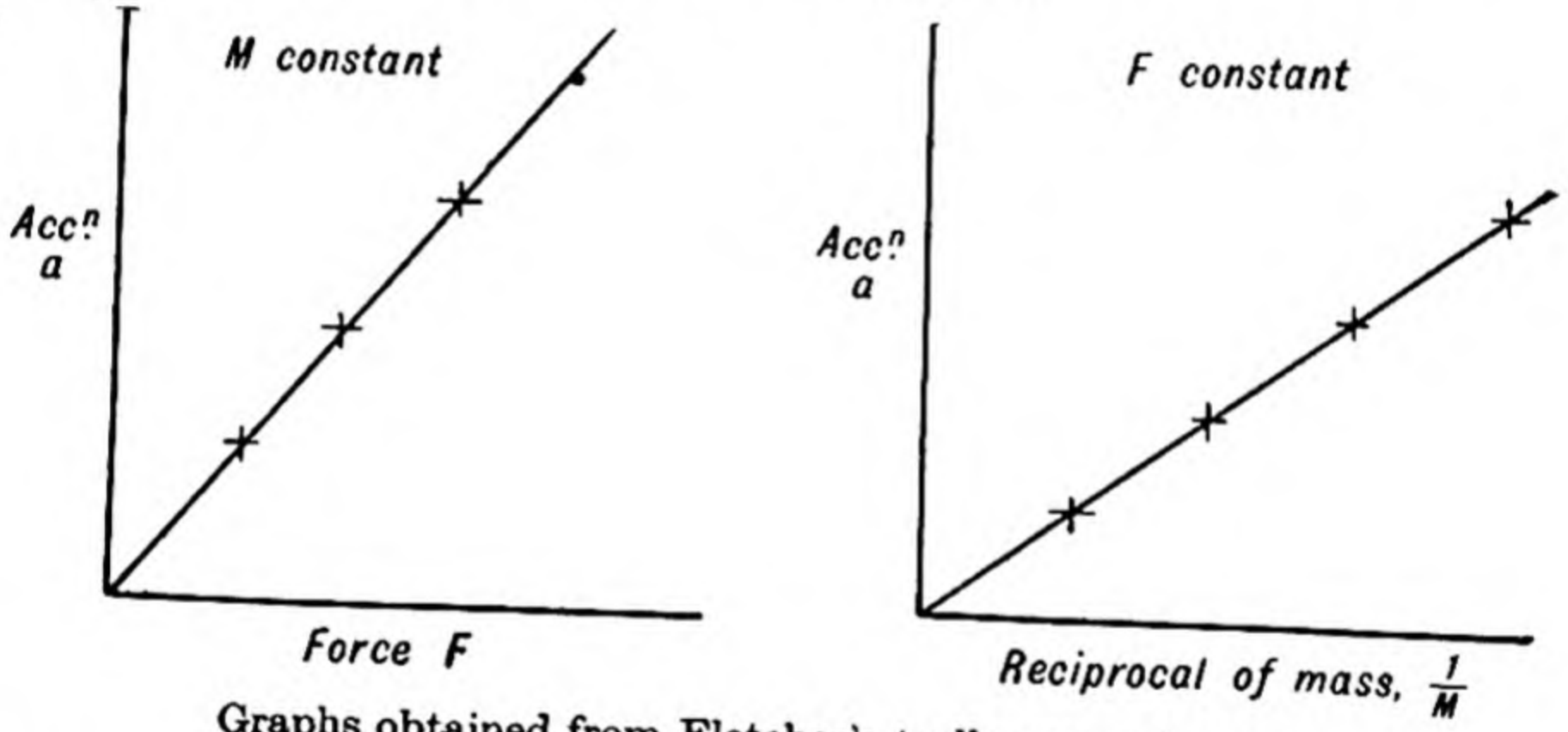
Length of successive "waves"	Average velocity	Gain in velocity in 1/5 sec.	Acceleration in cm. per sec. gained per sec.
AB 6·3 cm.	31·5 cm./sec.		
		4·0 cm./sec.	20 cm./sec./sec.
BC 6·7 cm.	33·5 cm./sec.		
		4·0 cm./sec.	20 cm./sec./sec.
CD 7·1 cm.	35·5 cm./sec.		
		4·0 cm./sec.	20 cm./sec./sec.
DE 7·5 cm.	37·5 cm./sec.		
		4·0 cm./sec.	20 cm./sec./sec.
EF 7·9 cm.	39·5 cm./sec.		

Verification of the equations for uniform acceleration. Using a trace obtained as above, the equations 2, 3, and 4 of page 60 may be verified. Thus, to test $s = ut + \frac{1}{2}at^2$, take the midpoints X and Y of the "waves" AB and EF: s is XY, u is AB/τ, t is 4τ, and a has been calculated. Substitution in the formula shows its applicability.

Relation between force and acceleration for a body of fixed mass. Turning again to the Fletcher Trolley, it is essential, since we require to measure the actual accelerating force, to overcome friction. Also,

it must be remembered that the mass hanging on the string must be included in the total mass that is accelerated by its weight, so this total mass must be kept constant.

One of the large metal cylinders on the trolley is removed from its slot, and its place is filled by several 20 gm. weights. The bed of the trolley is sloped, and a series of trials made until the trolley appears to run with uniform speed when it is gently pushed off. Final adjustment is made, taking a series of traces, until a trace is obtained in which the " waves " are all of equal length ; then, as there is no acceleration, no resultant force can be acting, so the resolved part of the weight of the trolley acting down the slope, together with the tension in the string due to the weight of the scale-pan, just combine to overcome friction.



Graphs obtained from Fletcher's trolley experiments.

Next, one of the 20 gm. weights is taken from the slot on the trolley and placed in the scale-pan. A trace is taken. This is repeated with several more weights. In each case the acceleration is worked out. A graph of acceleration against force (the weight of the hanging mass) is plotted. This is found to be a straight line passing through the origin (see graph), showing that the acceleration is proportional to the force acting. Hence,

for a fixed mass, the acceleration is proportional to the force producing it.

Acceleration produced in different masses by the same force. Rather more care is needed to demonstrate this, since alteration of the mass of the trolley by removing the metal cylinders alters the

effect of friction. The procedure for overcoming friction must thus be followed each time the mass is changed.

Keeping a fixed mass in the pan, traces are taken with different numbers of cylinders in the trolley, and the acceleration is worked out for each. The total mass of the trolley and suspended mass is written down each time. A graph of acceleration against $1/mass$ is found to be a straight line going through the origin (see graph), showing that for a fixed accelerating force, the accelerations produced in different masses are inversely proportional to those masses.

Thus,

for a fixed force, acting on different masses, acceleration is proportional to the reciprocal of the mass.

The dynamical equations. Let F stand for the magnitude of the force acting on a body of mass m, and producing acceleration a.

Then, as $a \propto F$ if m is constant,

and $a \propto 1/m$ if F is constant,

$a \propto F/m$,

or, $F \propto ma$.

Writing this as an equation,

$$F = kma,$$

where k is a numerical constant the value of which depends on the units chosen for measuring the quantities. If we define unit force as that force which produces unit acceleration in unit mass, $k = 1$.

In the C.G.S. system, the absolute unit of force is the **dyne**, which gives a mass of 1 gm. an acceleration of 1 cm. per sec. per sec.; the corresponding British unit, the **poundal**, gives a mass of 1 lb. an acceleration of 1 ft. per sec. per sec. (p. 5). So, provided the forces are measured in dynes (or in poundals), we can write

$$F = ma.$$

Now, supposing the force, acting for t sec., changes the velocity of the body from u to v cm./sec. The product (*mass* × *velocity*) is called the **momentum** of the body, and the momentum is thus changed from mu units to mv units; the momentum change being $mv - mu$, or $m(v-u)$ units. As $v = u + at$, then $a = (v-u)/t$,

$$F = m(v-u)/t,$$

so

or, rearranging, $$Ft = m(v - u).$$

The quantity Ft is called the **impulse** of the force; its units are sec. dynes (or sec. lbal.); the units in which momentum is measured are also thus sec. dynes (or sec. lbal.).

In words, if we are measuring all our quantities in absolute units,

$$Force = mass \times acceleration$$

$$Impulse\ of\ force = change\ in\ momentum.$$

These two equivalent statements contain the fundamental principle known as Newton's Second Law of Motion.

Mass and weight. The weight of a body of mass m gm. when acting alone on that body so that it falls freely, is found to give it a uniform acceleration g cm. sec.$^{-2}$. So, using $force = mass \times acceleration$,

$$weight = mass \times acceleration\ of\ gravity, \text{ or}$$

$$weight = mg \text{ dynes.}$$

Similarly, in British units, for a mass of m lb., when g is in ft. sec.$^{-2}$,

$$weight = mg \text{ lbals.}$$

Newton's Laws of Motion. These laws, the axioms on which the whole of dynamics is founded, are usually worded as follows:

First Law. Every body continues in its state of rest, or of uniform motion in a straight line, unless acted upon by an external force.

This is equivalent to defining force as "that which causes acceleration."

Second Law. Rate of change of momentum is proportional to the applied force, and takes place in the direction of the force.

This summarises the information illustrated experimentally using the trolley, and of course leads to the equations

$$Ft = m(v - u) \text{ and } F = ma,$$

in absolute units.

Third Law. Action and reaction are equal and opposite.

That is, if two bodies A and B interact, the force exerted by A on B is equal and opposite to the force exerted by B on A.

Action and reaction are equal and opposite *and act on different bodies.*

This is, perhaps the most difficult of all principles to understand thoroughly and to apply.

The "Principia". Newton's formulation of the laws of motion has been criticised from two points of view. The first is, that the "quantity-mass" of his definition below is an obscure and unnecessary concept. The second is that they say in three instalments what might well be put in one comprehensive statement; this is justified, but it must be remembered that the first two laws had been known for fifty years, since the time of Galileo, and that the third law was Newton's own extension of an existing frontier. It should be realised that the laws are really one, and can only be applied as a whole.

They form the opening chapter of the "Philosophiaea Naturalis Principia Mathematica", which was first published in 1686. This was translated into English in 1729 by Andrew Motte, and a revised translation was edited by Prof. F. Cajori (University of California Press and Cambridge University Press, 1934). The following quotations are from Cajori's version, and show the clearness of Newton's own expressions, which emphasise the experimental basis of his statements. The laws are preceded by definitions, which themselves imply some of the axiomatic content of the laws.

"DEFINITION I.

The quantity of matter is the measure of the same, arising from its density and bulk conjointly.

Thus air of a double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction, and of all bodies that are by any causes differently condensed.... It is this quantity that I mean hereafter under the name of body or mass. And the same is known by the weight of each body, for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shown hereafter."

"Is known" is the translation of the Latin *innotescit*, which means "becomes known", so the last sentence means that you can compare masses by comparing weights; it does not mean that "weight" is an appropriate synonym for "mass".

By the examples given, the concept named " mass " seems fairly clear. It has been said that the definition of mass in terms of density (which we now define as " mass of unit volume ") is logically unsound, but *relative density* is an idea which had for long been familiar, and it is in this sense that the term density is chiefly used by Newton. Thus (Book III, Prop. X) in discussing the density of the earth and the planets, he compares them with one another and with water. We can, using this definition alone, obtain masses bearing any chosen ratio to one another, if we have homogeneous materials of uniform and constant density available. For the masses of different specimens of any such material are proportional to their volumes.

Newton appears (Book III, Prop. VI, Cor. III and IV) to have had a simple " kinetic theory " picture in his mind, and to have considered the possibility that different materials were all made of solid particles of the " same density " (with " *inertias* in proportion to their bulks ") but differently spaced. A similar question concludes the 1717 edition of his " Opticks ". To-day we might say that the Newtonian " quantity-mass " of a body is something that represents the sum of all the atoms (or of all the sub-atomic particles) that it contains. This is the sense in which the term mass is understood by the chemist, who works with both balance and burette, prepared to use either the inertia or the volume of a body as a means of estimating its " quantity". This quantity is a constant for the body as long as no part of it is removed.

The " inertia-mass " is a physical property characteristic of a body, which it exhibits in its interactions with other bodies. Mechanical experiments using the laws of motion enable the inertia-masses of different bodies to be compared. Such experiments reveal that the inertia-mass appears to be constant for a given body ; and in Newtonian mechanics it is supposed that this is so. Assuming this, and also that inertia-masses are additive as scalars, then the inertia-mass of a body made up of a large number of similar particles is the sum of the inertia-masses of all the particles—or, the inertia-mass is proportional to the quantity-mass. It thus appears that quantity-mass is something which we *suppose* bodies to possess because we are physically aware of their inertia-masses. The term " mass " in physics means inertia-mass ; whether or

not it is always constant for a given body is a question that experiment alone can answer.

The pendulum experiments referred to are described under Proposition VI of Book III of the "Principia". Having proved earlier (Book II, Prop. XXIV) that if pendulums of the same length are found to swing with equal times, the quantities of matter in each must be proportional to the weights, Newton made pendulums of which the bobs were hollow boxes, of similar dimensions so as to experience similar air resistance. Equal weights (as found by "weighing") of different materials were put in the boxes; and it was found that (at least to an accuracy of one part in a thousand) the periods were the same, showing that the masses were equal.

"DEFINITION II.

The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly." That is, momentum is mass × velocity.

"DEFINITION III.

The vis insita *or innate force of matter is a power of resisting, by which every body, as much as in it lies, continues in its present state, whether it be of rest, or of moving uniformly forwards in a right line.*

"DEFINITION IV.

An impressed force is an action exerted upon a body, in order to change its state, either of rest or of uniform motion in a right line.

"This force consists in the action only, and remains no longer in the body when the action is over. For a body maintains every new state it acquires by its inertia only. But impressed forces are of different origins, as from percussion, from pressure, from centripetal force."

The celebrated definitions of absolute time, which "of its own nature flows equably without relation to anything external," and of absolute space, "always similar and immovable," then follow, with the suggestion that there may be some body at absolute rest among or beyond the fixed stars, though it would be impossible to discover this even if it were so. All observations of motion are relative; it may be that there is such a thing as absolute motion, to be reckoned by absolute time in absolute space, though rectilinear motion of this kind could never be detected as such; while absolute

rotation could be discerned by observing the centripetal force maintaining it. (This means that the laws are laws of *relative* motion.)

The operation of the laws of motion should hence be unaffected by rectilinear motion of the system in which the observations are made, or of the observer within the system. This again is a matter that only experiment can decide; and the experimental work of recent years has shown that it is true only if the relative velocities concerned are small compared with the velocity of light. But the theory of relativity does not conflict with Newton's laws for relative velocities which are within the range of any ordinary observation, and Newton's laws are now regarded as statements of experimental relations which are true as limiting cases of a more general law.

Newton's statement of the laws, and his own explanations of them, then follows.

"LAW I.

Every body continues in its state of rest or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Projectiles continue in their motion, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. . . .

"LAW II.

The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force) is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

"LAW III.

To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also

pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the rope; for the distended rope, by the same endeavour to relax or unbend itself, will draw the horse as much towards the stone as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other. If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of the bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are inversely proportional to the (masses of the) bodies."

The first two laws are, then, clear restatements of definitions or experimental truths already agreed to about the motion of "a body". The third law states how all this is relevant to the practical problem of one body acting on another. Newton proceeds to deduce as corollaries of these laws:

(1) The parallelogram of velocities rule.

(2) The rules for the composition and resolution of forces.

(3) The law of conservation of momentum: "The quantity of motion, which is obtained by taking the sum of the motions directed towards the same parts, and the difference of those that are directed to contrary parts, suffers no change from the action of bodies among themselves." This is deduced, using his argument, on page 84.

(4) The common centre of gravity of a system of bodies not subject to external forces is either at rest, or moves uniformly in a straight line.

(5) The motions of bodies within a given space are the same whether the space is at rest, or moving uniformly without rotation in a straight line.

(6) The relative motions of a system of bodies are unaltered if each is given the same acceleration in the same direction.

The laws are also applied to the motion of simple machines, and it is deduced that in the absence of friction the mechanical advantage of a machine (p. 139) should equal the velocity ratio.

Newton emphasises several times that the three laws are based on experiment, and he follows by describing experiments (first

performed by Wren and others) which he did with a pair of pendulums 10 ft. long to test the equality of action and reaction, by observing that momentum was conserved on collision. The experiments are of the same type as that described on p. 83, but it should be noted that Newton made his measurements with much greater care than we should use in a demonstration, and corrected scrupulously for the effects of air resistance. He used " unequal as well as equal bodies ", and this raises an interesting point ; if it is only possible to compare masses by " the changes of the velocities made towards contrary parts being inversely proportional to the masses of the bodies ", it is doubtful whether anything more than self-consistency can be taken as demonstrated. Again, that mass is proportional to weight follows from the first two laws, and if we use this to compare masses by weighing we compare the forces they exert on some other body and thus invoke the third law ; but if Newton is using his " quantity-mass", which means that the masses of, say, two lead spheres are proportional to their volumes, then this difficulty disappears.

These extracts from the " Principia " may well conclude with Newton's account of his investigation of the behaviour of different materials on impact.

" But to prevent an objection that may perhaps be alleged against the rule, for the proof of which the experiment was made, as if this rule did suppose that the bodies were either absolutely hard, or at least perfectly elastic (whereas no such bodies are to be found in Nature), I must add, that the experiments we have been describing, by no means depending upon that quality of hardness, do succeed as well in soft as in hard bodies. For if the rule is to be tried in bodies not perfectly hard, we are only to diminish the reflection in such a certain proportion as the quantity of the elastic force requires. By the theory of Wren and Huygens, bodies absolutely hard return from one another with the same velocity with which they meet. But this may be affirmed with more certainty of bodies perfectly elastic. In bodies imperfectly elastic the velocity of return is to be diminished together with the elastic force ; because that force (except when the parts of bodies are bruised by their impact, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes the bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried with balls of wool, made up tightly, and strongly compressed. For, first

letting go the pendulous bodies, and measuring their reflection, I determined the quantity of their elastic force ; and then, according to this force, estimated the reflections that ought to happen in other cases of impact. And with this computation other experiments made afterwards did accordingly agree ; the balls always receding one from the other with a relative velocity which was to the relative velocity with which they met as about 5 to 9. Balls of steel returned with almost the same velocity ; those of cork with a velocity something less ; but in balls of glass the proportion was as about 15 to 16. And thus the third Law, so far as it regards percussions and reflections, is proved by a theory exactly agreeing with experience."

The result here stated, that the relative velocity after impact is proportional to the relative velocity before impact, is known as *Newton's experimental impact law.*

Some Problems on the Laws of Motion

First it is necessary to be clear as to the exact meaning of the terms, "reaction" of Law III, and "external force" of Laws I and II.

A body experiences a *weight*, which is the pull of the earth on that body. The equal and opposite reaction to this is the pull of the body on the earth, acting at the earth's centre of gravity. If its weight $W\downarrow$ is the only external force acting on a body, that body of course is accelerated downwards. If the body rests on a table or the floor or other support, it may remain at rest because another external force—the upthrust $F\uparrow$ of the floor—is acting, and because it happens that this exactly balances the other external force—the body's weight. We may *call* $F\uparrow$ a reaction, but we deduce its value by applying Law I and Law II to the body, not by applying Law III to the system body-floor ; $W\downarrow$ and $F\uparrow$ both acting on the body are *two external forces*, and not "action and reaction", nor need they in general be equal and opposite. They will not be equal and opposite if the body is accelerating, as may happen if the floor is the floor of a lift in motion, for then the resultant of $F\uparrow$ and $W\downarrow$ (either $(F-W)\uparrow$ or $(W-F)\downarrow$) is the external force that accelerates the body.

Action and reaction are equal and opposite, and act on different bodies. The floor experiences a downward force $F\downarrow$, which is in all circumstances equal and opposite to the upthrust $F\uparrow$ on the body. If we wish to consider the behaviour of the floor, this is an external force acting on it. If we consider the body and floor together as a

single system, then the two forces $F\uparrow$ and $F\downarrow$ cancel one another as *internal forces* ; that is, they are forces arising within the system between its parts, and not external forces impressed on the system from without. But the weight $W\downarrow$ of the body is still an *external force*, and always will be, except in cases where we are considering the motion of the earth in the solar system, when the attraction between the earth and a fragment of its surface do cancel out as internal forces.

It may seem that, if $F\uparrow$ and $F\downarrow$ cancel out as equal and opposite, it is futile to consider them at all. This is quite true, unless we are concentrating on either the body or the floor. But if we are considering the motion or rest of the body, we must consider only the forces that act *from without* on the body, and we must take care to consider *all* the external forces of which $F\uparrow$ is one.

It will be easier to understand the real point of the foregoing after one or two examples have been considered.

Laws of motion applied to the trolley. Let M gm. be the mass of a trolley, actuated by the weight of a mass m gm. on a string over a light " frictionless " pulley (as on page 63).

The forces acting on the system from without are the weight of the trolley, M gm-wt., acting downwards, and the upthrust of the plane on the trolley, which is equal and opposite to this ; and the weight of the suspended mass, which is not opposed, and is m gm. wt. $= mg$ dynes acting on m.

The total mass accelerated is $(M+m)$ gm.

Using the equation $F = ma$,

$$mg = (M+m)a.$$

Then
$$a = \frac{mg}{(M+m)} \text{ cm. per sec. per sec.}$$

Here the system has been treated as a whole, and the external force acting on the system (the weight of the hanging mass) has been the only force it is necessary to consider.

If we wish to consider the motion of either m or M separately, it is necessary to take into account the tension in the string. This string exerts an upward force of T dynes on the mass m ; an equal and opposite reaction, the effect of m on the string, is transmitted through the string as a tension round the pulley, and applied to M horizontally. Considering the two bodies separately :

For m, $mg - T = ma$, ...(1)

For M, $T = Ma$. .....(2)

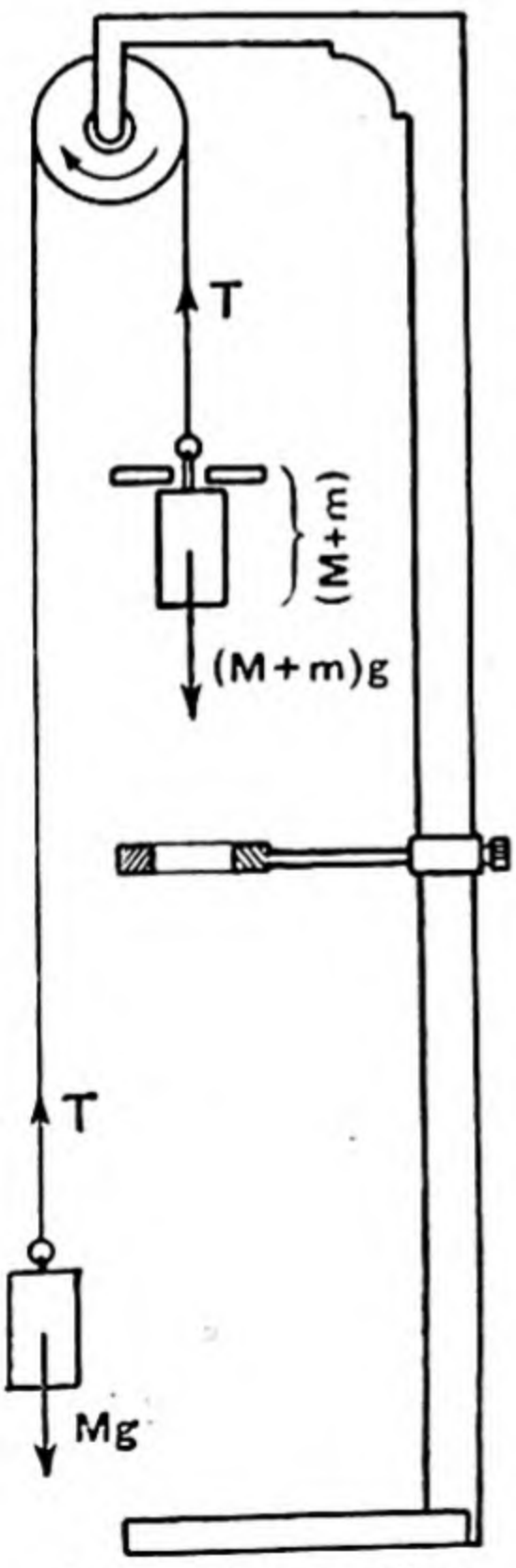
Now, as both bodies have the same acceleration, eliminating T between equations (1) and (2) gives as before,

$$a = \frac{mg}{(M+m)} \text{ cm. per sec. per sec.,}$$

while substituting for a in either equation gives

$$T = \frac{Mmg}{(M+m)} \text{ dynes.}$$

Equations (1) and (2) can also be considered with reference to the Third Law. It is often said that the product (mass × acceleration) of an accelerated body is a " kinetic reaction ", or a reaction due to the inertia of the body. This concept is of great use in some more advanced work, where it often appears rather as a mathematical device ; the reaction is real enough, but it does *not* act on the body.



Atwood's machine.

The reaction to the pull T of the string on the trolley is an equal and opposite force T exerted by the trolley on the string. *Action and reaction are equal and opposite, and act on different bodies.*

We can say quite correctly, when applying the Second Law, either :

(*a*) considering the *trolley*, an external force $T \rightarrow$ gives an acceleration $a \rightarrow$ such that $T = Ma$; or,

(*b*) considering the *string*, the inertia of the trolley M gives a reaction $T \leftarrow$ on the string, such that $Ma = T$. This reaction is experienced and felt if we try to accelerate the trolley by hand—and is the basis of our concept of inertia. But it does not act on the trolley from without ; it is directed *on to* an external agent *from* the trolley.

Laws of motion and Atwood's machine. The diagram illustrates the principle of Atwood's

machine, an apparatus which has the same essential features as the trolley (p. 63). Two masses M, M, of equal value, are connected by a light string passing over a light "frictionless" pulley. A small rider m rests on one of the large masses; this is so shaped that it can be removed by a platform placed at any chosen distance from the start of the motion. The idea is that after the removal of the rider, the velocity acquired while it has been causing acceleration persists steadily, and can be measured, whence the acceleration can be calculated. Thus, with means of applying a known force to a system of known mass, and of measuring the acceleration, the laws of motion can be investigated. Details of the experiments possible with this machine will be found in the text-books on Practical Physics; here it is only proposed to show how the laws of motion are exemplified by it.

As with the trolley, the system can be treated as a whole, when the external force is the weight mg dynes of the rider, and the acceleration is found (using $F = ma$) to be

$$a = \frac{mg}{2M + m} \text{ cm. per sec. per sec.}$$

Or, the tension T dynes in the string leads to

$$(M + m)g - T = (M + m)a,$$

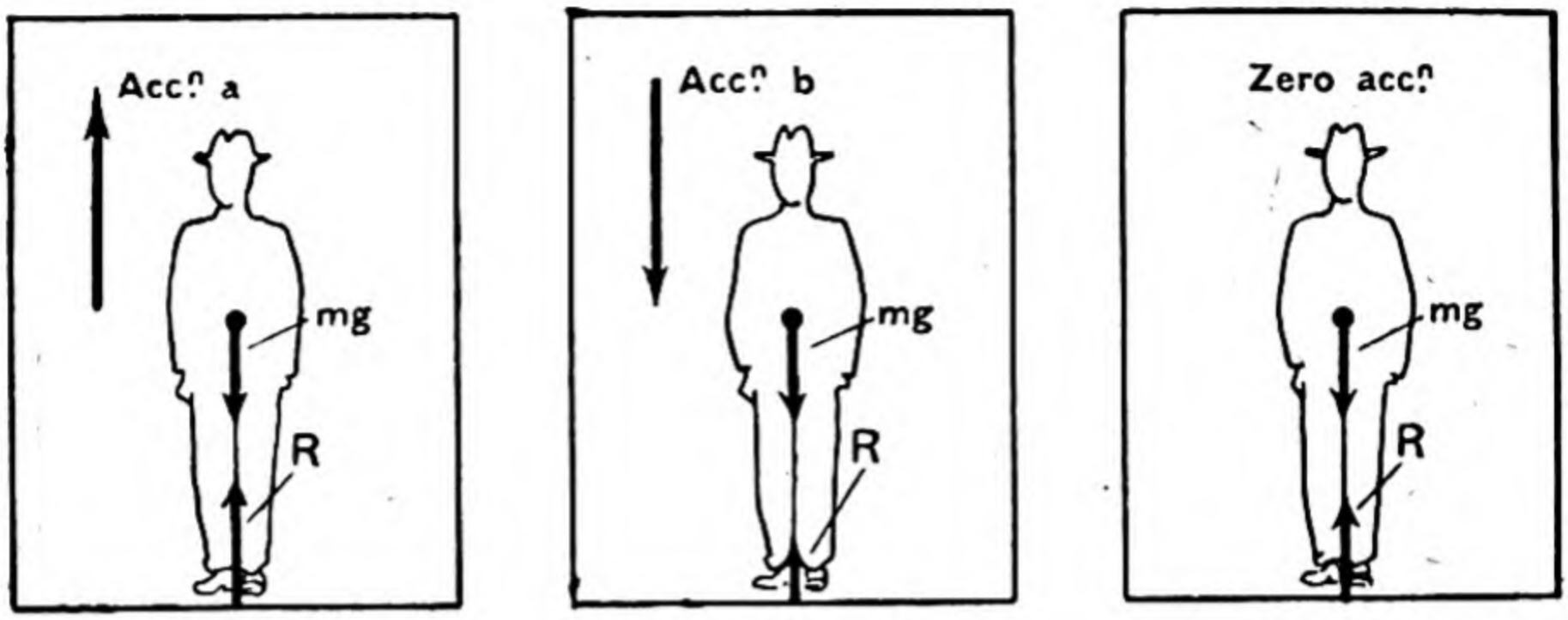
and

$$T - Mg = Ma,$$

giving the same result for a when T is eliminated, and enabling T to be calculated by eliminating a.

The man in the lift. The previous examples applied the laws to motion relative to the room. An external observer will apply them similarly to the motion of a man in a lift. The passenger may also apply them to his observations of motion relative to the lift. Both treatments give consistent results, for the laws are laws of relative motion. Consider a man of mass m gm., standing on the floor of a lift which is moving with acceleration a cm. per sec. per sec. with respect to its surroundings, reckoned as positive if upwards. Let R dynes be the upward pressure of the floor on the man (see diagram on p. 78).

The forces acting on the man are then mg dynes and R dynes, and the resultant is $(R - mg)$ dynes upwards.



Experiences of an observer travelling in an accelerated lift.

Using $$F = ma,$$

$$(R - mg) = ma,$$

so $$R = mg + ma = m(g + a) \text{ dynes.}$$

If a is negative, say $(-b)$, then R is $m(g-b)$ dynes.

If a is zero, that is, the man is either at rest or moving with uniform velocity, then $R = mg$.

The laws in the accelerating lift. What will be the experience of an observer in the lift, who takes the term "rest" to mean "rest with respect to the lift"? He drops something, and finds that it falls freely with acceleration $(g+a)$ instead of g cm. sec.$^{-2}$. Applying the laws of motion, he says its weight is $mass \times (g+a)$ dynes. His own weight becomes $m(g+a)$ dynes. Thinking in terms of statics, he regards the upthrust of the floor on his feet as the external force applied to balance his weight and keep him at "rest".

Similarly, if the acceleration of the lift is b cm. sec.$^{-2}$ downwards, his "g" becomes $(g-b)$; and his weight $m(g-b)$ dynes. In particular, if $b = g$, he will observe no acceleration of a freely falling body, which will appear to be weightless.

Problems on the behaviour in the lift of a pendulum, a mercury barometer, or an Atwood's machine or other device, are conveniently solved from the point of view of the observer in the lift, by using, in place of g, his observed value for the acceleration relative to him of a freely falling body. This is one of the few cases in which it is wise to abandon the viewpoint of the "fixed" external observer.

Motion in a Circle

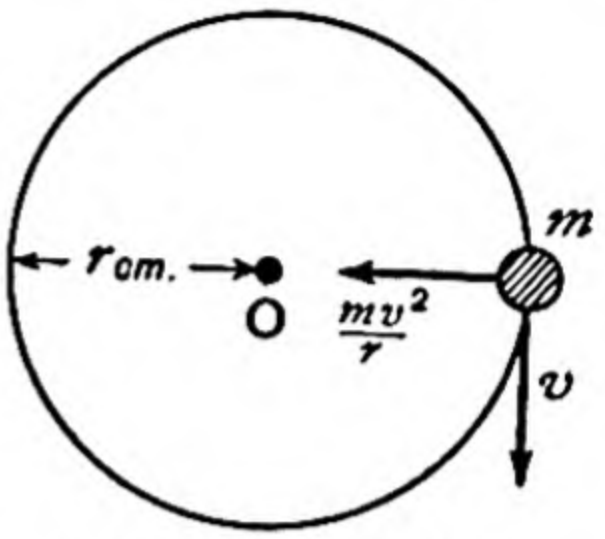
A body describing a circle with uniform speed. Consider a body of mass m gm. describing a circle of radius r cm. with uniform speed v cm. per sec.

The acceleration of the body towards the centre O of the circle is

$$v^2/r \text{ cm. per sec. per sec.}$$

Using $F = ma$, the body must be experiencing a force

$$F = m\frac{v^2}{r} \text{ dynes towards the centre } O.$$



Motion in a circle.

Thus, in order to maintain the motion of the body in a circular path, an external force mv^2/r dynes must act upon the body *from without*, and this force acting on the body is directed towards the centre of the circular path (Newton's "centripetal force", p. 71).

There must also, of course, be no resultant external force in the direction of the path, which is at each instant tangential to the circle. So the centrally-directed force must be the only force, or must be the resultant of all the forces, acting upon the body from without.

Note carefully the steps of this argument. We know the body has an acceleration towards the centre of the circle; we know that the force responsible for this acceleration is in the same direction as the acceleration; so the force acting on the body is directed towards the centre of the circle. The Second Law equation gives the magnitude of this force as mv^2/r absolute units, towards the centre of the circle.

Some people speak of a "centrifugal force" mv^2/r *acting outwards*. There is no objection to this, if the object on which this force acts is specified; but no such force can possibly act on the body describing the circle. Consider, say, the moon going around the earth; the *moon* experiences a gravitational pull mv^2/r (where m is the mass of the moon, and r the radius of its orbit) towards the centre of the path; the *earth* experiences an equal and opposite reaction mv^2/r directed outwards from the centre of the orbit towards the moon. Or, consider a railway engine taking a curve; the *engine* experiences

a force mv^2/r towards the centre of the curve; the *rails* experience an equal and opposite reaction mv^2/r directed outwards from the centre of the curve.

Part of the confusion to which this matter gives rise is due to faulty analysis of our own experience. If we hold a stone on a string, we exert an upward pull, which we can feel. As explained on p. 74 this is not a reaction to the weight of the stone, but a deliberately applied external force which, together with the weight, maintains the stone at rest. If we whirl the same stone round so that it describes a circle, we exert an inward pull, which we can feel. This is *not* a reaction to any action of the stone, but a deliberately applied external force which is not balanced, and which maintains the circular motion.

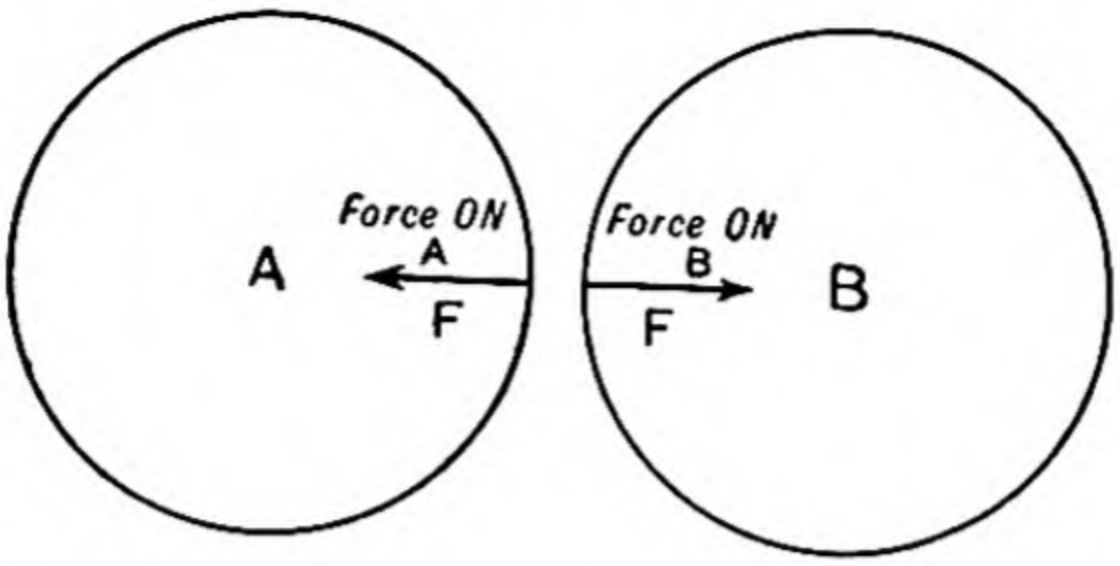
A second cause of difficulty is, that some problems (as, for example, on the stability of a vehicle) are readily solved by the methods of statics (p. 104) for which the first requisite is that the body concerned shall be at rest. This is conveniently achieved by supposing that the external force responsible for maintaining motion is annulled by an equal and opposite force for the sake of the problem—that is, the motion is "arrested" by the application of an imaginary force mv^2/r acting outwards. There is no objection to this, except that in simple problems it is an unnecessary artifice; but we cannot have it both ways. If a force mv^2/r acting outwards is applied to the body its revolution ceases; so long as it is moving in its circular path the resultant external force acting upon it is mv^2/r directed towards the centre of the orbit.

These points are considered in greater detail on p. 155.

Law of Conservation of Momentum

Interactions between the members of a system produce accelerations in the individual members, but the system as a whole, if it is a "closed system"—that is, not acted on by any external force—is unaffected by them, as the First Law of Motion states.

Consider as the closed system, two bodies A and B (see diagram) both in motion in the same line. Let them collide, the duration of contact during the impact being τ sec. At any instant during this impact, A experiences from B a force equal and opposite to the force that B experiences from A (by the Third Law); the magnitude



of these forces will not be constant over the whole interval, but we can represent it by the average value F, being sure that as action and reaction are equal and opposite at each instant, the average values are also equal and opposite.

The force experienced by A due to B, is F dynes $\leftarrow$;
the force experienced by B due to A, is F dynes $\rightarrow$.
The impulse experienced by A is $F\tau$ sec. dynes $\leftarrow$;
the impulse experienced by B is $F\tau$ sec. dynes $\rightarrow$.

By the Second Law,

the change of momentum of A is $F\tau$ sec. dynes $\leftarrow$, or
$-F\tau$ sec. dynes $\rightarrow$, taking the $\rightarrow$ direction as positive ;
the change of momentum of B is $F\tau$ sec. dynes $\rightarrow$.

The total change of momentum of the system A and B together, in the $\rightarrow$ direction, is $(F\tau - F\tau) = 0$.

The same argument, based entirely on the Laws of Motion and not necessarily involving the nature of the bodies or the actual type of interaction, must apply to any kind of interaction between two bodies, or to any system made up of pairs of bodies such as A and B, or to any number of bodies all acting simultaneously on one another.

Thus ;

For any closed system, the total momentum of all the parts in any given direction is unaltered by any interactions between those parts. This is the Law of Conservation of Momentum.

Note that this applies to the system as a whole ; the individual members do of course undergo momentum changes, for the action and reaction which are internal forces for the system as a whole are external forces for the individual parts on which they act.

The Law of Conservation of Momentum is a very convenient extension of the Laws of Motion, for three reasons :

(i) The exact nature of the interaction need not be specified ;

(ii) the actual value of the force involved in the interaction between two bodies is not required ; and

(iii) this Law is universally applicable wherever the Laws of Motion are themselves relevant.

The shot-and-gun problem. A gun of mass M gm. fires a shot of mass m gm. The velocity of the shot is v cm. sec.$^{-1}$ →, and that of the recoil of the gun V cm. sec.$^{-1}$ ←. Before firing, the total momentum in the → direction is zero. After firing, momentum of shot is mv →; momentum of gun is MV← or $-MV$→ ; so, as the total momentum in the → direction is unchanged, $mv - MV = 0$.

Experimental test of the Law of Conservation of Momentum. The ballistic balance (or ballistic pendulum) consists of two rectangular platforms, A and B (see diagram), each carried by four strings of equal length, so that they are effectively two pendulums constrained to swing in the same plane. Each carries a pointer which moves over a horizontal scale. Two points about the apparatus, both involving theory which is dealt with later, should be noticed now.

(1) The period of a pendulum of given length is the same whatever the arc of swing, if this is not too large ; thus both platforms, when pulled aside and released, take exactly the same time to reach the lowest part of the swing. Hence, if they are pulled aside to different distances, and released simultaneously, they both reach the zero, O, of the scale together, and collide there. *We know exactly where they will collide if they are released together.*

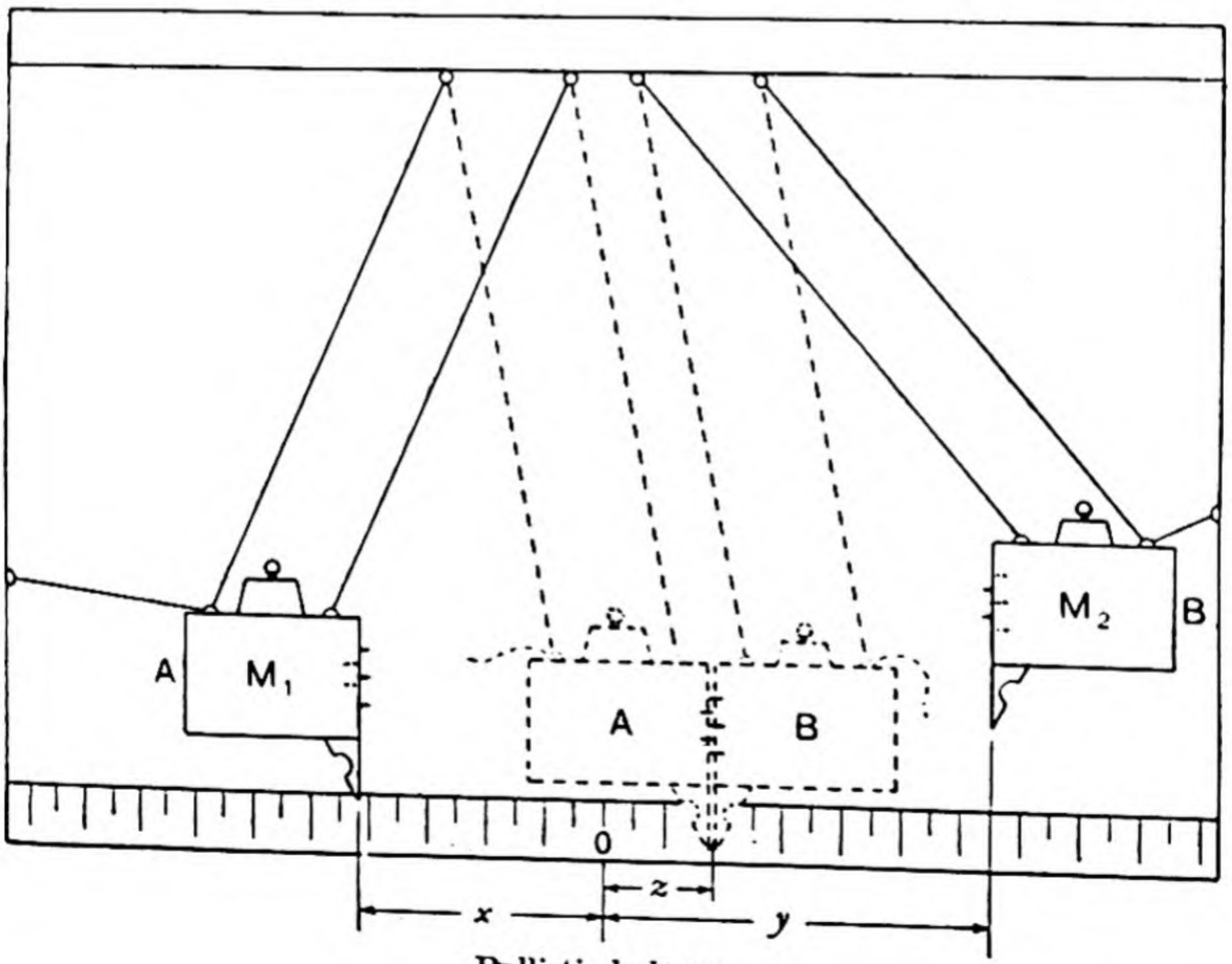
(2) It is shown later (p. 124) that the horizontal velocity, v, with which a pendulum-bob moves at the bottom of its swing is proportional to the square root of the vertical height, h, through which it has descended, for $v^2 = 2gh$. If l is the length of the string, and x the horizontal distance through which the bob is displaced prior to release, then $h\,(2l - h) = x^2$, by the geometry of the circle. If h is small compared with l, then approximately $2lh = x^2$; so $v^2 = x^2 . g/l$ and $v = x\sqrt{g/l}$. The velocity v at the moment of impact is nearly proportional to the distance x through which the bob has been drawn aside. Thus, if one of the platforms is pulled aside to a horizontal distance x_1, the velocity with which it encoun-

ters the other at O is proportional to x_1 ; conversely, if after impact the platform swings out to a horizontal distance x_2 before coming momentarily to rest, the velocity with which it set out from O is proportional to x_2.

The collision-surfaces of the two platforms may be fitted with needle-points so that on colliding the two adhere, or may be faced with rounded metal studs so that they rebound after collision.

Let M_1 be the mass of the left-hand platform A and its load, M_2 the total mass of the right-hand platform B. Both are pulled aside by means of a cotton thread, which runs through two loops in the frame, and is pulled tight. The distances x and y of the indexes of A and B from the scale zero are recorded. The thread is burned through ; the platforms collide at O, and if the surface is prepared for this purpose, they stick together ; the pair swing out to the right to a distance z.

As horizontal displacements are proportional to velocities at the lowest point of swing, x, y, and z can be taken to represent (in un-



Ballistic balance.

stated units) the horizontal velocities of A and B before collision, and of the combined AB after collision.

Let the positive direction be that outwards to the right of O.

Before impact :

momentum of $A = M_1x$ units $\rightarrow$;

momentum of $B = M_2y$ units $\leftarrow$, $= -M_2y$ units $\rightarrow$;

So, total momentum of the system in the $\rightarrow$ direction is

$$(M_1x - M_2y) \text{ units.}$$

After impact :

momentum of A and B together $= (M_1 + M_2)z$ units $\rightarrow$.

If the Law of Conservation of Momentum is true,

$$(M_1x - M_2y) - (M_1 + M_2)z = 0.$$

Changing the collision surfaces so that rebound occurs, the horizontal displacements x' and y' of A and B after collision can be observed. Suppose, as will often happen in an experiment, that these displacements are on opposite sides of O, say x' to the left and y' to the right.

In this case, the momentum of the system before impact, reckoned as before, is $(M_1x - M_2y)$ units $\rightarrow$.

After impact, momentum of A is M_1x' units $\leftarrow$, $= -M_1x'$ units $\rightarrow$;

momentum of B is M_2y' units $\rightarrow$.

So, total momentum after impact is

$$(M_2y' - M_1x') \text{ units } \rightarrow,$$

and, if the Law of Conservation of Momentum is true, we should find

$$(M_1x - M_2y) - (M_2y' - M_1x') = 0.$$

or, rearranging this expression, $(y + y')/(x + x') = M_1/M_2$.

As mentioned on p. 73 our view of this experiment depends on whether or not we are able to evaluate the ratio M_1/M_2 independently. If we suppose that this can be done, then the law of conservation of momentum has been demonstrated.

Some teachers approach the matter in this way : If we assume the laws of motion, we can find M_1/M_2 using a beam balance ; if we assume the laws of motion, we can find M_1/M_2 by the ballistic balance ; the two values for M_1/M_2 agree, so the laws operate consistently when we apply them.

Comments on the Laws of Motion. Sir Horace Lamb (following Clerk Maxwell) says that the real evidence for the correctness of these fundamental principles lies in their self-consistency. They cannot be tested singly, only as a whole, and Lamb suggests the following imaginary chain of operations :

1. Suppose we have the means of applying a constant force.
2. We observe this gives a certain acceleration to a body A.
3. We observe this gives different accelerations to different bodies $B, C, \ldots$, etc. Hence we are aware that the bodies differ in some respect which we call mass.
4. By finding two bodies to which it gives the same acceleration we obtain equal masses. If masses are additive as scalars (and here we are assuming Newton's definition really), we can make multiples of 2, 3, . . . times the mass of A.
5. And then show that the accelerations produced in different bodies by the same force are proportional to 1/mass.

Sir Edmund Whittaker, in " Analytical Dynamics of Particles and Rigid Bodies " (Cambridge University Press), states the laws of motion in the following form :

" If any set of mutually connected particles are in motion, the acceleration of any one particle is the resultant of that which it would have if it were perfectly free, and the acceleration along the lines joining it to the other particles which constrain its motion.

To several particles A, B, C . . . numbers m_A, m_B, m_C are to be assigned such that the acceleration along AB due to the action of B on A and the acceleration along BA due to the action of A on B are in the ratio $m_B : m_A$. The ratios of these numbers are invariable physical constants of the particles. Only the ratios are determinable by this law ; it is convenient to take some definite article A as standard, calling it the unit of mass, and then to call the ratios m_B/m_A, m_C/m_A, etc., the masses of the other particles X.

The ratio m_B/m_A is in fact equal to the ratio *weight of B/weight of A*.

The mass of the compound particle formed by uniting two or more particles is found to be equal to the sum of the masses of the separate particles.

In general, if an acceleration represented by a vector f is induced in a particle of mass m by any agency, the vector mf is called the force due to this cause acting on the particle."

It thus appears that there are three simple direct mechanical ways of comparing the masses of two bodies A and B.

(1) m_A/m_B = volume of A/volume of B,

which follows from Newton's definition, and is only true if they are of the same material (or of the same density).

(2) m_A/m_B = acceleration of B/acceleration of A,

which follows from the laws of motion, and is only true if they are experiencing the same force (or equal forces).

(3) m_A/m_B = weight of A/weight of B,

which follows from the laws of motion and of gravitation, and is only true if they are at the same place on the same planet.

That all three methods do give the same value for the *ratio* is, of course, what ordinary experiments all reveal. And if the equivalence of the ratios (2) and (3) is held to demonstrate the consistency of Newton's laws, reinforcement by the ratio (1) surely justifies his "quantity-mass" concept.

EXAMPLES ON CHAPTER 3

(Take g as 981 cm. sec.$^{-2}$ or 32 ft. sec.$^{-2}$.)

1. Prove that for uniformly accelerated motion in a straight line, the distance s covered in time t by a body moving with initial velocity u and with uniform acceleration f is given by

$$s = ut + \tfrac{1}{2}ft^2.$$

A lift starts upwards from rest with uniform acceleration. After 4 sec. it has acquired a velocity of 30 ft. per sec. and proceeds at this velocity for 2 sec. The motion is now uniformly retarded and the lift finally comes to rest 9 sec. after starting. Plot a velocity-time graph of the motion and calculate from the graph (or otherwise) the total distance travelled by the lift. (O. & C.)

2. State Newton's Laws of Motion.

Two bodies, each moving in the same straight line, collide. Show how the laws of motion lead to the result that there is no change in total momentum.

A body of mass 15 lb. is placed on a smooth table at a distance of 9 ft. from its edge, and is connected by a string passing over the edge with a body of mass 1 lb.; find (*a*) the common acceleration of the two bodies, (*b*) the time which elapses before the heavier body reaches the edge of the table. (O. & C.)

3. Describe an experiment to show that the acceleration of a body is proportional to the force acting on it.

A man weighing 10 stone stands in a lift. Calculate the force exerted on the man by the floor of the lift when the lift is moving (*a*) upward, (*b*) downward, with an acceleration of 2 ft. per sec. per sec. (O. & C.)

4. Distinguish between the average velocity of a moving body and its actual velocity at a given instant.

Draw the shape of the velocity-time graph for a body moving with uniform acceleration and deduce from it the distance travelled by the body in a given time.

The speed of a train is reduced from 60 m.p.h. to 30 m.p.h. in 880 yd. If the retardation remains constant, how much further will the train run before coming to rest? (O. & C.)

5. Define *acceleration*. How is it related to the magnitude of the force producing it? How would you demonstrate this relationship?

A barrel weighing 100 lb. is being pulled up a smooth plane inclined at 30° to the horizontal by means of a rope which is parallel to the surface of the plane. If the tension in the rope is 130 lb. wt., what is the acceleration of the barrel? (O. & C.)

6. State Newton's Second Law of Motion, and show how it can be used to define a unit of force.

A balloon and contents of total mass 2000 lb. are falling freely with an acceleration of 2 ft. per sec.^{2}. Find the change in the vertical motion after 125 lb. of ballast have been thrown out. Neglect air friction. (O. & C.)

7. What do you understand by the acceleration due to gravity?

How may the acceleration due to gravity at different places be compared?

From a balloon ascending with a vertical velocity of 16 ft./sec., a stone is let fall which reaches the ground in 10 sec. Neglecting friction, find the height of the balloon when the stone is released. (O. & C.)

8. Define *force*, *acceleration*, *momentum*, and state any relationships between them.

A train of mass 100 tons, initially at rest, is subjected to a steady driving force of 700 lb. wt. for 2 minutes, and subsequently to a steady retarding force of 200 lb. wt. Draw a velocity-time graph for its motion, and determine the total time during which it is in motion and the distance travelled in this period. (O. & C.)

9. State the principle of *conservation of momentum* and show how it follows from Newton's Laws of Motion.

If a force of 10 lb. wt. acts steadily on a mass of 1 ton, how long will elapse before the mass has moved 14 ft., if it starts from rest? (O. & C.)

10. What do you understand by *uniform acceleration*?

Describe an experiment to show that a body under the action of a constant force moves with uniform acceleration.

An aeroplane flying horizontally at 320 ft. per sec. releases a bomb at a height of 1,600 ft. Find the time taken for the bomb to reach the ground, and also the magnitude and direction of the velocity with which it strikes the ground. (O.)

11. State Newton's Laws of Motion.

A wooden block, of mass 5 lb., is pulled along a rough horizontal table by means of a string parallel to the surface of the table, which passes over a light frictionless pulley at the edge of the table and carriés a weight of 3 lb. The resistance offered by the surface of the table to the motion of the block is 2 lb. wt. Calculate the acceleration with which the block moves. (O.)

12. Two masses, each of 200 gm., are attached permanently to the cord (or ribbon) of an Atwood's machine, so as to hang on opposite sides of the pulley. Calculate the acceleration of the system when a rider of mass 10 gm. is placed on one of the permanent masses. (O.)

13. A trolley trace [similar to that of the figure of p. 63] was taken with the spring making one complete vibration in $\frac{1}{5}$ second. Starting with the point A, some distance from the beginning of the trace, the following measurements were taken :

Measurement	*AB*	*AC*	*AD*	*AE*	*AF*	*AG*
Distance in cm.	10·0	18·5	25·5	31·0	35·0	37·5

Show that the retardation is uniform, and find its value. (O.)

14. Explain what is meant by *uniform acceleration*.

Prove that the distance s travelled in time t by a body which starts from rest and moves in a straight line with uniform acceleration a is given by the equation $s = \frac{1}{2}at^2$.

A body starts from rest and moves in a straight line with a uniform acceleration of 20 ft. per sec. per sec. Plot a distance-time graph for the first 5 secs. of its motion, and from the graph find (*a*) the time taken to travel 100 ft., (*b*) the velocity after it has travelled 100 ft. (O.)

15. What is meant by *uniform acceleration*?

Describe an experiment to show that a freely falling body moves with uniform acceleration.

Two metal cylinders, A and B, each of mass 490 gm., are attached to the ends of a light string which passes over a light frictionless pulley as shown in the diagram. The cylinder A carries a rider of mass 1 gm. After A has fallen 50 cm. from rest, a ring R removes the rider but allows A to travel on. Find the time A takes (i) to reach the ring, (ii) to travel the *next* 50 cm. after the removal of the rider. (O.)

16. Explain how Newton's first and second laws of motion respectively define *force* and the way in which it is to be measured. Show that the third law leads to the principle of conservation of momentum, illustrating your answer by *two* practical examples.

A chain hanging vertically and weighing 3 lb. per ft. is being wound on a drum of 3 ft. diameter, the free end of the chain being coiled on the floor. The drum rotates uniformly with a speed of 90 revolutions per min., its axis being horizontal and 20 ft. above the floor. Draw a diagram to scale showing how the tension in the chain, at the point

where it first makes contact with the drum, varies with the length of the unwound chain as the last 40 ft. are being wound up. Use the diagram to deduce the total energy required for this operation. Show on the diagram how much of this is potential energy, and explain what form the remainder of the energy takes. (N.)

17. Sketch (*a*) a velocity-time graph and (*b*) a distance-time graph for a body moving with uniform acceleration. Explain how the distance travelled in a given time can be found from the first graph.

Starting from rest a body slides down a smooth inclined plane. It acquires a velocity of 8 ft. per sec. after travelling 2 ft. What is its acceleration and what is the inclination of the plane to the horizontal? (O. & C.)

18. Show that, under certain ideal conditions, the path of a particle projected near the surface of the earth is a parabola. Find the range of a particle projected from the surface of the earth with velocity v at an angle α to the horizontal. What factors affect the result in practice?

An aeroplane flies at a height h at a constant speed u in a straight horizontal line, so as to pass vertically over a certain gun. At the instant when the aeroplane is vertically over it, the gun fires a shell which hits the aeroplane. Neglecting the effects of the atmosphere, deduce the minimum muzzle velocity of the shell and the correct angle of elevation at this velocity. (O. & C.)

19. A coach is slipped from a train travelling uniformly at 60 miles per hour when the train is one mile from a station. If the coach is uniformly retarded so as to stop at the station, where is the train at the moment the coach stops at the station? (C. S.)

20. An aeroplane flying horizontally at a uniform speed drops a bomb at a target one mile away horizontally. The bomb when it strikes the target is moving in a direction making an angle of 60° with the horizontal. Find the height and speed of the aeroplane. (C. S.)

21. Describe the ballistic balance (or ballistic pendulum), and explain how you would use it (*a*) to compare the masses of the suspended bodies, (*b*) to verify Newton's experimental impact law.

22. In an experiment with the ballistic balance, the angular displacements from the vertical of the strings supporting A are α before and β after collision, both to the left ; for B the corresponding angles are θ and ϕ, both to the right.

Show that the ratio $\dfrac{\text{mass of } A}{\text{mass of } B} = \dfrac{\sin\frac{\theta}{2}+\sin\frac{\phi}{2}}{\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}}$, and find the ratio of the relative velocity after impact to the relative velocity before impact.

CHAPTER 4

EQUILIBRIUM

A BODY at rest or in uniform rectilinear motion with respect to the surroundings, which we suppose to be fixed, is said to be **in equilibrium.**

Considering the body for the moment as equivalent to a single massive particle located at its centre of gravity, it follows that if it is at rest there can be no acceleration of the centre of gravity in any direction, and hence no resultant force in any direction. It would be easy to say that equilibrium is simply a state in which no external forces act on the body ; but we cannot imagine a body on which no external forces act. What happens must be that *the external forces acting must together combine to give a zero resultant.*

However, the external forces acting on an actual body are not (except in certain important particular cases) necessarily concurrent, and it is only rarely that their lines of action will all pass through the centre of gravity. But it has been seen (p. 50) that a force applied anywhere on a body can be regarded as applied at the centre of gravity if it is supplemented with an appropriate couple to compensate for this shift. Let us suppose this done, and again examine the body, still in equilibrium with no angular acceleration (p. 153). The moments of the introduced couples must just balance one another. Any point can be regarded as a possible axis about which rotation might conceivably occur, but does not. So *the algebraic sum of the moments about any axis of all the external forces acting on a body in equilibrium must be zero.*

Whenever equilibrium occurs, these two conditions

(1) resultant force zero,

(2) resultant moment about any axis zero,

must always be satisfied by the external forces. The second really includes the first, for if a body rotates about any fixed axis not passing through the centre of gravity, the latter must necessarily be displaced by a resultant force ; absence of resultant moment about *any* axis therefore implies that the resultant force must be zero.

The foregoing should be thoroughly understood before any attempt is made to solve problems on equilibrium. It is not pretended that such problems are ever easy, for as a rule they require concentration on two things at once, namely, the essential physics of the problem and the mathematical calculations resulting therefrom. The former is by far the more difficult to achieve ; and the student is recommended to think carefully what is happening, to consider the problem in terms of the principles outlined above, and to bring his own common-sense to bear on it, before writing down any equations.

Two simpler cases will be considered in detail before discussing the general problem of a body at rest under a number of external forces.

Two Forces

Equilibrium under only two forces. For a body to be in equilibrium under two external forces :

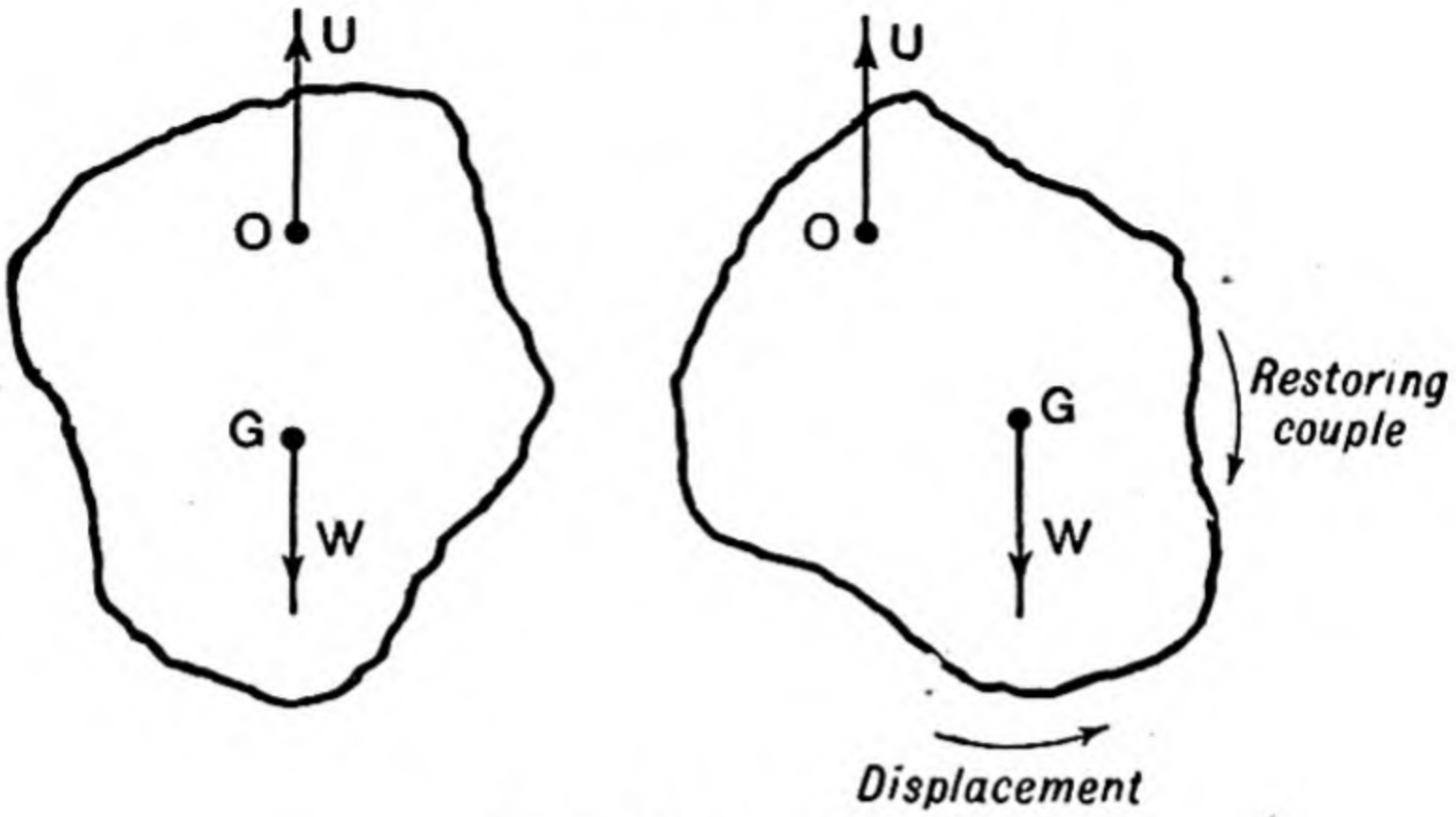
(*a*) they must be equal and opposite, otherwise they have a resultant which is not zero. (They can *not* be the "action and reaction" of Newton's Third Law, of course, since action and reaction in that case act on different bodies.)

(*b*) they must have the same line of action (though directed in different senses along it), otherwise they would, if satisfying (*a*), constitute a couple, and so their resultant moment about any point would not be zero.

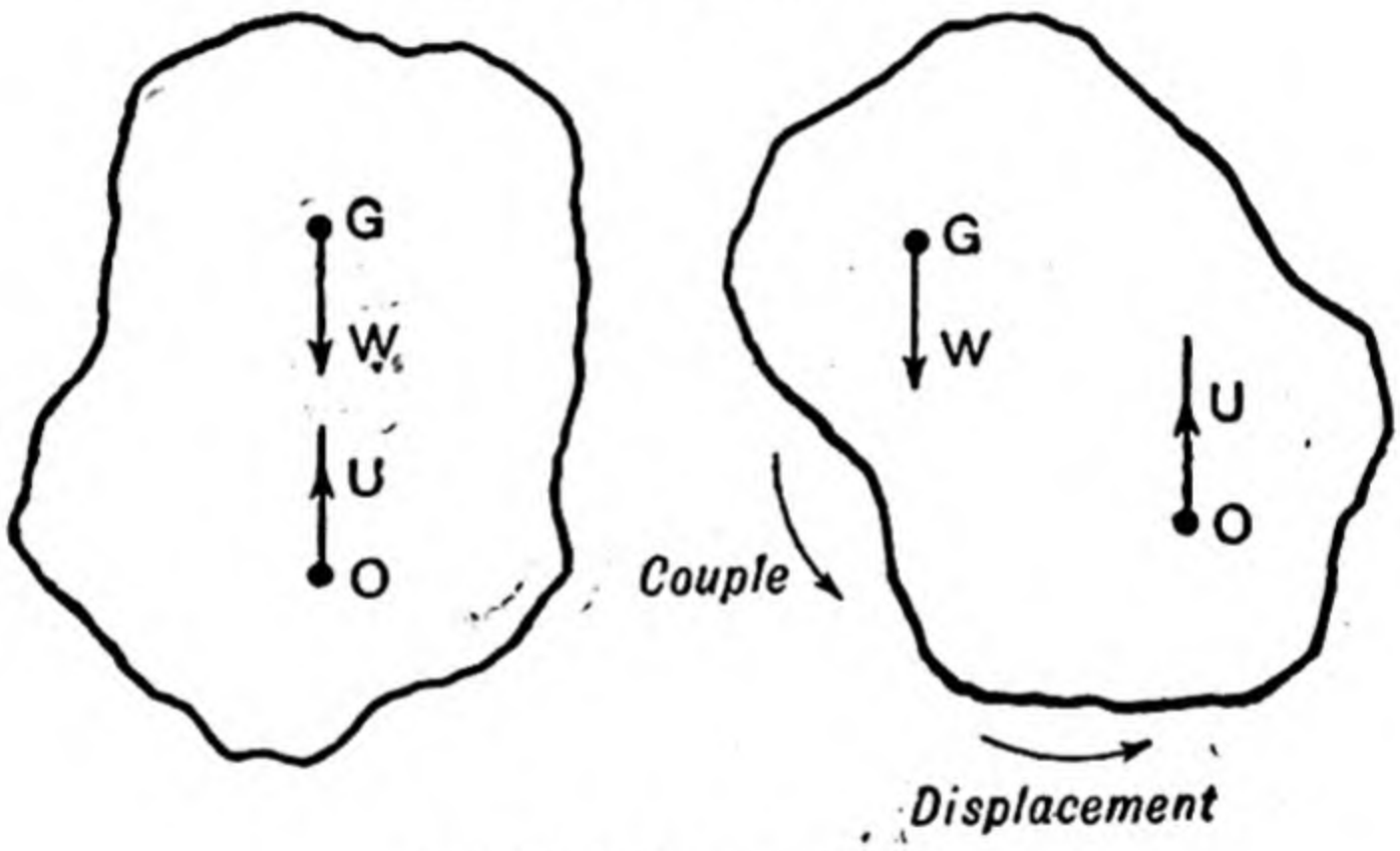
Consider the very important practical case of a body held at rest by an external upward force U. By (*a*), this must be equal in magnitude and opposite in direction to the other external force acting on the body, namely, its weight W ; and by (*b*) the line of action of U, like that of W, must pass through the centre of gravity G, of the body.

This can be achieved in two ways. The first (see p. 92, Fig. *a*), has the external upward force applied at a point O vertically above G, by resting the body on a smooth pivot at O or hanging it from a string attached to O. In this case, if the body is displaced slightly from the equilibrium position, W and U are no longer collinear, and form a couple tending to restore equilibrium. Equilibrium in this case is said to be **stable.**

In the second way, U is applied at a point below the level of G.

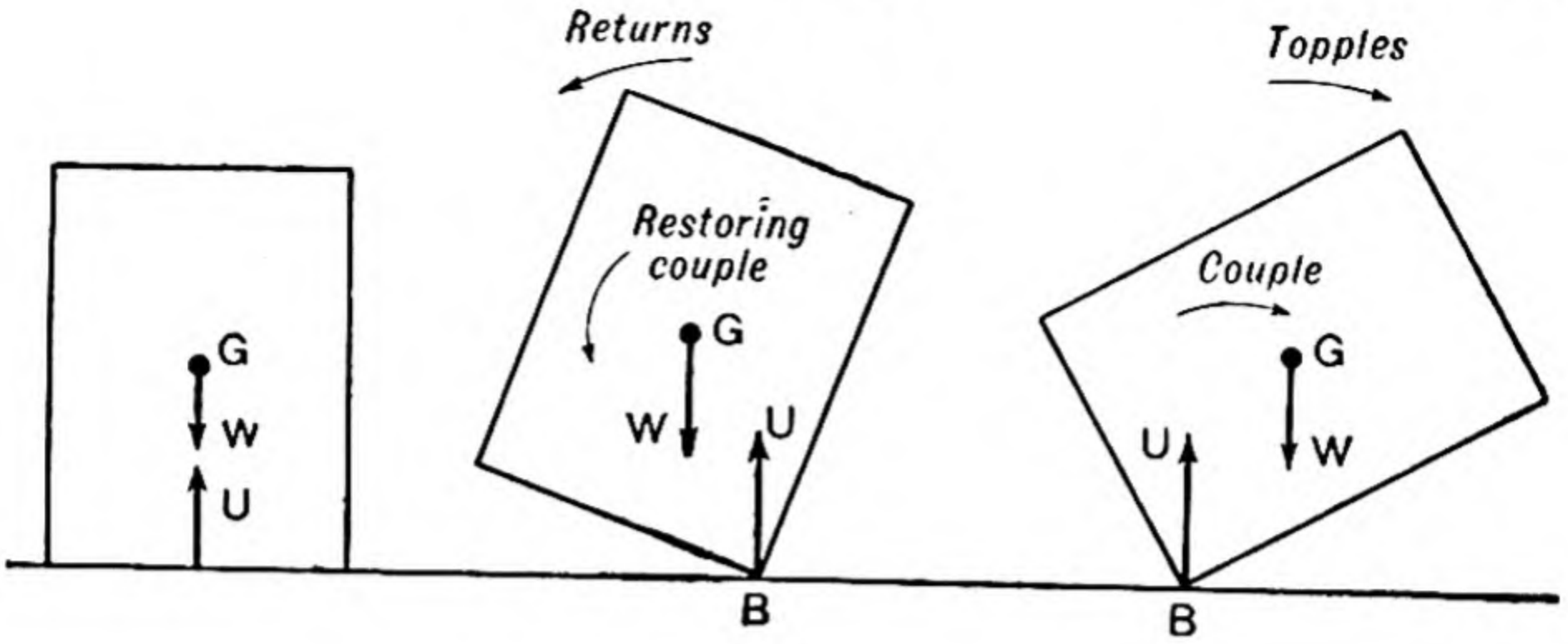


(*a*) Stable equilibrium



(*b*) Unstable equilibrium

In Fig. *b* is shown a case satisfying both conditions for equilibrium ; but (if *O* is a *smooth* pivot) a very small displacement causes *W* and *U* to form a couple which increases the displacement. This equilibrium is called **unstable**. If we say, as is usual, that unstable equilibrium exists when the condition can be destroyed by a small displacement, the question arises, how small? Mathematically, the test for unstable equilibrium is an *infinitesimal* displacement, but in all practical cases a small finite displacement is needed. If we can set a lower limit to the displacement required to destroy equilibrium, this is also an upper limit to the range of displacements for which equilibrium is not unstable—in other words, we are dealing with **limited stability**, and not instability.

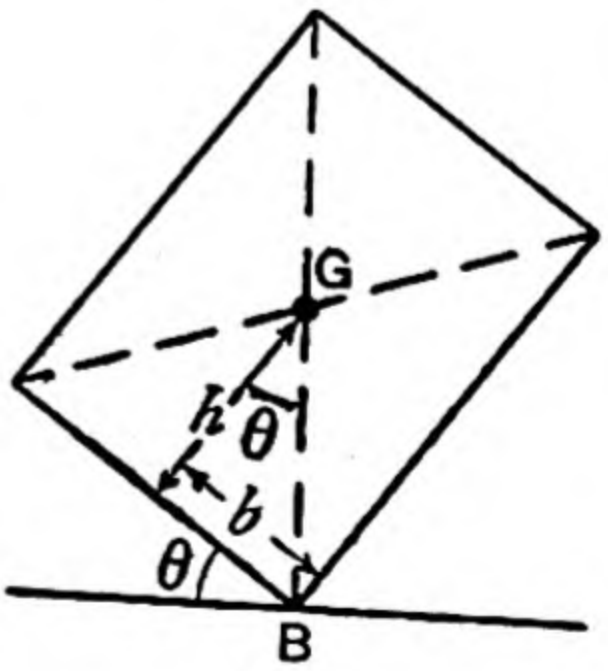


(c)

Fig. *c* represents a body resting on a horizontal plane surface. Contact between the body and the plane takes place at a limited number of points, at each of which there is some contribution to the *resultant* upthrust U, the line of action of which passes through G. If the body is slightly displaced, the only points of contact lie in the edge B, and U is at once transferred to pass through B, so that U and W constitute a couple. The body will return to the original position when released if the displacement has not taken G up to, or to the right of, the vertical line through the edge B, if G has passed to the right of the vertical line through B, then the body continues to turn in the direction of the displacement until a new equilibrium position is reached.

The body as it rests with its base on the plane is in stable equilibrium, and we can assess the degree of stability ; the greater the angle through which it must be displaced before equilibrium is destroyed, the greater the stability.

For a symmetrical body, if h is the height of G above the base, and $2b$ the width of the base, the angle θ through which it must be displaced before G is above B is given by $\tan\theta = b/h$ (Fig. *d*). We can speak of the situation of a body for which b is large and h small as being more stable than that of one for which b is small and h large ; but we can scarcely call the latter strictly unstable. When G is actually vertically

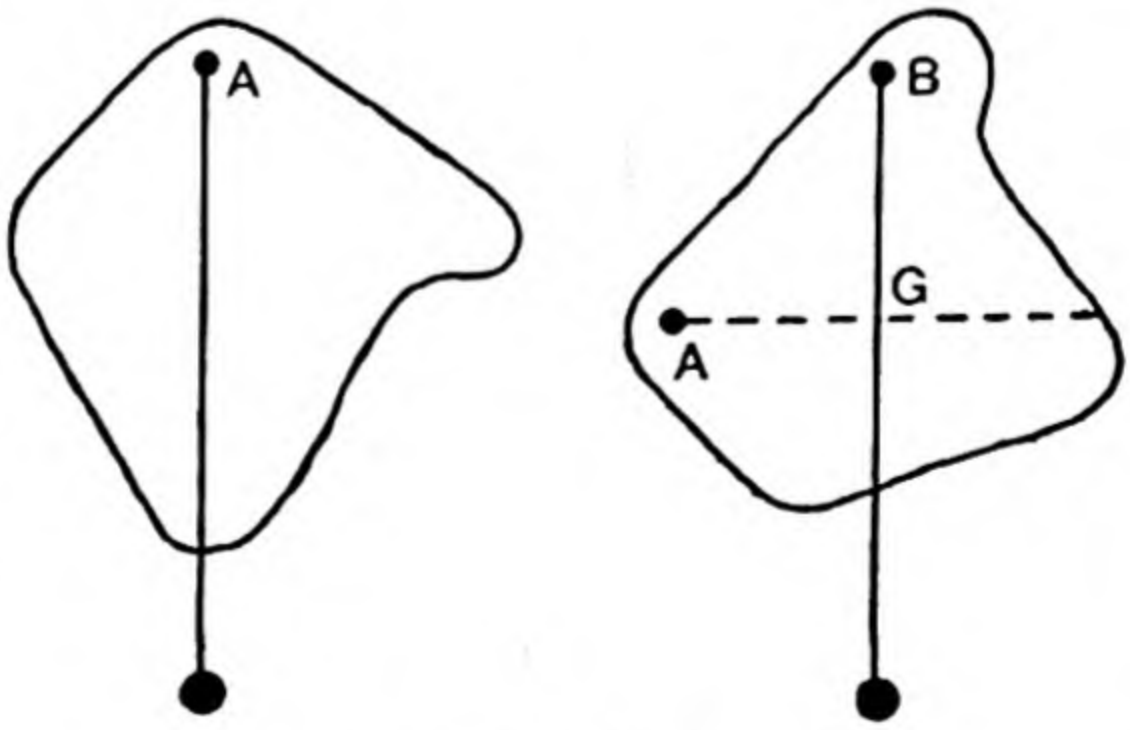


(*d*)

above B, the *conditions* for unstable equilibrium obtain ; but in this case, unless the edge B has a finite area, equilibrium is not practicable ; while if the area is finite we get equilibrium of a precariously limited *stability*.

A further case is possible, in which the point of application of U is transferred so that after displacement U still acts through G ; for example, a sphere, cylinder or cone of uniform material on a horizontal plane surface, or a body supported on a smooth pivot passing through the centre of gravity. Here, the body neither returns to the original position after a small displacement, nor proceeds to a new position ; it stays displaced. This is called **neutral equilibrium.**

Determination of the position of the C.G. of a thin sheet. Several small holes are pierced in various positions round the edge of the sheet, and the body is suspended from a suitable smooth pivot through one hole A ; a plumb-line of fine thread is attached to the same pivot. Both are allowed to swing freely, and come to rest. The direction of the plumb-line is marked on the sheet. As the sheet is in stable equilibrium, the centre of gravity is vertically below the support, and thus somewhere along the plumb-line trace. The process is then repeated, using a second hole B, about a quarter of the way round from the first. The two lines intersect at the centre of gravity, which must lie on both. As a check, one or more additional suspension points can be taken, giving further lines, all of which will pass over the centre of gravity.



Locating C.G. of a thin sheet.

Three Forces

Equilibrium under three forces. If three external forces maintain a body in equilibrium, we can imagine that two of them (P and Q) are replaced by their resultant R, so that the problem can be considered in terms of only two forces, namely, this resultant R and the third force S.

The resultant R of any two of the forces must thus be equal and opposite to the third force, and must act in the same line. From this it follows that the lines of action of the three forces must all lie in the same plane, for otherwise R cannot act along the same line as S.

The lines of action of all three forces must *either* all be parallel, *or else* pass through one point.

If P and Q are parallel, their resultant R is parallel to them ; so S, acting along the same line as R, must be parallel to P and Q.

If P and Q are not parallel, their lines of action must intersect at some point O ; their resultant R must also pass through O (see p. 40) ; but the third force S is equal and opposite to R, so its line of action must also pass through O. Thus all three lines of action are concurrent at O.

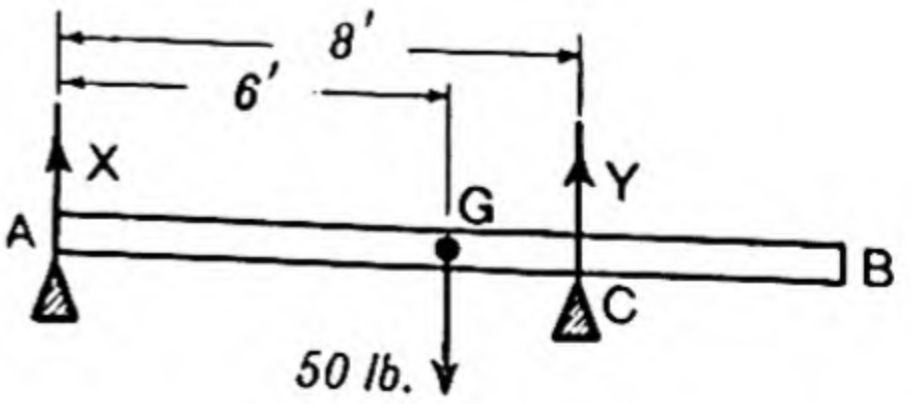
The two cases will be considered separately.

Three parallel forces. The condition for the *resultant force to be zero* is that the algebraic sum of the forces must be zero ; that is, any one force is equal and opposite to the sum of the other two.

The condition for the *resultant moment about any point to be zero* is that the algebraic sum of the moments of all the forces about any point in the plane shall be zero.

EXAMPLE. A uniform beam AB, 12 ft. long, weighing 50 lb., is supported at the end A and at a point C which is 4 ft. from B. Find the reactions of the supports at A and C.

We are considering the equilibrium of the *beam*, so the first thing is to draw a diagram showing all the forces acting on it ; thus we put in the weight, acting through the centre of gravity G at the middle of the beam, and the upthrusts at A and C acting on the beam ; we do not consider the downward thrusts



by the beam on the supports at A and C, for these act on the supports and *not* on the beam.

Let X and Y be the upthrusts at A and C in lb. wt.

Then, by the first condition,

$$X + Y = 50. \text{ lb. wt.}$$

Taking moments about A,

moment of 50 lb. wt. at $G = 50 \times AG \curvearrowright = 300$ lb. $\times$ ft. $\curvearrowright$.

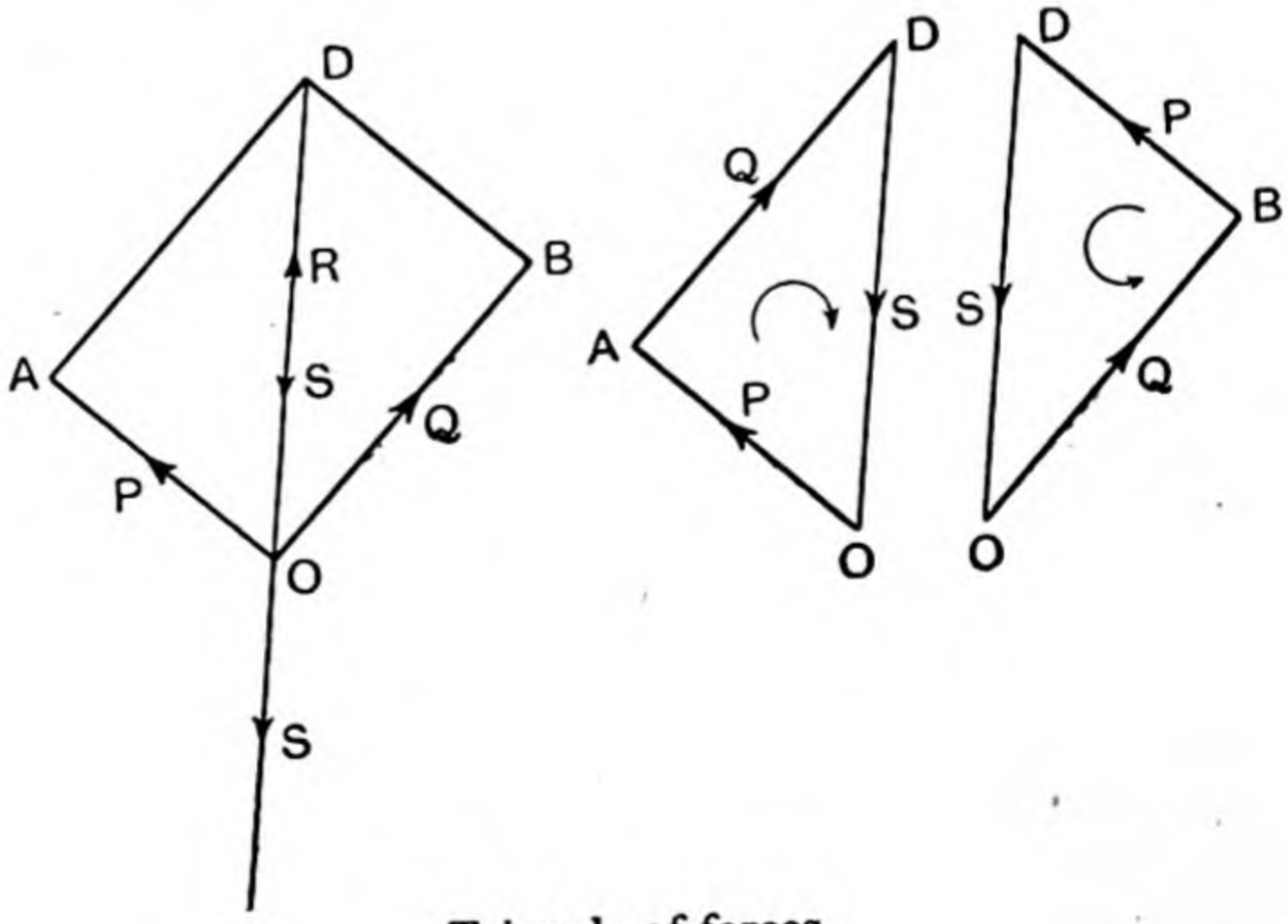
moment of Y at $C = Y \times AC \curvearrowleft = 8Y$ lb. $\times$ ft. $\curvearrowleft$.

So the algebraic sum of the moments about A is $8Y - 300$, which is zero. Hence $Y = 300/8 = 37{\cdot}5$ lb. wt., and $X = 50 - Y = 12{\cdot}5$ lb. wt.

Three non-parallel forces. Consider the three forces P, Q, S in equilibrium, their lines of action intersecting at one point O.

S must be equal and opposite to the resultant R of P and Q, represented by the diagonal OD of the parallelogram of forces $OADB$ (see diagram); and as OD completely represents R, DO must completely represent S. If we sacrifice the convenience of being able to represent the actual point of application of the forces, we can use *half* this parallelogram to represent P, Q, and S; either of the triangles OAD or OBD will do this. Whichever of them is taken, it will be seen that *the arrows all follow the same way round.*

Thus, **three non-parallel forces in equilibrium can be represented in size and direction** (though not in actual point of application) **by the three sides**



Triangle of forces.

of a triangle taken in order (that is, with the arrows following consecutively round). This is the **triangle of forces rule.**

The condition for the resultant moment to be zero is automatically satisfied. For (p. 43) the moment of R about any point is equal to the algebraic sum of the moments of P and Q about that point; and as S is equal and opposite to R, the moment of S about any point must be equal and opposite to the sum of the moments of P and Q about that point. From the point of view of the solution of a problem, therefore, the moment condition gives no information not already available in the triangle rule, for it simply restates the essential data in another form. Thus, *the triangle rule alone* enables any problem on three non-parallel forces in equilibrium to be completely solved; but the use of the zero-moment condition gives a useful alternative approach.

Problems on three non-parallel forces. The first essential is to read the problem carefully and note:

(i) the body whose equilibrium is in question;
(ii) that only three forces act *on that body*;
(iii) exactly what these forces are;
(iv) the point through which the three lines of action pass; this may be obvious, or may have to be determined by finding the point at which *two* of the lines of action meet.

Failure is usually due to misconception of the problem; once these preliminaries have been properly thought out, difficulty should not occur.

The experienced student will then pause and consider whether it will be simpler to solve by taking moments; or if, using the triangle rule, the triangle is to be solved trigonometrically. In both these cases diagrams need only be drawn with the accuracy sufficient to illustrate the working. It is best at first, though, to acquire some proficiency in purely graphical solutions, when two careful diagrams must be drawn. These are:

(1) **The space diagram.** This should be drawn with all the angles correct, and to scale if this is needed in order to find the angles from the information supplied. The point O through which all the lines of action pass must be located and shown on the figure. All the three forces acting on the body must be shown, and correctly arrowed.

(2) **The force triangle for the point *O*.** A scale is chosen, and, starting with the known force, a triangle is constructed with its sides parallel to the forces of the space diagram. The force triangle may be superposed on the space figure, or drawn close by the side of it, constructing the parallel lines with straight-edge and set-square. The sides of the completed triangle should be arrowed in the directions of the forces they represent, as a check that the triangle has been correctly drawn with the sides taken in order. The lengths of the sides then represent the magnitudes of the three forces.

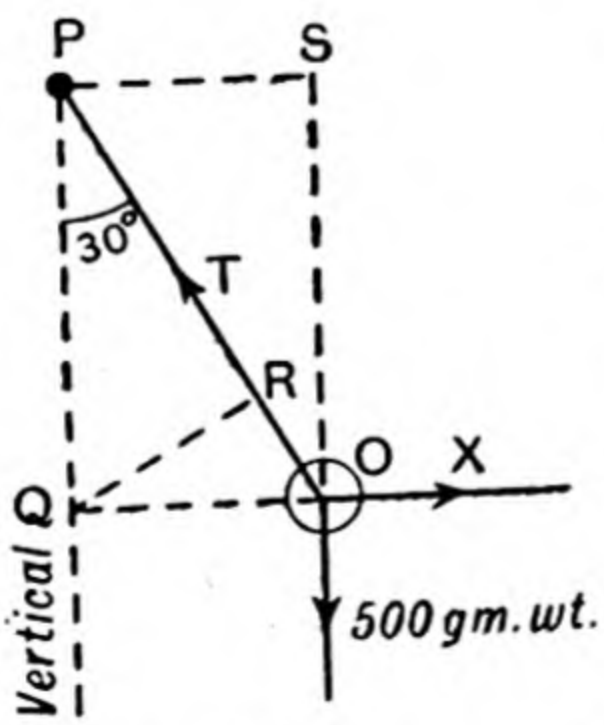
EXAMPLE. A small ball, weighing 500 gm., is suspended by a thread from a fixed support ; a horizontal force X is applied to the ball, so that the thread is inclined at an angle of 30° to the vertical. Find the value of X, and also the tension in the string.

The ball is in equilibrium under three forces : its weight, 500 gm. wt., acting vertically downwards ; the horizontal force X ; and the tension T directed along the string. These all meet in a point O.

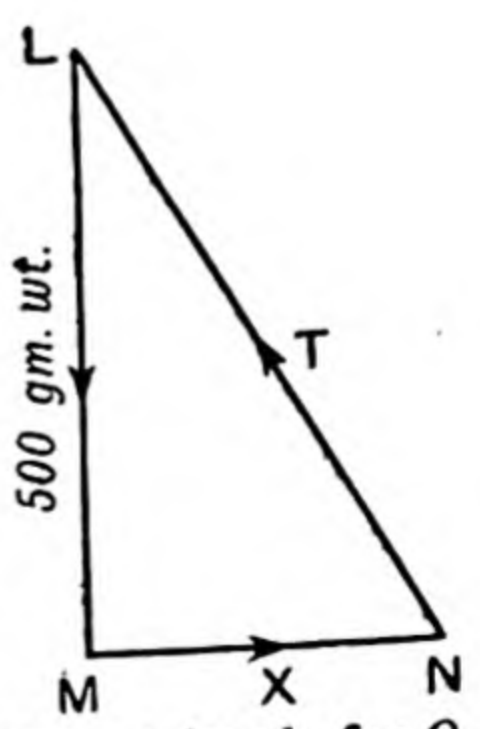
The diagram shows the completed exercise. It will be noted that the force triangle is started with the known force, LM, and that the point N is found by drawing from L and M respectively lines parallel to the other two forces. The sides MN and NL represent X and T respectively, and these are measured and interpreted according to the chosen scale.

Solving the triangle LMN trigonometrically, as $MN/LM = \tan 30°$, and $LM/NL = \cos 30°$, it is seen that $X = 500/\sqrt{3} = 289$ gm. wt., and $T = 2 \times 500/\sqrt{3} = 577$ gm. wt., to three significant figures.

Alternatively, referring to the space diagram, by taking moments about P (through which T passes and about which it thus has no moment),



Space figure.



Force triangle for *O*.

$$X \times PQ \curvearrowleft - 500 \times OQ \curvearrowright = 0,$$

whence X is determined; and taking moments about Q (through which X passes, and about which it thus has no moment),

$$T \times QR \curvearrowleft - 500 \times OQ \curvearrowright = 0,$$

so T is calculated.

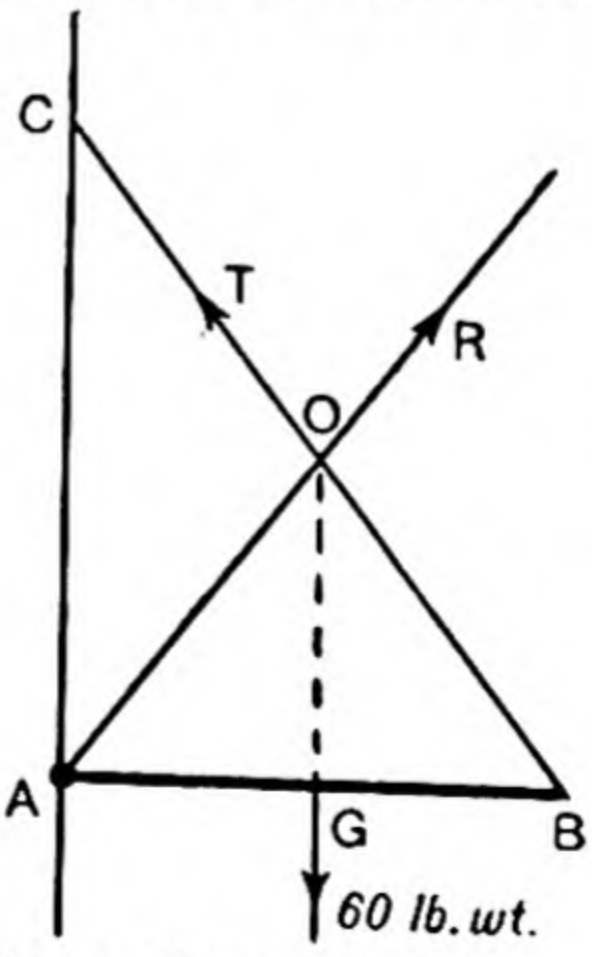
It will be seen that, as $OQ/PQ = \tan 30°$, and $QR/OQ = \cos 30°$, the calculation is the same essentially as if the force triangle were solved trigonometrically. If the question had asked for the determination of either T or X alone, instead of for a complete solution, this would have been the quicker method. It is a useful practice to look over the finished problem, after it has been solved graphically, and note the points that would have been selected to take moments about if this alternative had been used.

EXAMPLE. A uniform beam AB, 3 feet long, weighing 60 lb., is hinged to a wall at A, and held horizontally by a string attached at B and joined to a point C in the wall 4 ft. vertically above A. Find the tension of the string and the reaction of the hinge.

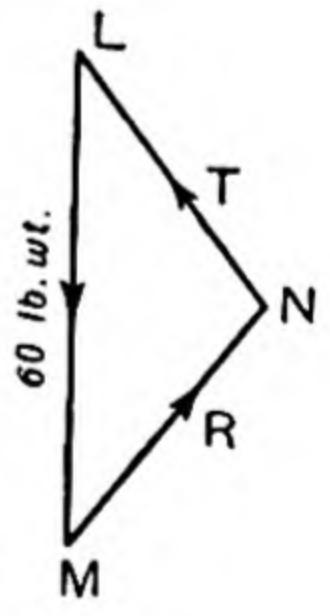
Considering the equilibrium of the beam, the three forces acting are: the weight, 60 lb. wt., acting vertically downwards through the mid-point G; the tension T in the string, directed along BC; and the reaction R of the hinge on the beam, the direction of which is to be determined. Hinges are so designed that they can prevent motion in any direction at right angles to the axis of rotation, and hence the reaction may be in any direction required to preserve equilibrium with the other forces. But the lines of action of the weight, of T, and of R must all meet in one point; that is, the line of action of R passes through the point O at which the string BC and the vertical line through G intersect.

The diagram shows the graphical solution. It has been necessary here to draw the space diagram to scale, so as to get the directions correct. As before, the force triangle for O has been drawn to scale starting with the known force.

By calculation, it is seen that the force triangle LMN and the triangle CAO of the



Space diagram drawn to scale.



Force triangle for O.

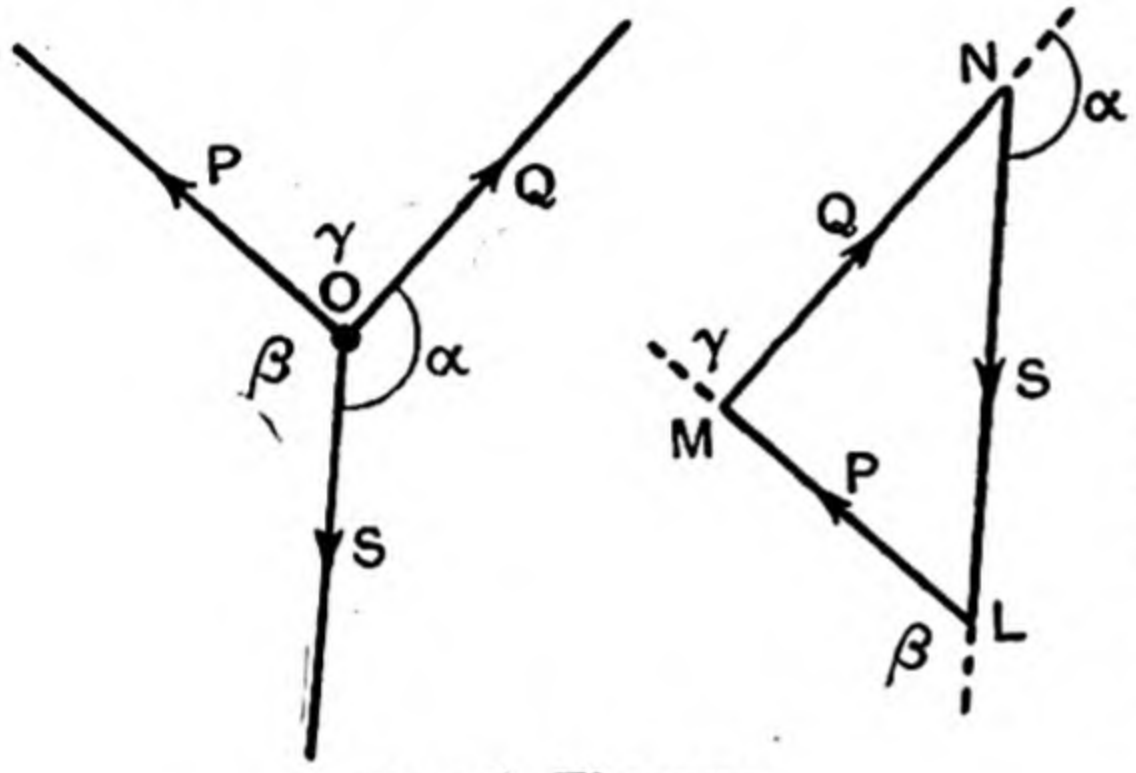
space diagram are similar, whence

$$\frac{T}{60}=\frac{LN}{LM}=\frac{OC}{CA}=\frac{2{\cdot}5}{4}$$

so that $T=37{\cdot}5$ lb. wt. ; similarly $R=37{\cdot}5$ lb. wt.

To solve by moments, taking moments about A gives an equation for T, while, taking moments about B, an equation for R is found. The appropriate perpendicular distances are obtained from the space figure.

Lami's Theorem. In both the preceding examples, the triangle, having been drawn, was easily solved by calculation. In one case the properties of a right-angled triangle were used, in the other case similar triangles. **Lami's Theorem** is a rule for applying the "sine formula" to solving the triangle without actually constructing it. The diagram represents the space diagram for three forces P, Q, S in equilibrium at O; α is the angle between Q and S, β that between S and P, and γ that between P and Q. In the force triangle for O, with LM representing P, MN representing Q, and NL representing S, α is the exterior angle at N, β that at L, and γ that at M. By the sine formula,



Lami's Theorem.

$$\frac{LM}{\sin(180^\circ-\alpha)}=\frac{MN}{\sin(180^\circ-\beta)}=\frac{NL}{\sin(180^\circ-\gamma)};$$

whence, as $\sin(180^\circ-\alpha)=\sin\alpha$, etc.,

$$\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}$$

or, each force is proportional to the sine of the angle between the other two forces.

General Conditions for Equilibrium

Proceeding to the general case of any number of forces keeping a body in equilibrium, two of the restrictions found for three forces no longer operate; for

(1) the lines of action *need not necessarily all lie in one plane* ; we shall, however, restrict discussion to the case when all the lines of action are coplanar ;

(2) if the forces are not parallel, the lines of action *need not necessarily pass through one point.*

Any number of parallel coplanar forces acting on a body give equilibrium when

(1) **the resultant force is zero** ; that is, the sum of the forces acting in one sense is equal to the sum of the forces acting in the opposite sense, or the algebraic sum of all the forces is zero ;

(2) **the resultant moment is zero** ; that is, taking any point in the plane, the algebraic sum of the moments of all the forces about that point is zero.

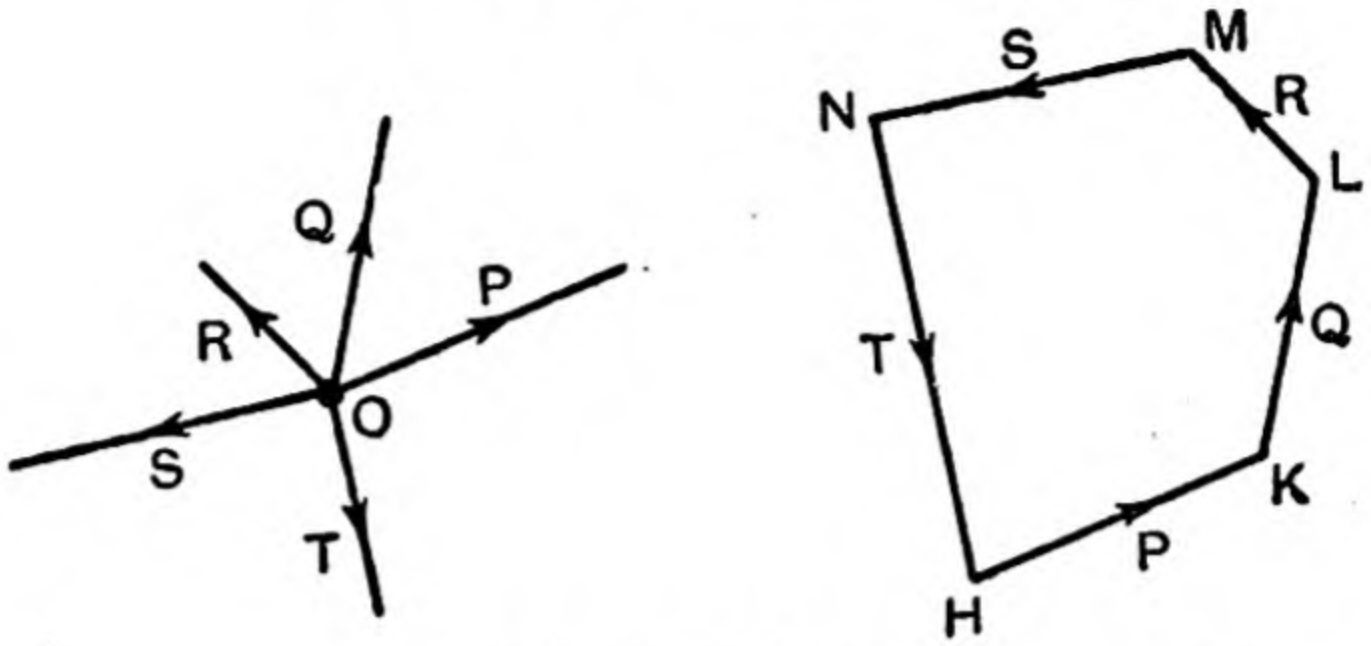
Any number of non-parallel coplanar forces will not, in general, have their lines of action concurrent. But we can, as on p. 50, consider each force displaced so that the line of action passes through a chosen point, provided an appropriate couple is introduced to compensate for the shift. Suppose this done for all the forces, so that we have a set of non-parallel forces the lines of action of which all pass through one point O, together with a compensating couple for each force.

Consider the forces acting at O first. For equilibrium, *their resultant must be zero* : we have seen (p. 37) that the resultant of a set of vectors drawn successively to scale is given by the closing side which completes the figure as a polygon. If the resultant is zero, this closing side is zero ; that is, the forces must *themselves* be represented in size and direction by the sides of a closed polygon, taken in order.

This is the Polygon of Forces rule : **any number of non-parallel forces acting at a point and producing equilibrium can be represented in size and direction by the sides of a closed polygon, taken in order.**

Thus the forces P, Q, R, S, T acting at O are represented to scale in size and direction by the sides HK, KL, LM, MN, NH of the closed polygon $HKLMN$ (p. 102).

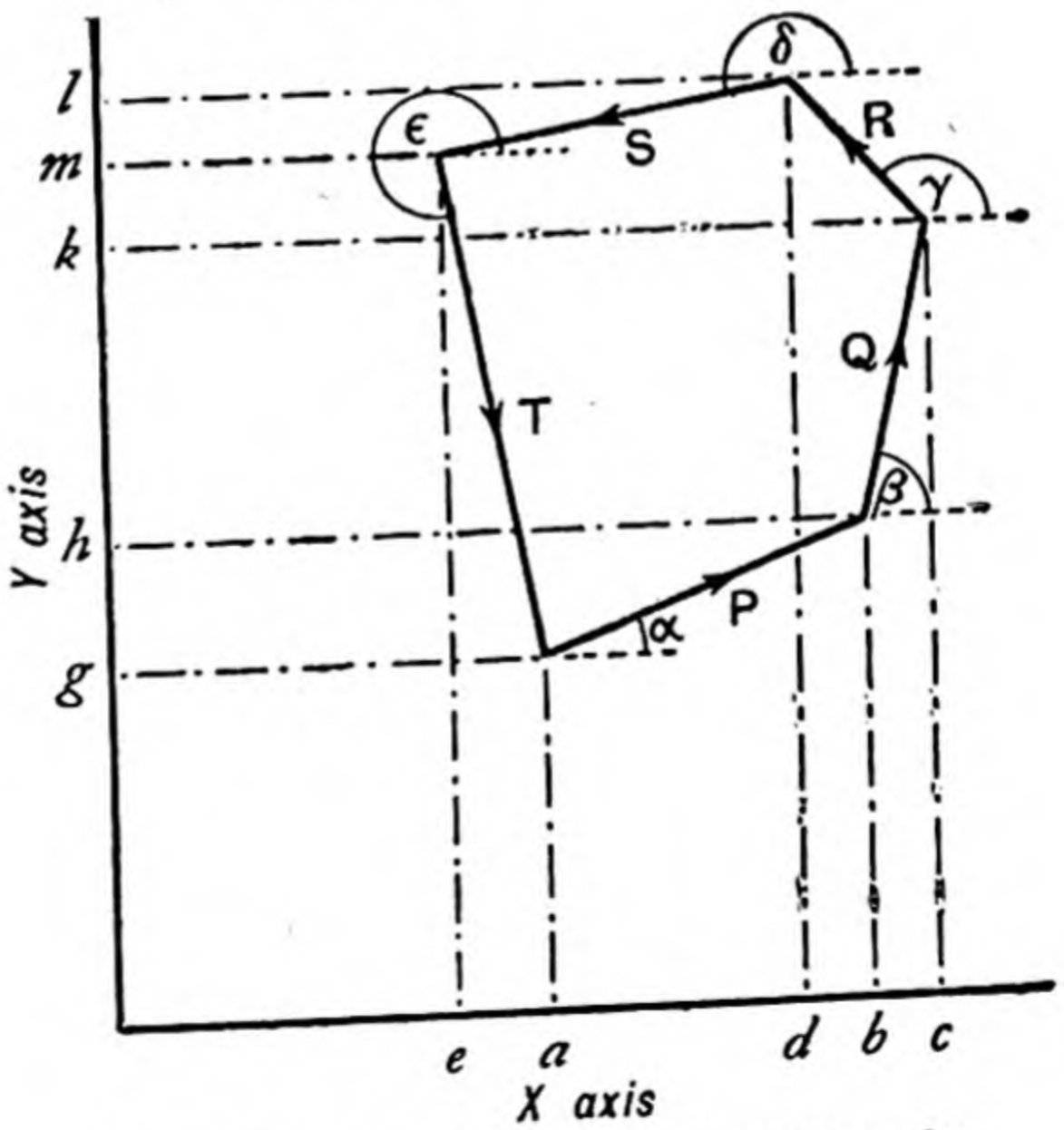
It is not always convenient to apply this condition graphically, and it may be put in another form. Choose any two axes X and Y at right angles to one another ; suppose the forces make angles



Polygon of forces.

α, β, γ, δ, ϵ, respectively with the X-axis. Resolve each force into its two components along the X-axis and the Y-axis; these can conveniently be marked off along the axes for reference. Thus, along the X-axis:

Component of P is $P \cos \alpha$, represented by ab
Q $Q \cos \beta$, bc
R $R \cos \gamma$, cd
S $S \cos \delta$, de
T $T \cos \epsilon$, ea



Resolution along two axes at right angles.

Starting from a, and marking each component off, we end up again at a, so the algebraic sum of the components is zero. Similarly, along the Y-axis :

Component of P is	$P \sin \alpha$,	represented by gh
Q	$Q \sin \beta$,	hk
R	$R \sin \gamma$,	kl
S	$S \sin \delta$,	lm
T	$T' \sin \epsilon$,	mg.

Marking off from g, we return to g at the end, so the algebraic sum of the components is zero.

This must inevitably happen for *any closed figure*, whatever the number of sides. So the zero-resultant condition of the polygon rule can be restated in this form :

Take any two axes at right angles to one another, and resolve each force into its components in these two directions. Then, for each of these directions separately, the algebraic sum of the resolved parts of all the forces in that direction is zero.

The condition that the resultant moment about any point must be zero, must now be considered. First, for the forces now collected at O, as the resultant itself is zero, the algebraic sum of the moments of the forces at O about any point in the plane must be zero ; for the algebraic sum of the moments of a set of forces is equal to the moment of their resultant about any point. But there are the compensating couples to consider. If there is to be equilibrium, the algebraic sum of the moments of these couples about any point in the plane is zero. Or, remembering that each force at O together with its couple is exactly equivalent to the force itself in its actual line of action, *the algebraic sum of the moments of all the forces about any point in the plane must be zero.*

Thus, the general conditions for equilibrium for any number of non-parallel forces in the same plane, acting on a body, are :

(1) **the algebraic sums of the resolved parts of all the forces in each of any two directions at right angles are separately equal to zero ;** and,

(2) **the algebraic sum of the moments of all the forces about any point in the plane is zero.**

Necessary and sufficient conditions. It should by now be evident that the essential physics of equilibrium for coplanar forces is summed up in the two statements :

(1) *resultant force zero,*
(2) *resultant moment about any point zero,*

with the understanding that (2) must essentially include (1). We have seen the conditions that must *necessarily* hold with two, three, and any number of coplanar forces ; but the *minimum* number of conditions required to ensure the satisfaction of (1) and (2) will be *sufficient* to enable us to describe the equilibrium completely and to solve any practicable problem. In other words, it must be remembered that the necessary conditions contain a wealth of information, and that if we try to employ all possible data all the time, we shall have more at our disposal than suffice for the task in hand.

For example, for equilibrium with three non-parallel forces, the forces must *necessarily* be coplanar and concurrent ; must *necessarily* be represented by a closed triangle ; and must *necessarily* have the moments about any point giving an algebraic sum zero. But the *sufficient* conditions are

either concurrence, and the triangle rule
or, lying in one plane, and the moments condition.

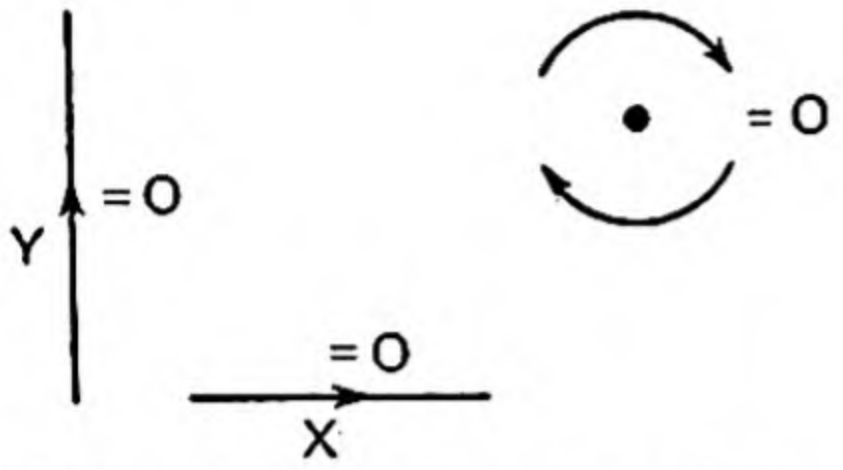
For a triangle is a plane figure, and there is no need to say that the sides of a plane figure are coplanar ; and it is easily shown that three non-parallel forces satisfying the moments condition must, if they lie in one plane, be concurrent. Thus the full list of necessary conditions includes complete and separate sets of conditions that are both necessary and sufficient.

With any number of forces, acting in any direction in the same plane, while a *necessary* condition is that for *any pair* of perpendicular axes in any directions, the sums of the resolved parts of all the forces are separately zero, it is *sufficient* if this is done for only *one* such pair of axes ; and while *necessarily* the algebraic sum of the moments of all the forces about *any point* must equal zero, it is *sufficient* if, coupled with resolution, moments are taken about only *one chosen point.*

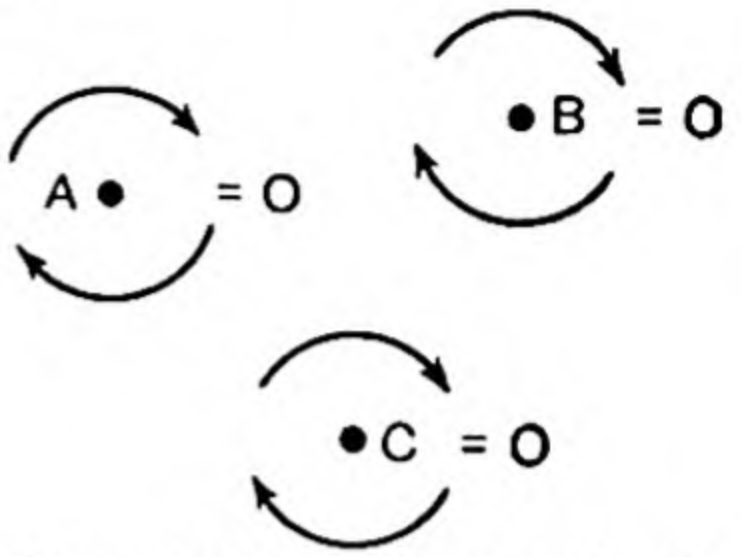
Thus, the necessary and sufficient conditions can be stated :

(1) **Choose one pair of directions at right angles to one another. The sum of the resolved parts of all the forces in each of these two directions is zero.**

(2) **Choose one point in the plane. The algebraic sum of the moments of all the forces about that point is zero** (see diagram (*a*)).



(*a*) One set of necessary and sufficient conditions for equilibrium.



(*b*) Another set of necessary and sufficient conditions for equilibrium.

Another set of necessary and sufficient conditions can be found. Consider any three points A, B, C, not in the same straight line. If the algebraic sum of the moments of all the forces about each of these three non-collinear points is separately zero, then there must be equilibrium. For certainly not more than two of these points, say A and B, can lie on the line of action of the resultant; so that if the moment of the resultant about C, which is not on its line of action, is zero, the resultant must itself be zero. Hence we have, as an alternative useful set of necessary and sufficient conditions,

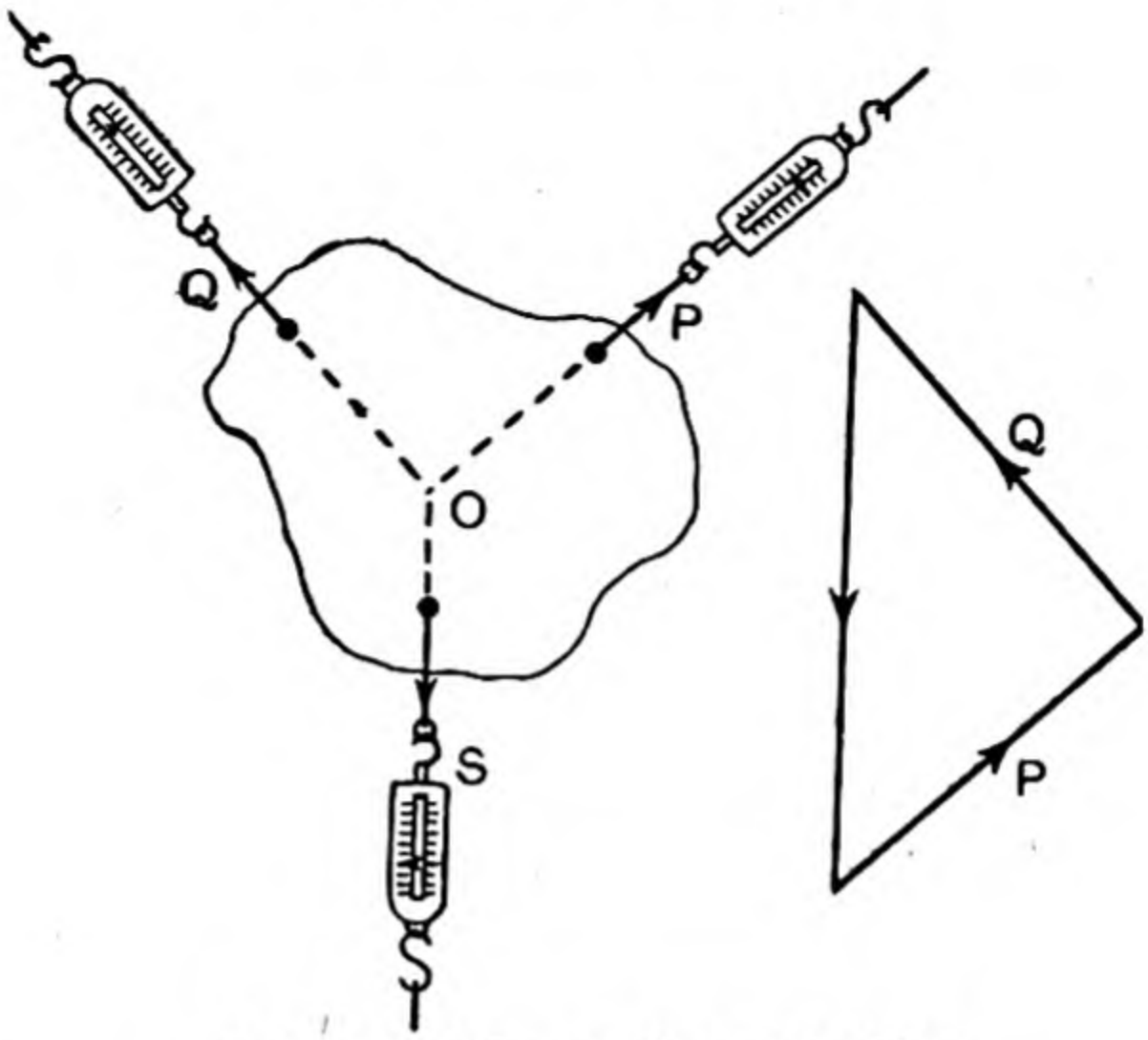
Choose any three points, not in the same line, and take moments about each of these points. Then for each point, the algebraic sum of the moments of all the forces is zero (see diagram (*b*)).

There is a third set: that the algebraic sums of the resolved parts along any line is zero, and if the algebraic sums of the moments about two points in that line separately equal zero.

Principle of Moments. It appears that the necessary and sufficient conditions for equilibrium are included in the moments condition which, in its most general form, is known as the **Principle of Moments.** This states that, when a body is in equilibrium under any number of coplanar forces, then about *any point* in the plane, the algebraic sum of the moments of all the forces is zero.

Experimental Verification of Equilibrium Conditions

Three non-parallel forces. A light sheet of thin plywood is held by strings attached at suitable points to three spring balances (see p. 106).



Triangle of forces demonstration.

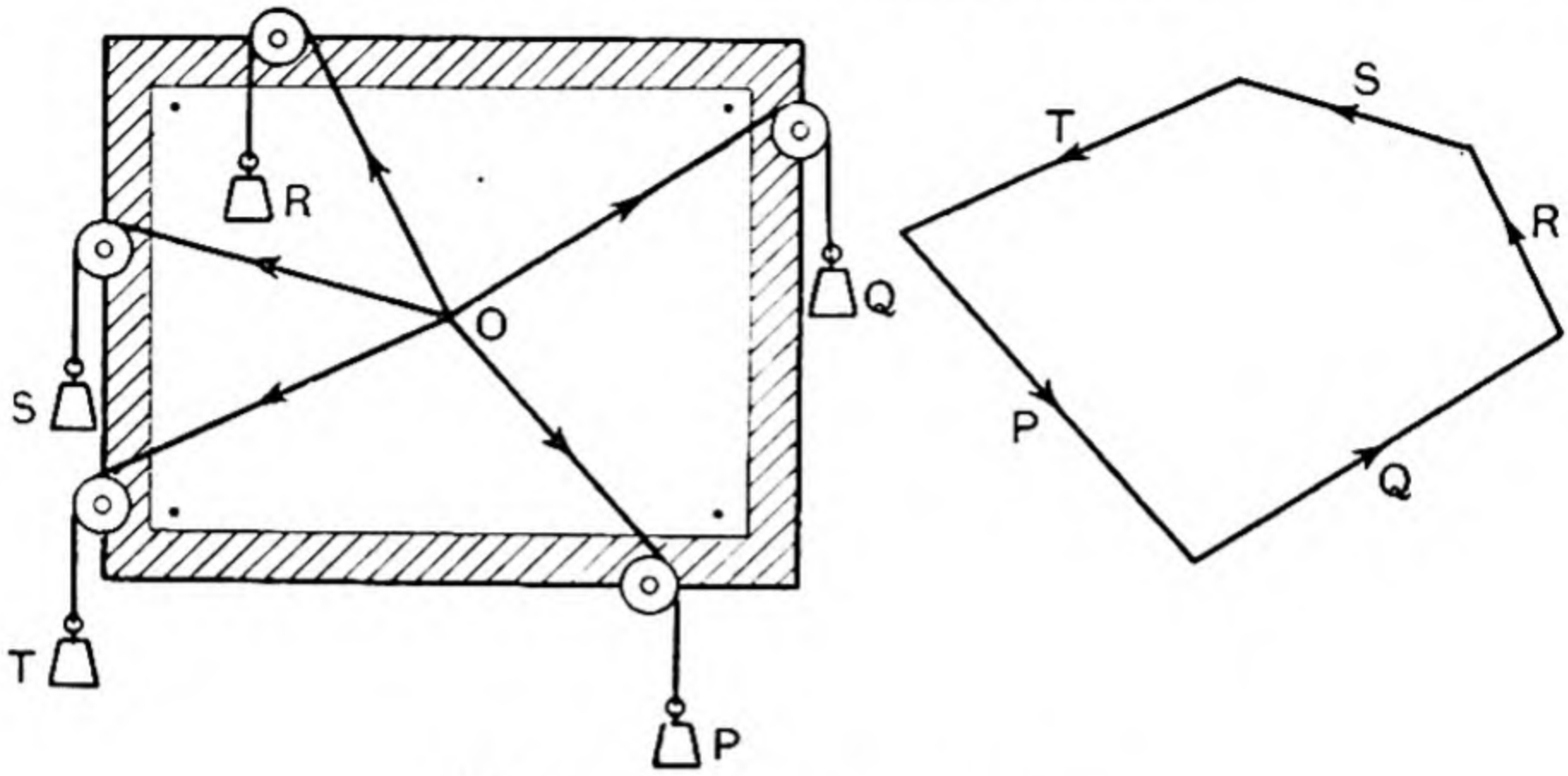
It is found that :

(i) The lines of the three strings all lie in one plane—that of the sheet ; equilibrium with the strings not coplanar can only be achieved by the intervention of a fourth force.

(ii) The lines of action of the strings, produced on to the sheet, all meet in one point *O* (Note : this is *not* the centre of gravity of the sheet).

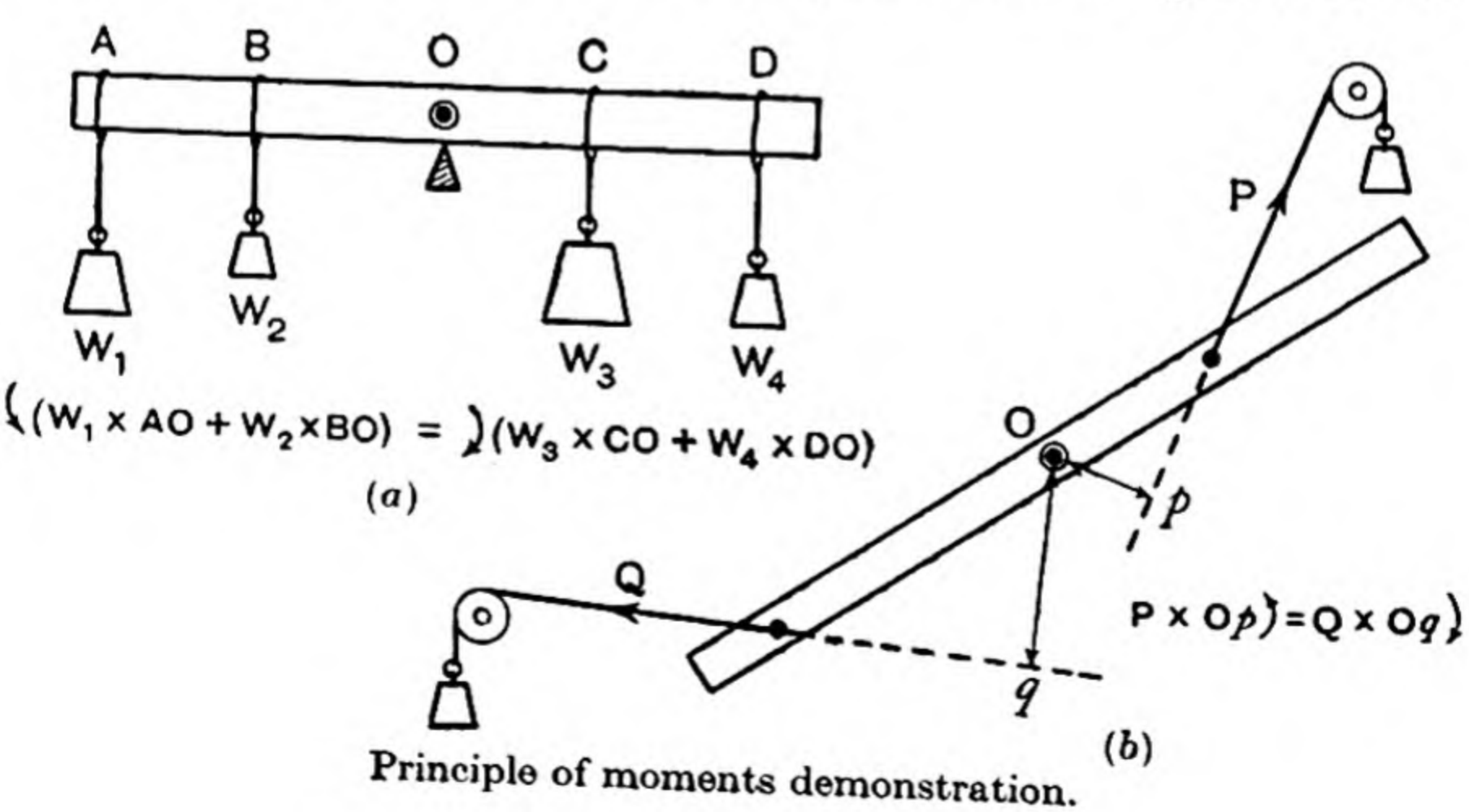
(iii) Reading the balances, and noting forces *P*, *Q*, and *S*, the lines on the sheet can be used as a space diagram ; constructing from this a force triangle, it is found that the lines representing the three forces *P*, *Q*, and *S* in size and direction do form a closed triangle with the arrows following round.

Any number of forces in any direction. The polygon of forces rule, for forces acting at a single point, can be tested with the apparatus illustrated. When the system has settled into equilibrium, the lines of the strings are traced on the paper behind them, and a space diagram for the point at which they meet is drawn ; from this a force diagram is constructed to scale starting with one force *P*, and working steadily round the space diagram. The lines representing the individual forces in size and direction form a completed polygon.

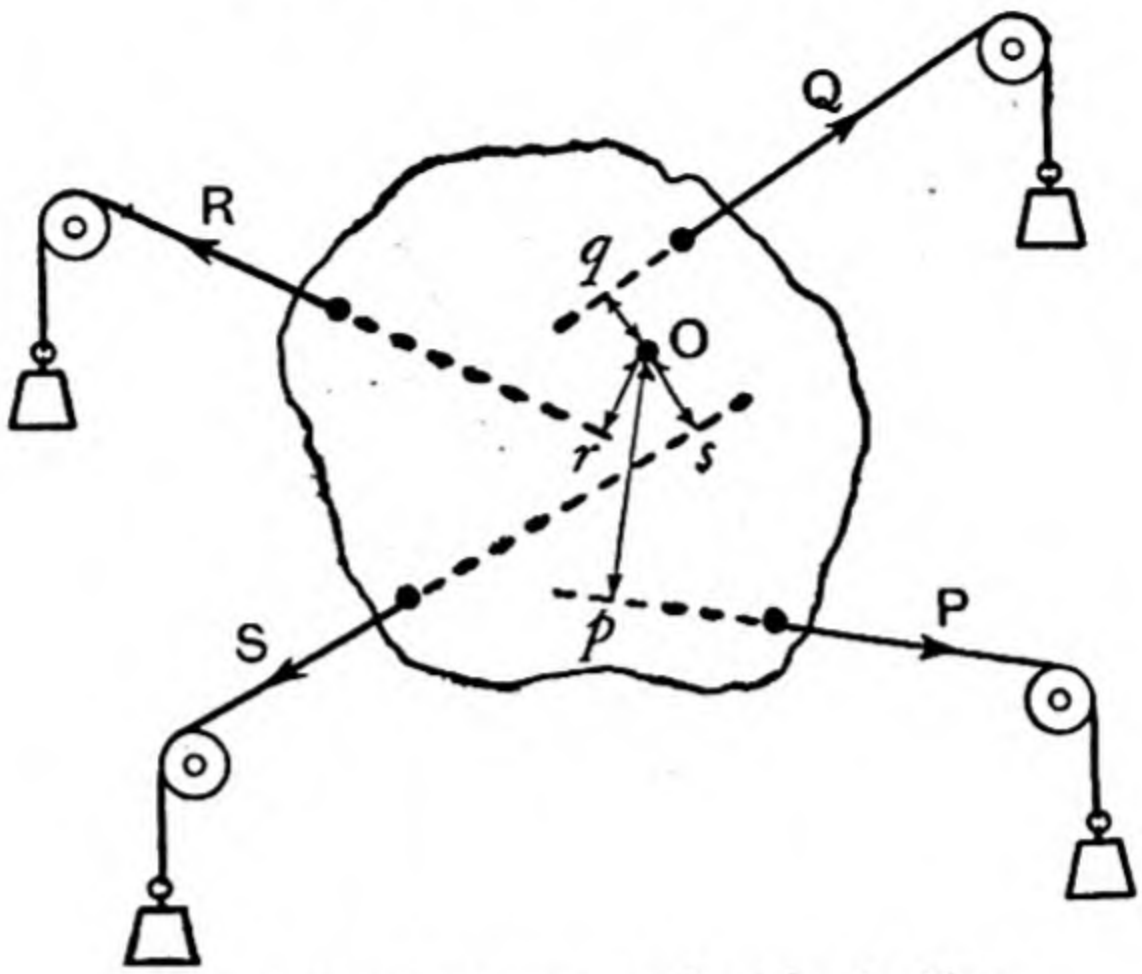


Polygon of forces demonstration.

The " Principle of Moments ", that when a body is in equilibrium under any number of forces in one plane, the algebraic sum of the moments of all the forces about any point in the plane is zero, is often tested in a rather half-hearted way. Figs. *a* and *b* show two experiments with a pivoted bar, in which it is found that, taking moments about the pivot, the sum of the clockwise moments balances the sum of the anticlockwise moments (or the algebraic sum of the moments of the forces not passing through the pivot about that pivot is zero). The objection to this is that, unless one is very careful, the existence of the upthrust at the pivot and the



Principle of moments demonstration.



Principle of moments demonstration.

weight of the metre-stick also passing through the pivot are ignored ; it is true that their moments are zero as they pass through the pivot, but they are external forces acting on the stick and will have to be considered if moments are taken about any other point. Also, fixing one's attention on an actual pivot about which there is no rotation tends to obscure the fundamental point, that any point in the plane is a possible imaginary pivot about which there is no rotation.

The best way of demonstrating the principle is with a light plywood sheet held in equilibrium by measured forces as shown. The lines of the strings are marked on the sheet, and it is found that for any chosen point O, the algebraic sum of the moments ($P \times Op$), etc., is zero, within the limits of experimental error.

EXAMPLES ON CHAPTER 4

1. Distinguish between *stable* and *unstable* equilibrium.
Explain why the tendency of a body to topple over generally increases with the slope of the surface on which the body rests.

A straight uniform glass tube of length 40 cm. contains a mercury thread of mass equal to that of the glass tube. When one end of the mercury is flush with one end of the tube, the system can be balanced on a horizontal pivot situated 12 cm. from that end. Calculate the length of the mercury thread. (O. & C.)

2. Explain what is meant by the moment of force about an axis.
Describe an experiment to illustrate the law of moments.

A uniform ladder 16 ft. long and weighing 100 lb. rests on a horizontal floor. A man lifts one end of the ladder until it is inclined at an angle of 30° to the floor, the other end remaining at rest. Calculate the force necessary to maintain the ladder in this position, assuming that the force is exerted at right angles to the ladder, (O. & C.)

3. Explain the term *moment of a force.* State the conditions under which a rigid body remains in equilibrium under the action of three coplanar forces which are not parallel.

A uniform ladder of mass 200 lb. rests on a pavement against a smooth wall at an angle of 60° to the horizontal. A boy of mass 100 lb. climbs up the ladder, which is prevented from slipping by a force of 100 lb. applied horizontally at the foot of the ladder. By what fraction of the ladder's length can the boy ascend before the ladder slips? (O. & C.)

4. What is meant by the moment of a force about an axis? If a number of parallel forces in one plane act on a rigid body, what are the conditions for equilibrium?

A uniform rod of length L, and weight W lb., will balance on a fulcrum 2 in. from one end if a weight of 18 lb. is hung from that end, and on a fulcrum 3 in. from the end if a weight of 8 lb. is hung on that end. What are the values of L and W? (O. & C.)

5. Explain the terms *force, resolved part of a force, moment of a force.* State the conditions under which a body will remain in equilibrium under the action of three coplanar forces which are not parallel.

A mass of 50 lb. is supported by two threads inclined respectively at angles of 60° and 30° to the vertical. Calculate the tensions in the threads. (O. & C.)

6. Show that if three forces, acting in one plane upon a rigid body, keep it in equilibrium, they must *either* meet in a point *or* be parallel.

A heavy uniform smooth sphere rests on two smooth planes, whose inclinations to the horizontal are 30° and 60° respectively. Find the proportion of the weight of the sphere supported by each plane. (O. & C.)

7. Explain what you understand by *force, tension, equilibrium.*

A straight uniform rod BC of length 5 ft. and of mass 10 lb. is suspended from a peg A by two threads AB and AC which are at right angles to one another. The rod rests in equilibrium at an angle of 30° to the horizon.

Find by a graphical construction or otherwise (*a*) the lengths of the threads, (*b*) the tensions in the threads. (O. & C.)

8. Define *moment of a force,* and show that the algebraic sum of the moments of any two forces about a point in their plane is equal to the moment of their resultant about the same point.

A uniform rod AB of weight 10 lb. can rotate in a vertical plane about a hinge at A, and is pulled aside by a force $5\sqrt{3}$ lb. acting horizontally

through B. Calculate (a) the inclination θ at which the rod will rest, (b) the magnitude and direction of the reaction at the hinge. (O. & C.)

9. Distinguish between (a) the moment of a force about an axis, and (b) the work done by a force. State the units in which these quantities are measured in the C.G.S. system.

A uniform box-lid AB, hinged at A, and of mass 20 lb., is held at an angle of 30° to the horizontal by inserting a prop CD at right angles to AB. Given that the distances AB and AC are equal, find the thrust in the prop. (O. & C.)

10. State the conditions of equilibrium of three coplanar forces acting at different points of a body.

A uniform rod AB of mass 2 lb. rests in equilibrium with its ends on two smooth planes which are facing each other and inclined at 60° and 30° respectively to the horizontal. By construction or otherwise, find the reactions between the rod and the planes. (O. & C.)

11. State the conditions of equilibrium of a number of coplanar forces which meet at a point.

A mass of 150 lb. hangs by two inextensible strings AC and BC inclined at 60° to the horizontal. A horizontal pull of 50 lb. wt. is also applied to the mass in the plane ABC. Find the tensions in the strings. Calculate also for what value of the horizontal pull the string BC becomes just slack. (O. & C.)

12. What conditions must be satisfied in order that three non-parallel forces acting on a body may keep it in equilibrium?

A pendulum bob weighing 200 gm. is suspended from a light thread. Find the size and direction of the least force required to pull the bob aside so that the thread makes an angle of 30° to the vertical. (O.)

13. State the conditions which must hold if three non-parallel forces acting on a body maintain it in equilibrium.

It is found that during a gale the mooring cable of a captive balloon is at an angle of 20° to the vertical, and that the tension in the cable is 200 lb. wt. Find the horizontal force exerted by the wind, and the net vertical upthrust experienced by the balloon. (O.)

14. State the rules for finding the magnitude and line of action of the resultant of a system of coplanar parallel forces.

Explain the term *centre of gravity*. Describe and explain how you would find experimentally the centre of gravity of a thin card of irregular shape.

A wooden stool 2 ft. 2 in. high consists of a square seat with a uniform vertical leg at each corner. The seat is 2 in. thick and of mass 2 lb. The total mass of the legs is $1\frac{1}{4}$ lb. Find the height of the centre of gravity. (N.)

15. Describe a force diagram method by which you would find the weight of an object (between 1/2 and 3/2 kgm.) if you were provided

with a piece of string, a kilogram weight, a sheet of paper, drawing instruments, and the necessary stands.

An aeroplane has to fly 100 miles due west while a 30 m.p.h. wind is blowing from the north-west. If the plane can travel at 150 m.p.h. in still air, find, graphically or otherwise, the direction in which the pilot must point the machine and how long the journey will take.

(N.)

CHAPTER 5

WORK AND ENERGY

WORK is done when a force moves the body to which it is applied. It is measured by the product (*force* × *distance moved in the direction of the force*).

Usual units for measuring work and their values are :

1 erg : the work done when a force of 1 dyne moves its point of application 1 cm. in its own direction ;

1 joule : which is 10^7 ergs ;

1 kilowatt-hour (kWh.) : which is 3,600,000 joules ;

1 cm.-gm. : the work done when a force of 1 gm. wt. (g dynes) moves its point of application 1 cm. in its own direction. This equals g ergs.

British System units are :

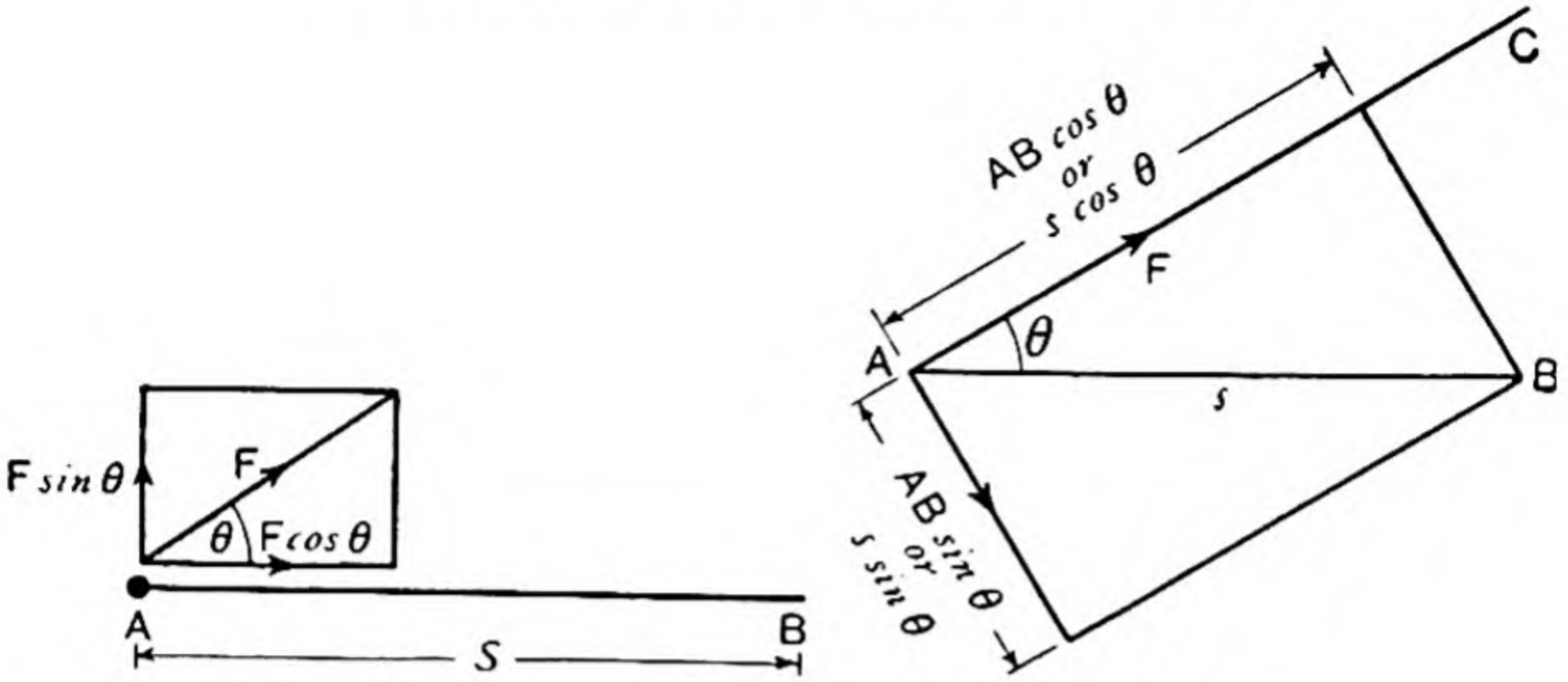
1 foot-pound (ft. lb.) : the work done when a force of 1 lb. wt. moves its point of application 1 ft. in its own direction ; and the corresponding absolute unit, $1/g$ foot-pound, or

1 foot-poundal : the work done when a force of 1 poundal moves its point of application 1 ft. in its own direction.

Problems are often posed in other possible units ; but these (for example, the foot-ton and kilogram-metre) are explained by their names. In Physics we endeavour to work as much as possible in terms of absolute units (either ergs or joules) for what may be called laboratory problems ; and in British System gravitational units—foot-pounds—for practical problems on machines, etc.

Force inclined to direction of motion. If a force F, acting in the direction AC inclined at an angle θ to the constraining direction AB in which the body moves a distance s, work is done.

Two methods can be used to measure the work done : (*a*) F can be resolved into two components (Fig. *a*), one of which, $F \cos \theta$, moves a distance s in its own direction, and the other of which,

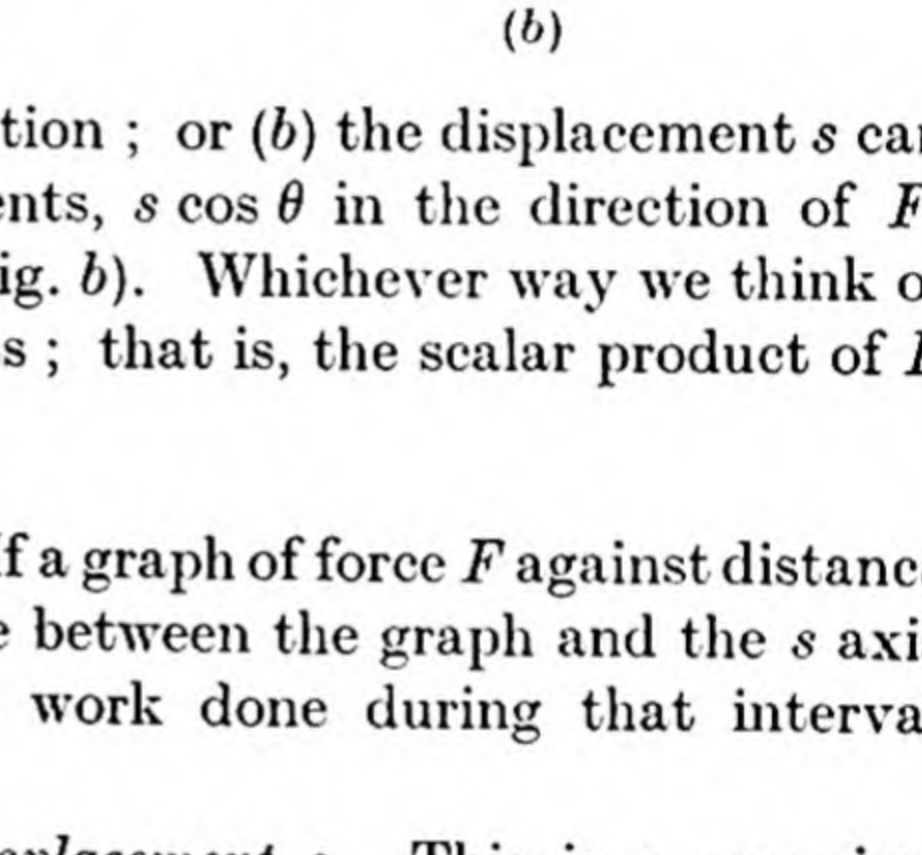


$F \sin\theta$, does not produce any motion ; or (*b*) the displacement s can be resolved into two displacements, $s\cos\theta$ in the direction of F, and $s\sin\theta$ perpendicular to F (Fig. *b*). Whichever way we think of it, the work done is $Fs\cos\theta$ units ; that is, the scalar product of F and s.

Work done by a varying force. If a graph of force F against distance s is plotted, the area of the space between the graph and the s axis for any interval represents the work done during that interval (p. 27).

(*a*) *Force F proportional to displacement s.* This is a very important practical case, applying to all elastic displacements in which Hooke's Law is obeyed. Here the graph is a straight line through the origin (see graph) ; if therefore a force F slowly applied gives an extension x, the work done is represented by the area of the triangle OAB, which is $\frac{1}{2}AB \times OB = \frac{1}{2}Fx$.

This is at first a little difficult to realise, for it seems as if surely a force F has moved a distance x ; but what actually happens in this case is that

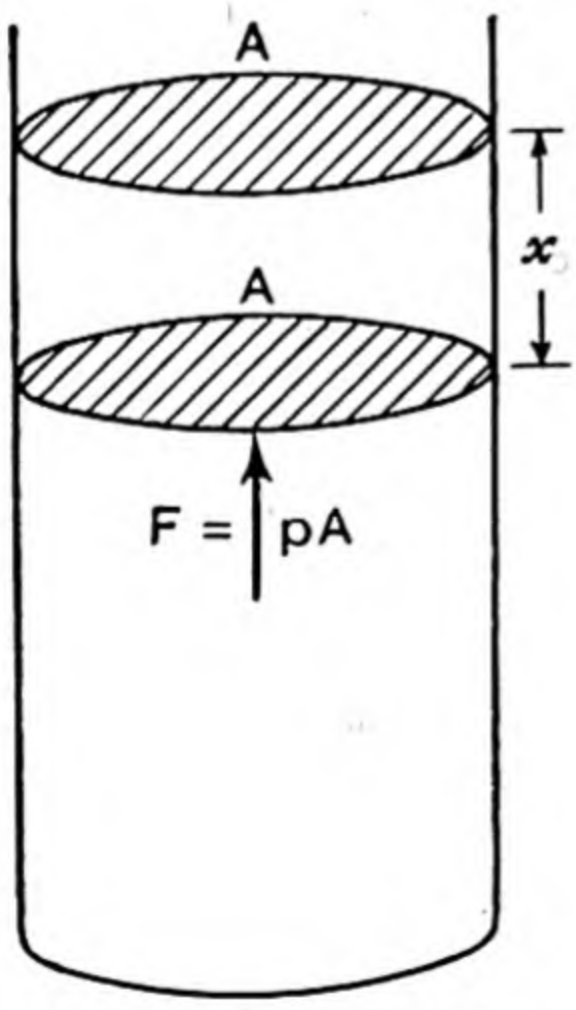


Work diagram.

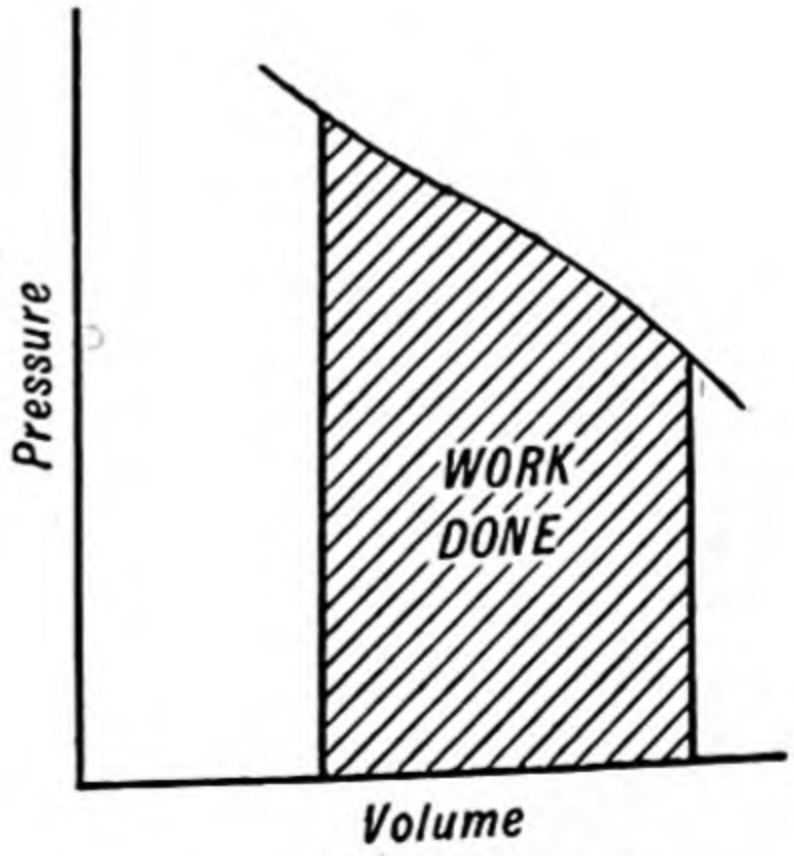
the spring, wire, cord, or whatever is being stretched, exerts and experiences, at any displacement s, only just that force needed to produce such displacement; the remainder during stretching being exerted by the hand or whatever means is adopted to apply F gradually. More precisely, we are supposing that the motion proceeds by a series of small departures from successive equilibrium stages.

(*b*) *Work done by an expanding gas*: Suppose for a moment that a certain mass of gas is enclosed in a cylinder of area of cross-section A, exerting on its walls a pressure p. The force exerted on the piston is pA, and the work done when the gas expands to raise the piston a distance x is pAx units. But Ax is the increase in volume; so the work done by the gas expanding at constant pressure p is (*pressure* × *volume change*) units.

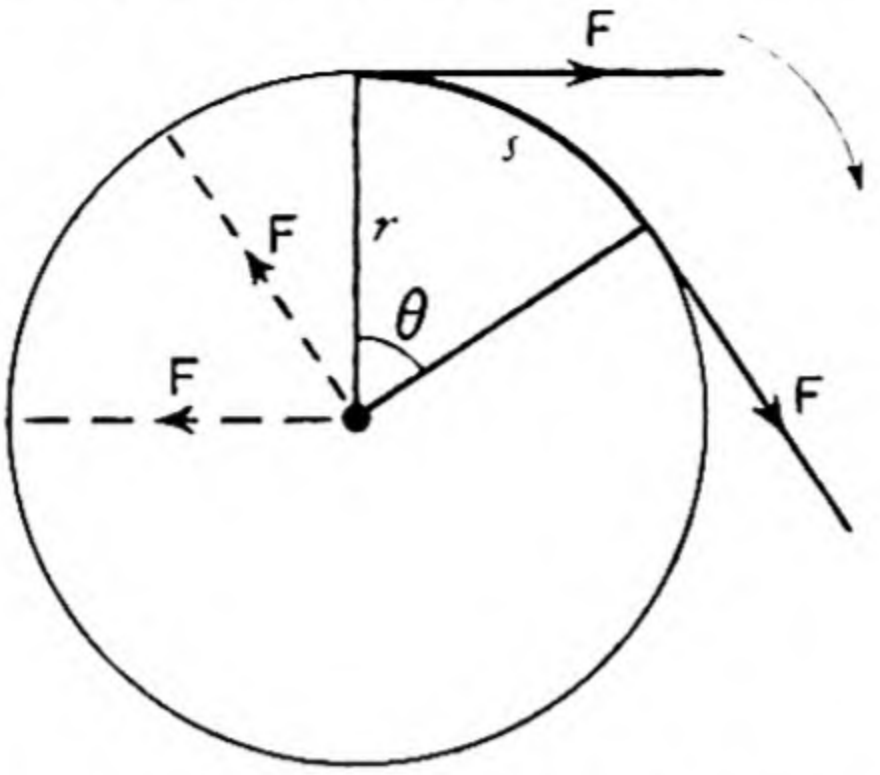
We can plot a graph of *force per unit area* (pressure p) against *displacement* × *area* or volume change, and this again will show the work done as the area between the graph and the "displacement" axis. The same will be true even if the gas does not expand at constant pressure—if, for example, Boyle's Law holds (p. 222) or the adiabatic equation (p. 234). However p changes, the work done during an expansion by the gas is represented by the area as indicated in the diagram.



Expansion of a gas.



Work diagram.



Work done by a tangential force producing rotation. In the diagram, the force F is applied always tangentially to the circumference of a wheel of radius r pivoted at the centre.

Let the wheel turn through θ radians, so that F moves through a distance $s = r\theta$ in its own direction.

Then, work done by $F = Fs = Fr\theta$

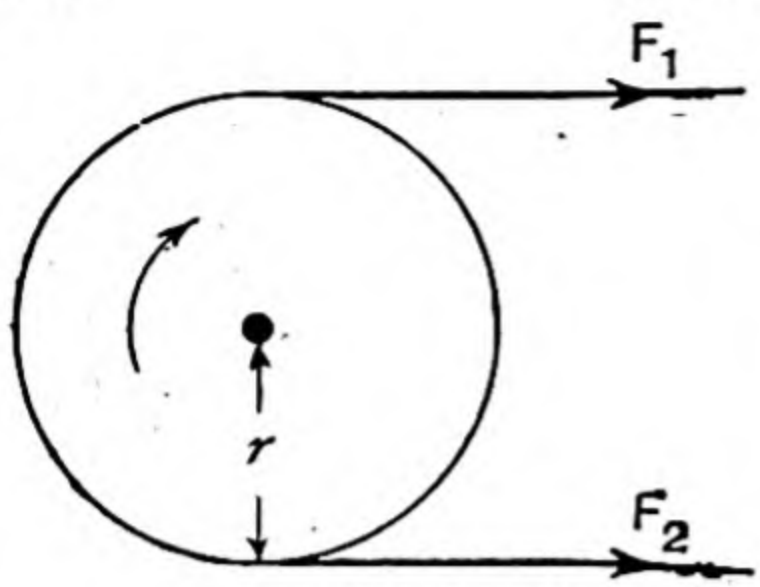
= (*moment of force about axis* × *angle turned through in radians*).

Work done by a couple. What really happens in the case above is that bodily translation of the wheel is prevented by a reaction F (dotted arrow) at the bearing, so that a couple of moment Fr is in reality acting. As the reaction, passing through the bearing, does not produce any motion, it does not do any work ; so the total work done by this couple is that already noted, namely, $Fr\theta$ units. But the moment of the *couple* is Fr, and it is true in general to say :

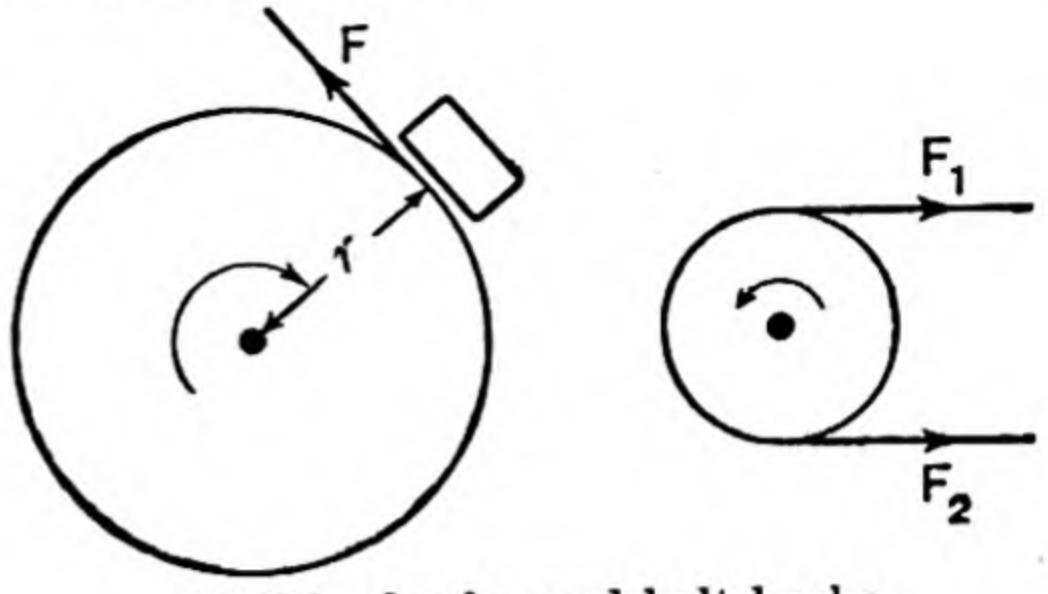
work done by couple = (*moment of couple* × *angle turned through in radians*).

Thus, if $C = Fr$ is the moment of the couple in dyne-cm., and θ the angular displacement in radians, the work done is $C\theta$ ergs.

Work done by a varying couple. If a graph of couple against angular displacement is plotted, the area of the space between the graph and the displacement axis for any interval represents the work done during that interval. In the important case when the couple is proportional to the angular displacement, the graph is a straight line through the origin, like that on p. 113 ; and the work done by a couple C in producing an angular displacement θ is then $\frac{1}{2}C\theta$.



(*a*) Belt drive.



(*b*) Rim brake and belt brake.

Work done by a belt drive. Let F_1 and F_2 be the tensions in the two portions of the belt (Fig. *a*). Then the work done by F_1 in turning through θ radians is $F_1 r\theta$, and that done by the wheel against F_2 is $F_2 r\theta$. The net work done on the wheel is $(F_1 - F_2) r\theta$ units.

Work done against a brake. If a single brake, for example, a cart wheel brake, applies a tangential frictional force F (Fig. *b*) then the work done by the wheel against the brake in turning through θ radians is $Fr\theta$; in the case of a belt brake, in which the tensions in the two parts are F_1 and F_2, this is $(F_1 - F_2) r\theta$.

Energy

What happens to the work done? In all practical examples, some (and in many cases all) of the work done in displacing a body is apparently lost for ever. If we exert a force of 1 lb. wt. to push a pile of books a distance of 1 ft. along a horizontal bench, the work we do goes in " overcoming friction ", and our foot-pound of work once done has gone for good ; further, if we want to get the books back to their original place we must do *another* foot-pound of work. Later it will be seen that this lost work can be traced and strictly accounted for—but it is no longer available to us as work.

If, however, we take a mass of 1 lb. and raise it 1 ft. vertically, matters appear more promising ; for if we are able to let it fall through 1 ft. vertically again, its own weight acting on the mass will perform 1 ft.-lb. of work—exactly the work that we did *on the mass* in raising it. This can be tested experimentally in a variety of ways, and can be done at any time we choose.

Imagining the lifted body to be rather like a reservoir of stored work, which can be released at will, it is said that the body possesses

energy, or a capacity for doing work. It is not quite true to say this is a property of the body alone, for obviously the system (earth + body) between them share responsibility for the weight.

Many other similar examples of recoverable work will come to mind ; for example, the work done in deforming the mainspring of a watch is recoverable as work done driving the mechanism ; work done in compressing air in a cylinder is recoverable when the air is allowed to expand again ; work done in extending the elastic of a catapult is recoverable as work done in accelerating the projectile, and so on. In all these cases, the system concerned is said to possess " potential energy ".

The Potential Energy of a system is the capacity for doing work possessed by the system on account of the relative positions of its parts.

The utilisation of this energy demands a rearrangement of the parts of the system, and the amount of work obtainable depends on the extent of this rearrangement. The potential energy made available when the system moves from arrangement A to arrangement B is the work done when the system moves from arrangement A to arrangement B, and is equal to the work which must be done on the system to move it from arrangement B to arrangement A.

For example, if we take a body of mass m gm. and lift it vertically through a height h cm. above the bench, we do mh cm. gm. of work on it, or mgh ergs of work on it. The potential energy in this arrangement of the earth-body system, considering only the possibility of the body returning to the bench again, is

mgh ergs.

Again, if we extend a spring by the application of a force F dynes through a distance x cm., assuming Hooke's Law (p. 202) holds, the work done on the system is $\frac{1}{2}Fx$ ergs (p. 113) ; this is the potential energy in the stretched spring, for this is the work that should be made available when it contracts again.

There is another important possibility. Suppose you take a mass of 1 lb. in one hand, apply a considerable vertical force on it for a short distance—say 16 lb. wt. for 1 ft.—and then release it. It sails up, continuing of its own accord to rise and do work in " raising its weight". Your dealings with it end the instant it leaves the hand. The work that is being done in raising it can only come from a

reservoir of work that the body itself possesses as it leaves the hand; or the body possesses energy *because it is moving*.

The capacity for doing work, possessed by a moving body on account of its motion, is called Kinetic Energy.

The kinetic energy of a body can be calculated from the amount of work the body can perform while being brought to rest, or from the amount of work expended on it while it was being set in motion.

Suppose the body, of mass m gm., rises to a vertical height h cm. on being projected upwards with a vertical velocity of u cm. per sec.

The work actually done *by* the body before it comes momentarily to rest is equal to the work done in raising it h cm. vertically, and is mh cm. gm., or

$$mgh \text{ ergs.}$$

Now, using $v^2 = u^2 + 2as$, putting $v = 0$, and remembering that $a = -g$ cm. per sec. per sec., then $u^2 = 2gh$; and $gh = u^2/2$. The work done raising the body is thus:

$$\tfrac{1}{2}mu^2 \text{ ergs.}$$

This is the kinetic energy with which it starts off; and clearly, however this energy is afterwards employed, any body of mass m with velocity u cm. per sec. has kinetic energy $\frac{1}{2}mu^2$ ergs.

Thus, *in absolute units*, the kinetic energy of a body is

$$\tfrac{1}{2}\ mass \times (velocity)^2.$$

A further point arises here. At the top of its flight, the body still has energy—potential energy of mgh ergs; as it returns, it accelerates, losing potential energy and gaining kinetic energy. When back at the start it has lost all its mgh ergs of potential energy and gained $\frac{1}{2}mu^2$ ergs of kinetic energy, where $u^2 = 2gh$, as before. This will presumably be expended in performing work in damaging the bench slightly and setting it vibrating. The whole operation can be regarded as a series of transformations thus:

$$work \rightarrow kinetic\ energy \rightarrow potential\ energy \rightarrow kinetic\ energy \rightarrow work.$$

Throughout this argument we have assumed a very important principle, that *energy is conserved*. This means that the kinetic energy of the body is equal to the work expended on accelerating it; that the potential energy of the body-earth system at the top of the flight

is equal to the kinetic energy of the body at the start; that the kinetic energy at the end of the downward journey is equal to the potential energy at the top; and that the work done by the body in being arrested is equal to the kinetic energy destroyed. All experiments support these assumptions, when due allowance is made for energy dissipated (to an extent which in this example is quite small) in ways yet to be discussed.

To sum up, it appears that when work is done it can only be used in one of two ways: either

(*a*) in " overcoming friction " (see p. 120), or

(*b*) in producing energy, which may be either

(i) *potential energy*, if a system in equilibrium is disarranged to a new equilibrium position, or

(ii) *kinetic energy*, if a body is set in motion.

It will be found that all cases that can be included in (*b*) merely delay the true settling of the account, namely (*a*).

Further consideration of potential and kinetic energy. When we speak of the potential energy of a system, we usually have in mind a particular change in its configuration which it is proposed to employ in order to do work. A lump of m gm. of coal, held h cm. above the bench, " collaborates " with the earth to yield mgh ergs of work when it drops to the bench. But a more usual way of obtaining work from coal is to burn it, for coal possesses *chemical potential energy*, thus liberated. Here we take a system of combustible organic molecules and atmospheric oxygen, and obtain work by a rearrangement of the parts—chiefly into molecules of carbon dioxide and water—which liberates heat, and this heat is the intermediary between the chemical potential energy of the coal and the performance of useful work.

Hence the definition of potential energy really covers energy due to the arrangement of atoms in molecules, to the arrangement of electrons in the outer parts of an atom, or to the arrangement of protons and neutrons within the nucleus of an atom.

In, say, a steam engine, heat provides the link which enables the chemical potential energy of the coal to be converted into actual work; heat must thus itself be a form of energy, since heat can be made to furnish work. Heat is believed to be *molecular* kinetic

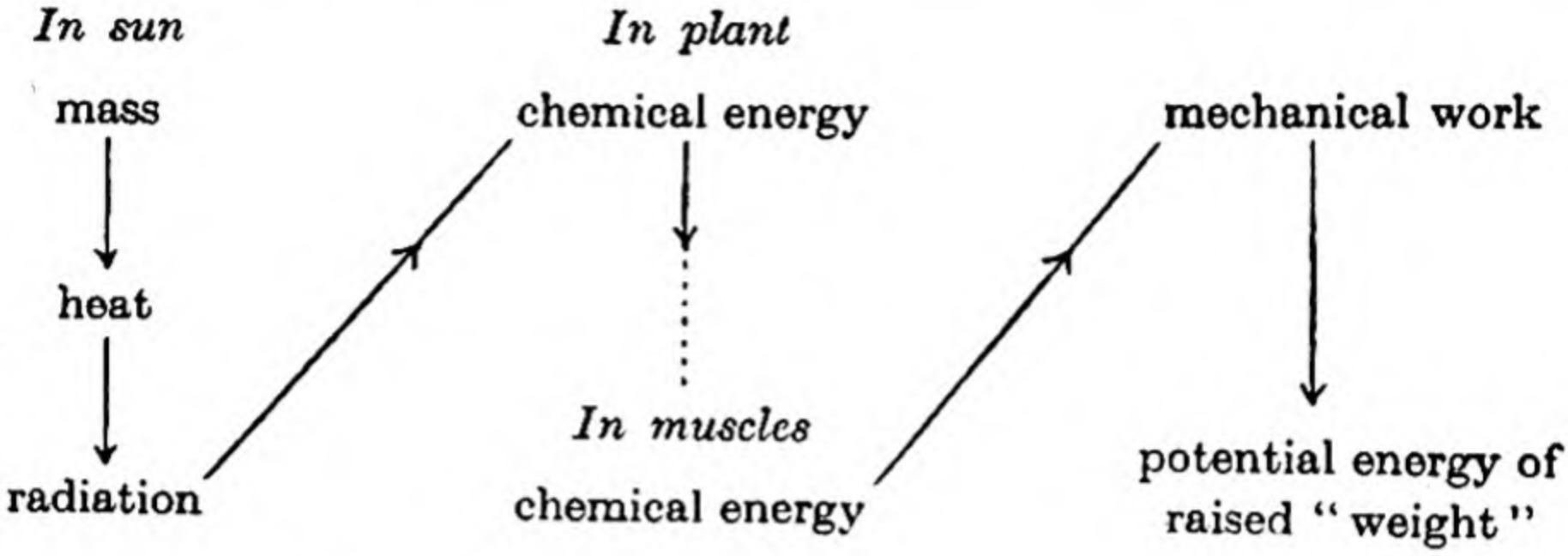
energy. Considering the case of a gas, in which the molecules are free to move about in all directions with very great speeds, it is supposed that the average speed and hence the average kinetic energy is greater at high temperatures than at low temperatures. To raise the temperature of a given mass of gas, heat must be supplied ; the result is that the average kinetic energy of the molecules is increased. We suppose that this increase in the molecular kinetic energy *is identical with* the heat supplied (not just numerically equal to it, as we find if the same units are used for both measurements). Heat *is* molecular kinetic energy. In the case of a solid, though the atoms and molecules are not free to move in all directions, yet they can vibrate ; and the kinetic energy they possess is due to their vibrations about their mean positions.

It can now perhaps be seen how " work done overcoming friction " fits into the general scheme of energy transformation. Friction between two dry solid surfaces may be due to a temporary engagement of minute surface projections, either by mechanical contact or by molecular forces (but see page 126). Relative motion between the two surfaces may cause vibrations of these minute projections, in much the same way as happens when the hand is drawn across the teeth of a comb ; these vibrations themselves constitute an *access of heat* to the surface layers. So, when work is done overcoming friction, and heat is developed, we really have a conversion of work into *molecular kinetic energy*, which is the one form most difficult to confine in a closed system, and the one form that is most difficult to utilise for the performance of work again.

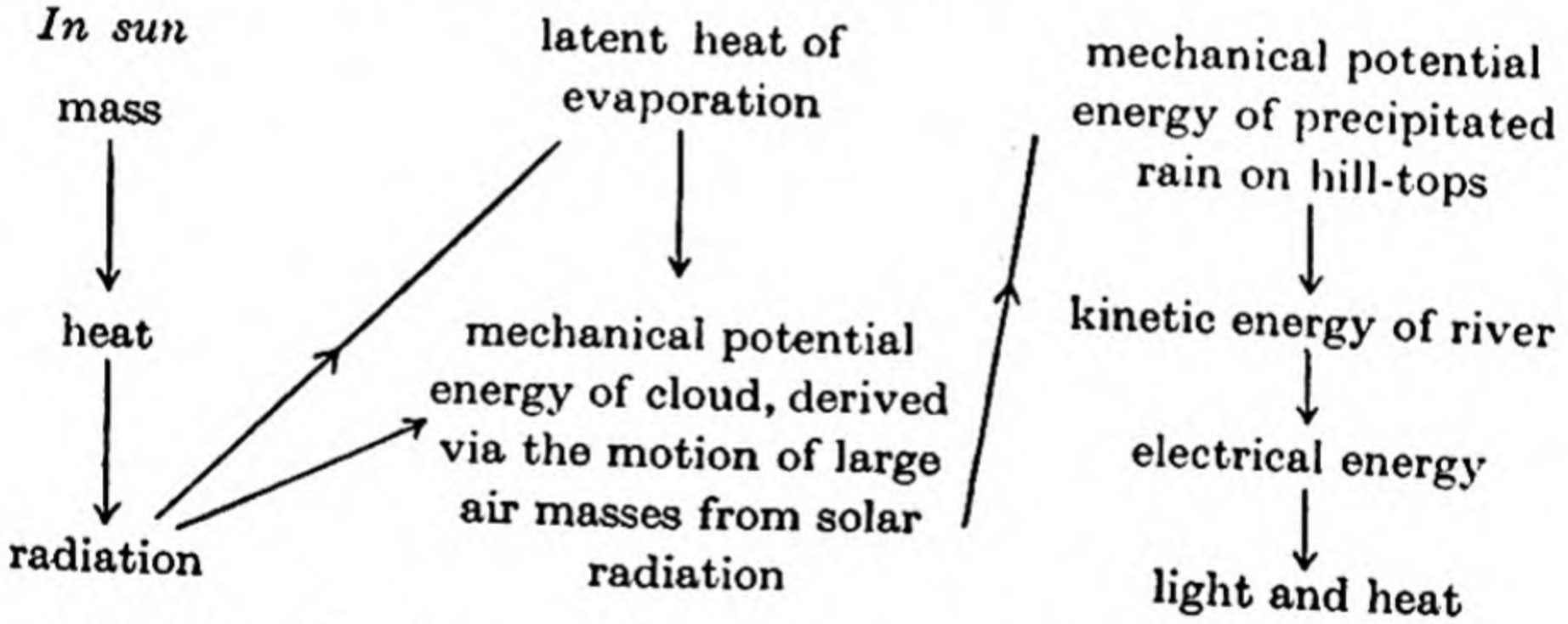
Conservation of Energy

The Law of Conservation of Energy. All physical events can be regarded as transformations of energy from one form to another, or as transferences of energy from one body or system to another. For example, if you do work in raising a weight, this is attributable to the consumption of chemical energy stored in your muscles, which was derived by a series of chemical changes from vegetable food, either directly or via the flesh of a herbivorous animal ; the vegetables built up their store of energy from that radiated by the sun ; and the continuous supply of solar radiation is furnished by the

destruction (or conversion) of mass as protons and neutrons combine to form helium nuclei. The chain of energy conversions can thus be represented :



Again, the electric lighting of a factory may be supplied by a dynamo driven by a river, the source of which a few miles away is several hundred feet higher than the factory. It is fed by the generous rainfall of the surrounding hills, and the chain of energy conversions, starting again with the sun, is :


In sun
mass
heat
radiation
latent heat of evaporation
mechanical potential energy of cloud, derived via the motion of large air masses from solar radiation
mechanical potential energy of precipitated rain on hill-tops
kinetic energy of river
electrical energy
light and heat

Similarly, all other physical changes can be analysed and (with the exception of a few cases in which the energy source is outside the solar system, which need not concern us) the energy can be traced back to the same prime source, the sun.

Various forms of energy have been named here—mechanical kinetic and potential energy, electrical energy, chemical energy, radiation, heat, light, and mass. With regard to mass, it perhaps seems curious to have insisted earlier on the difficulty of defining the concept adequately, and now to state that it is a form of energy—

but the experimental and theoretical work of recent years demonstrates that mass itself as such can really be annihilated; in its place other forms of energy appear. Note that we are not speaking of the mechanical energy *possessed by* a mass of material that can be increased or diminished by suitably varying its velocity or position, but of the energy that is furnished *in exchange for* the mass when that is destroyed. Again, the permanence of the mass possessed by a body was insisted upon earlier; this is, of course, true in its simplest sense for all bodies normally encountered—the box of "weights" in the laboratory is certainly not radiating itself away. But it is still true in its wider sense, for if mass is destroyed, it reappears in another form.

Energy cannot be created or destroyed. When conversion of energy from one form to another occurs, the energy that disappears from one form reappears in exactly equivalent amount in the other form.

This is the **Law of Conservation of Energy.**

The term "exactly equivalent" perhaps needs explaining. When a body falls to the ground, the kinetic energy gained is exactly equal to the potential energy lost, and we can write

P.E. lost = K.E. gained.

As an equation, that is arithmetically true, for both are measured in the same units. But heat, for example, is commonly measured in heat units; mass is measured in grams. The "rate of exchange" between heat units and energy units, and between mass units and energy units, must be determined before we can write down equations involving these forms of energy. It is found that:

one calorie (the quantity of heat which raises the temperature of 1 gm. of water through one centigrade degree) is equivalent to nearly $4{\cdot}2 \times 10^7$ **ergs**, and

one gram is equivalent to 9×10^{20} **ergs.**

If the necessary conversion of units is made, so that the quantities concerned are always expressed in energy units, we can say that the energy that disappears from one form reappears in exactly *equal* amount in another form.

Application of the Law of Conservation of Energy to mechanical problems. Any experiment performed, or any problem considered, is only one link in a continuous chain of energy changes. We do not

seek to follow the energy conversion past this particular link ; but in practice our wishes in the matter have no effect on the inevitable sequence. To take the case of the falling weight : some of the potential energy is imparted as kinetic energy to the molecules of the surrounding air, and thus dissipated ; we can write down that " work is done overcoming air resistance ", but there is no easy way of conserving this heat energy for measurement ; and the kinetic energy of the weight on reaching the ground is slightly less than the potential energy with which it started.

This kind of thing always happens, and in applying the law of conservation of energy to problems we must distinguish between three sets of conditions :

(*a*) *When the energy dissipated is relatively so small that it can be neglected*, as in the above example ; then almost the whole energy of the system concerned is preserved throughout as mechanical energy, and we can use the equation

$$\text{K.E.} + \text{P.E.} = \text{constant.}$$

(*b*) *When the energy dissipated is an important part, or possibly the whole, of the initial energy*, and we write this down as external work done, using the equation

$$\text{energy change} = \text{work done.}$$

(*c*) *When the dissipated energy is not negligible, and at the same time is not the main subject of inquiry.* For example, a body sliding down a rough incline expends some of its potential energy in overcoming friction ; its kinetic energy at the bottom can be calculated only if we know from previous experimental measurements what allowance to make for the work done against friction. When two ordinary bodies collide, some of their kinetic energy is expended in deforming the bodies during the act of contact ; we cannot calculate the kinetic energy after collision unless we have experimental information about the energy-loss in a collision between bodies of the materials concerned. It is often said that the law of conservation of energy cannot be applied to examples of this class ; this is, of course, quite untrue. The law is universal in its validity, but in drawing up an energy balance sheet it is necessary to account accurately for all energy expended in all ways, and unless the necessary information is available this cannot be done.

Simple energy problems.

(1) A pendulum bob is drawn aside so that it is h cm. vertically above its lowest point, and released. Find the velocity with which it passes through the lowest point.

Here it is legitimate to neglect the work done against air resistance. Let the mass of the bob be m gm., and the final velocity v cm. per sec.

Then, P.E. lost $= mgh$ ergs ;

K.E. gained $= \frac{1}{2}mv^2$ ergs.

So, as P.E. + K.E. is constant,

$$\frac{1}{2}mv^2 = mgh$$

and $v^2 = 2gh$, so

$$v = \sqrt{2gh} \text{ cm. per sec.}$$

(2) A vehicle of mass 1000 kgm. is travelling at 10 metres per sec. Find the average resistance that must be applied to arrest it in a distance of 20 metres.

Let F be the resistance in dynes.

Work done by vehicle against resistance

$$= F \times 20 \times 100 \text{ ergs} = F \times 2 \times 10^3 \text{ ergs.}$$

K.E. lost by vehicle $= \frac{1}{2} \times 1000 \times 1000 \times 100^2$ ergs.

$$= 5 \times 10^9 \text{ ergs.}$$

As energy change = work done,

$$F \times 2 \times 10^3 = 5 \times 10^9$$

$$\therefore \quad F = 2{\cdot}5 \times 10^6 \text{ dynes.}$$

Friction

When two solid surfaces in contact move, or tend to move, relatively to one another, a force called the **force of friction** acts in the plane of contact of the surfaces in such a direction as to oppose motion. *Friction always opposes motion.*

Three cases arise :

(1) *The applied force tending to cause motion is relatively small*; in this case, friction is equal and opposite to the applied force in the direction of motion, by which we mean the sum of the resolved parts in that direction of the external forces other than friction.

(2) *The surfaces are on the point of motion.* This happens when the applied force is just equal to the greatest force that friction can in this case exert, called the **limiting friction.**

(3) *Motion occurs.* Here, the force exerted by friction against motion is less than the value of the limiting friction, and is called

the **force of sliding friction.** The resultant force in the direction of motion is the difference between the applied force in that direction and the force of sliding friction.

The usually accepted " laws of friction " applicable to dry solid surfaces can be summarised as follows :

(1) When the surfaces are at rest, friction can take any value up to a certain maximum value called the limiting friction, F_L.

(2) For a given pair of surfaces, the limiting friction F_L is proportional to the normal force R between the surfaces, at right angles to the plane of contact. The ratio F_L/R is called the **coefficient of statical friction**, μ_L, between the two surfaces,

$$F_L = \mu_L R.$$

(3) With all ordinary surfaces, the limiting friction is independent of the area of the surface, within wide limits. This is to be expected, since close contact can only occur at a limited number of points, and the extent of the non-touching area will be immaterial. If the area of contact is so small as to cause obvious indentation of one surface, this rule (" Amontons' law ") cannot be true.

(4) When motion occurs, the sliding frictional force, F, is proportional to the normal force R between the surfaces. The ratio F/R is called the **coefficient of sliding friction** (or the dynamical coefficient of friction), μ, between the two surfaces ;

$$F/R = \mu,$$

$$\therefore\ F = \mu R.$$

(5) With all ordinary surfaces, the sliding friction is independent within wide limits of the area of contact, and is also independent of the speed within limits (" Coulomb's law ").

We shall not concern ourselves much with friction between surfaces at rest. For one thing, in the majority of cases of statical equilibrium where friction is involved it is rarely limiting ; again, experiments to determine μ_L really measure a local value for the particular parts of the surfaces in contact, and small local irregularities make it less of a " constant " for the two materials concerned than is usually stated ; finally, the chief practical concern with friction is *as an opposer of motion that is actually occurring*, and as a means of transforming energy when work is actually being done to maintain

motion. Hence, sliding friction is by far the more important in practice.

Facts and theories of friction. Amontons (1699) stated that the frictional force between two surfaces is proportional to the normal load and independent of the area of contact. He also believed that the coefficient of friction between all dry surfaces had the same value, 1/3, whatever the nature of the surfaces, and supposed that the mechanism of sliding consisted of lifting the slider over minute interlocking projections, the process being like moving the slider up a number of little inclined planes, presumably of slope $\sin^{-1} 1/3$.

Coulomb (1779) found that the frictional force is proportional to the normal load, and independent of the velocity of sliding; but he worked with relatively small loads and discontinued measurements when either surface showed signs of scratching.

Since the coefficient of friction is known to depend on the nature of the sliding surfaces, Amontons' mechanism is very difficult to accept. Ewing, Rayleigh, and others have suggested that the surface layer of a *polished* solid is more like a fluid, and that friction has its origin in molecular cohesive forces; that is, it is as if the surface layers adhere, and the adhering layer is then sheared elastically. But molecular cohesion would be expected to be proportional to the number of molecules involved, or to the area of actual contact. What is this area of contact?

The area of contact can be estimated in two ways, either by observing the very small signs of scratching carefully and measuring the area of the scratched track; or by measuring the electrical resistance between the surfaces. An ordinary steel slider on a steel surface is found to scratch an area of 3×10^{-6} sq. cm. for a normal load of 400 gm. wt., giving a stress of $1{\cdot}3 \times 10^{5}$ kgm. wt. per sq. cm., more than ten times that sufficing to cause the metal to flow plastically. Thus there can be no "little inclined planes"; the surface must be deformed. Is there actually plastic deformation?

This can be tested by measuring the radius a of the circular depression made by a sphere of radius r under a normal load of W. Hertz (1881) showed that if the deformation is elastic, then

$$a = 1{\cdot}1\,(Wr/Y)^{1/3},$$

where Y is Young's modulus for the material of the deformed plane.

But if plastic flow occurs, the sphere sinks until $W/\pi a^2 = S$, the stress required to give plastic flow, whence

$$a = (W/\pi S)^{1/2}.$$

It should be possible to distinguish between these formulae by measuring the electrical conductance (the reciprocal of the resistance), which is proportional to the area of contact and thus to a^2; the conductance should be proportional to $W^{2/3}$ for elastic deformation, and to W for plastic flow. Experiments suggest that the latter is nearer the truth.

With this information, it is possible to explain Amontons' law, and also to see why the coefficient of friction is different for different materials. Whatever the nominal area of contact, the real area (which may be as little as 10^{-5} to 10^{-6} the nominal area) is proportional to the load, as the material flows plastically to ensure this; molecular cohesive forces at the places of intimate contact bind the surfaces with a force proportional to the real area and thus proportional to the load; and shearing of this intimate layer requires a force proportional to the load.

What happens at sliding? Investigations of metal surfaces by Bowden and his collaborators show that sliding is not uniform, but a series of intermittent stickings and slippings, such as one experiences when the finger is drawn along the surface of a waxed table. Local temperature rises may be so considerable as to cause welding of the two surfaces at the "real" places of contact, and these welded junctions are then sheared and severed. The effective instantaneous coefficient of friction thus oscillates between a maximum and a minimum value. Ordinary experiments presumably measure the maximum value as the coefficient of sliding friction.

In any case, it appears that Coulomb's law can only be taken as a rough approximation to the facts. The experiments of Bowden and Leben show that the coefficient of friction for steel sliding on steel lies between 0·3 and 0·4 for slow relative motion, and between 0·3 and 0·5 for higher speeds, so that only as a very crude approximation can it be said that μ is independent of the relative velocity.

The bowing of a violin string is probably similar to the effects just described. Rayleigh, in his "Theory of Sound" says, "The sustaining power of the bow depends on the fact that *solid friction is less*

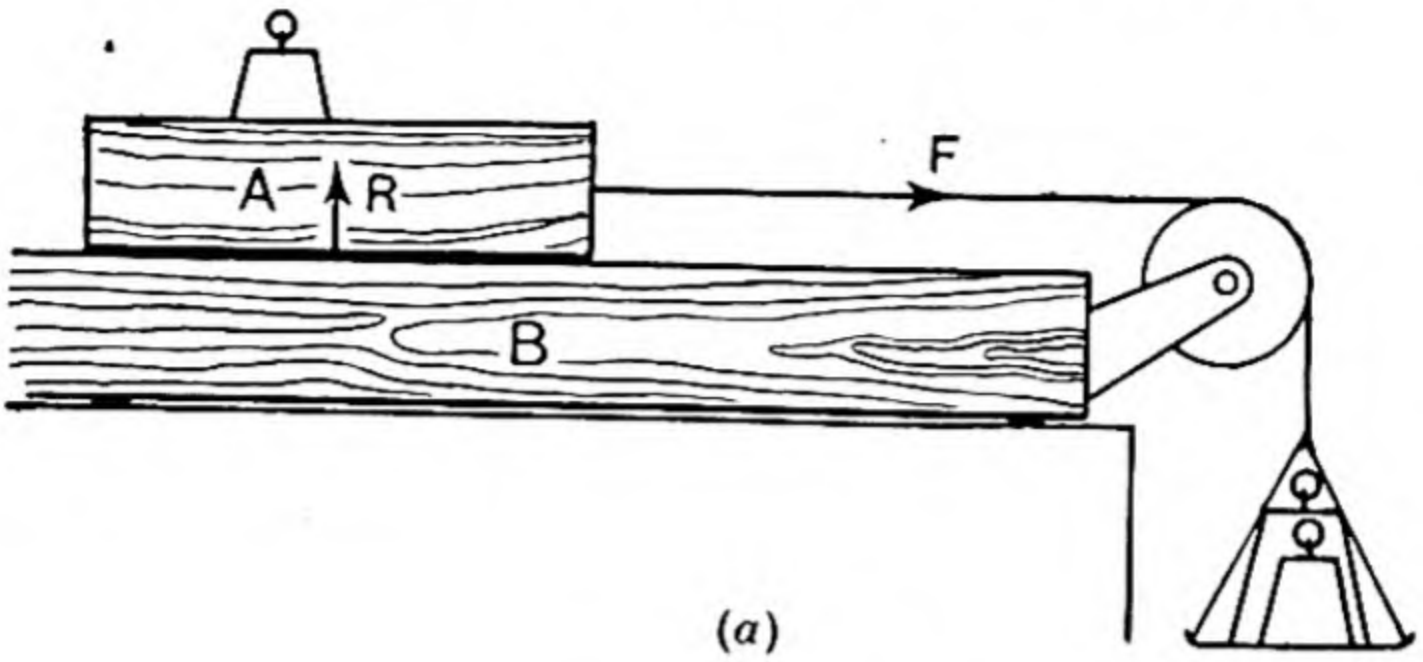
at moderate than at small velocities, so that when the string is moving with the bow (not improbably at the same velocity) the mutual action is greater than when the string is moving in the opposite direction with a greater relative velocity." Alexander Wood, in his " Acoustics ", says, " The string adheres to the bow and moves with it until the tension *overcomes the static friction*, when it slips past the bow until it once more comes to rest relative to the bow and is drawn aside once more ", and cites recent direct experimental evidence for this. If sliding motion is really a succession of stickings and slippings in every case, the violin bow can be regarded as a typical example of sliding, instead of a difficult case that needs an explanation of its own.

What is the real nature of the surfaces which we suppose to be in contact in ordinary experiments? Bowden and Hughes, experimenting with sliders in a high vacuum, found that for nickel on tungsten the coefficient of sliding friction was as high as 6·0 when all occluded gas had been driven from the metals ; the value in air is about 0·3. Admission of gases other than oxygen into the apparatus caused no diminution in the very high value of the coefficient, but when oxygen was admitted the value fell to that measured in air. It appears that, with metals at least, the surfaces are oxidised in the air, and we measure the friction between oxide films. But the properties of the materials in bulk play a part in the observed effects, since deformation extends beyond the surface.

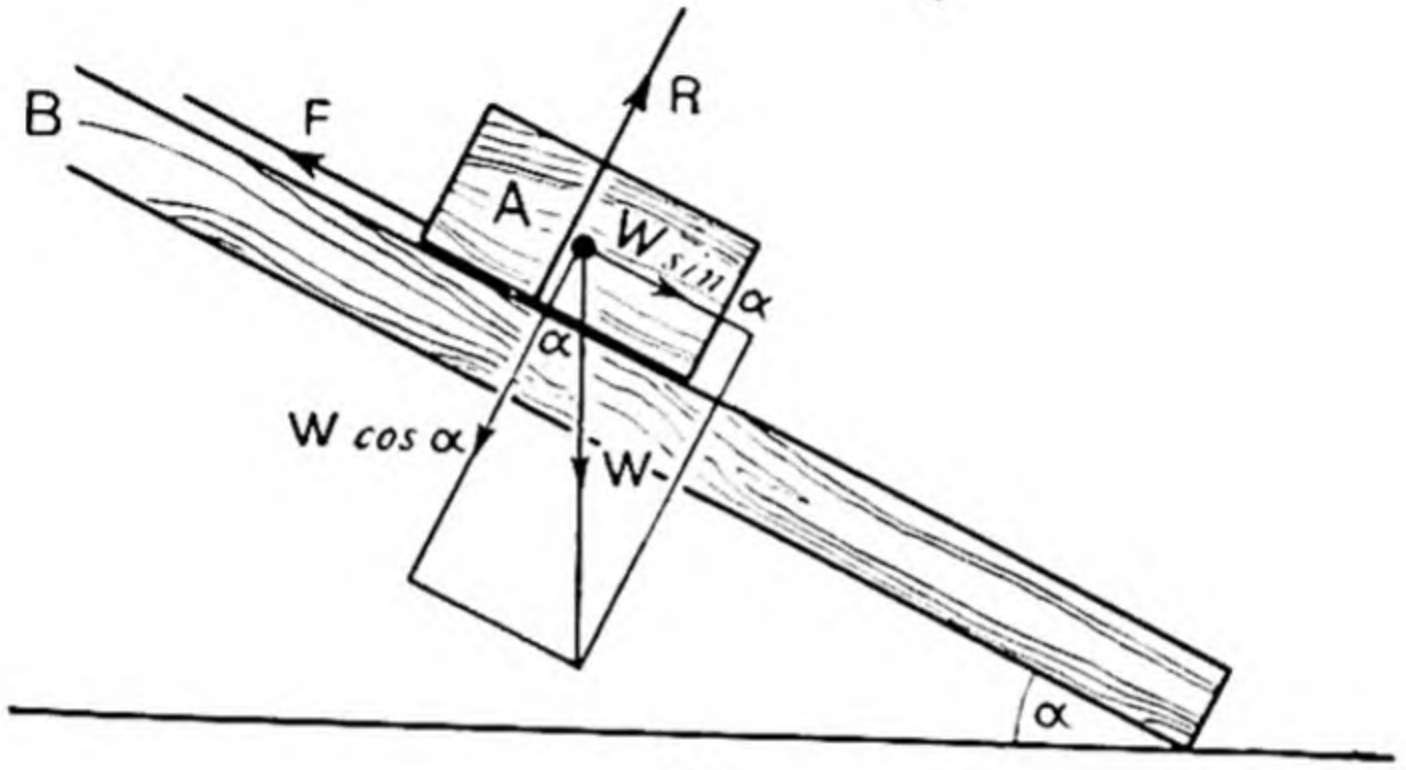
Lubricated friction is a different problem. Here, the chief factors involved are the adherence of the lubricating fluid to the surfaces, and the viscosity of the fluid, and the frictional force is known to vary with the velocity of sliding, being proportional to this velocity if it is not too great.

Measurement of the coefficient of sliding friction.

Method 1. A block of one material A (Fig. a) is pulled along a horizontal base or bed of the other material B, by means of a string passing over a pulley carrying a scale-pan to which weights can be added. The block A is weighed, and various loads can be placed on it. The load in the pan is adjusted until, when displaced slightly, A slides over B at a uniform rate ; then the load is just overcoming the force of sliding friction, F, and so just equals F. The normal force between the sliding surfaces, R, is equal to the total weight of



(*a*)



(*b*)

the block and the weights it carries. The coefficient of sliding friction is $\mu = F/R$. Several readings are taken, with different weights on A, and the average value of μ is taken.

Method 2. The block A is placed on the bed of material B, which it tilted until it is inclined to the horizontal at such an angle that A slides down it at a uniform speed when started from rest by a gentle push (Fig. *b*). This angle α is measured. The component of the weight W of the block acting along the slope is $W \sin \alpha$, and this just overcomes F, to which it must be numerically equal; the normal force between the surfaces is the resolved part of W at right angles to the slope, $W \cos \alpha$.

Thus,

$$F = W \sin \alpha,$$

$$R = W \cos \alpha,$$

and

$$\mu = F/R = \tan \alpha.$$

Method 3. A block of material A is sent sliding along a horizontal bed of material B, and the retardation a cm. sec.$^{-2}$ measured. If m gm. is the mass of the block, the normal load is mg dynes and the retarding force μmg dynes. As *force = mass × acceleration*,

$$\mu mg = ma,$$

so

$$\mu = g/a.$$

Simple energy problems involving friction.

1. A body is thrown at a speed of 10 metres per sec. along a rough horizontal surface, and comes to rest after travelling 6 metres. Find the coefficient of sliding friction.

Let m gm. be the mass of the body.

The normal force R between the surfaces is mg dynes; the frictional force opposing motion is μmg dynes, where μ is the coefficient of sliding friction. The work expended in overcoming friction is then

$$600\mu mg \text{ ergs.}$$

The kinetic energy lost $= \frac{1}{2} \times m \times 1000^2$ ergs.

Equating the kinetic energy lost by the body to the work done by it,

$$600\mu mg = \tfrac{1}{2} m\ 1000^2,$$

$$\therefore\ \mu = \frac{1000 \times 1000}{2 \times 600 \times 981},$$

$$= 0{\cdot}86.$$

2. A body slides down a rough plane inclined at 30° to the horizontal. If the coefficient of sliding friction between the body and the plane is 0·4, find the velocity after it has travelled 6 ft. along the plane.

Let m lb. be the mass of the body. The vertical height fallen is $6 \sin 30° = 3$ ft., and the P.E. lost $= m \times g \times 3 = m \times 32 \times 3$ ft. lbal.

The normal force between body and plane is

$$m \times \cos 30° \text{ lb. wt.} = m \times 0{\cdot}866 \text{ lb. wt.} = m \times 32 \times 0{\cdot}866 \text{ lbal.}$$

The sliding frictional force is μ times this, or

$$0{\cdot}4 \times m \times 32 \times 0{\cdot}866 \text{ lbal.}$$

The work done against friction travelling 6 ft. against it is thus

$$6 \times 0{\cdot}4 \times m \times 32 \times 0{\cdot}866 \text{ ft. lbal.}$$

Let the final velocity be v ft. per sec. The kinetic energy of the body is then $\frac{1}{2} mv^2$ ft. lbal.

Then,

$$\begin{matrix}\text{P.E. lost} \\ \text{by body}\end{matrix} = \begin{matrix}\text{K.E. gained} \\ \text{by body}\end{matrix} + \begin{matrix}\text{work done} \\ \text{by body}\end{matrix}$$

so $$m \times 32 \times 3 = \tfrac{1}{2}mv^2 + 6 \times 0{\cdot}4 \times m \times 32 \times 0{\cdot}866,$$

$$\therefore \tfrac{1}{2}v^2 = 96 - 66{\cdot}5 = 29{\cdot}5,$$

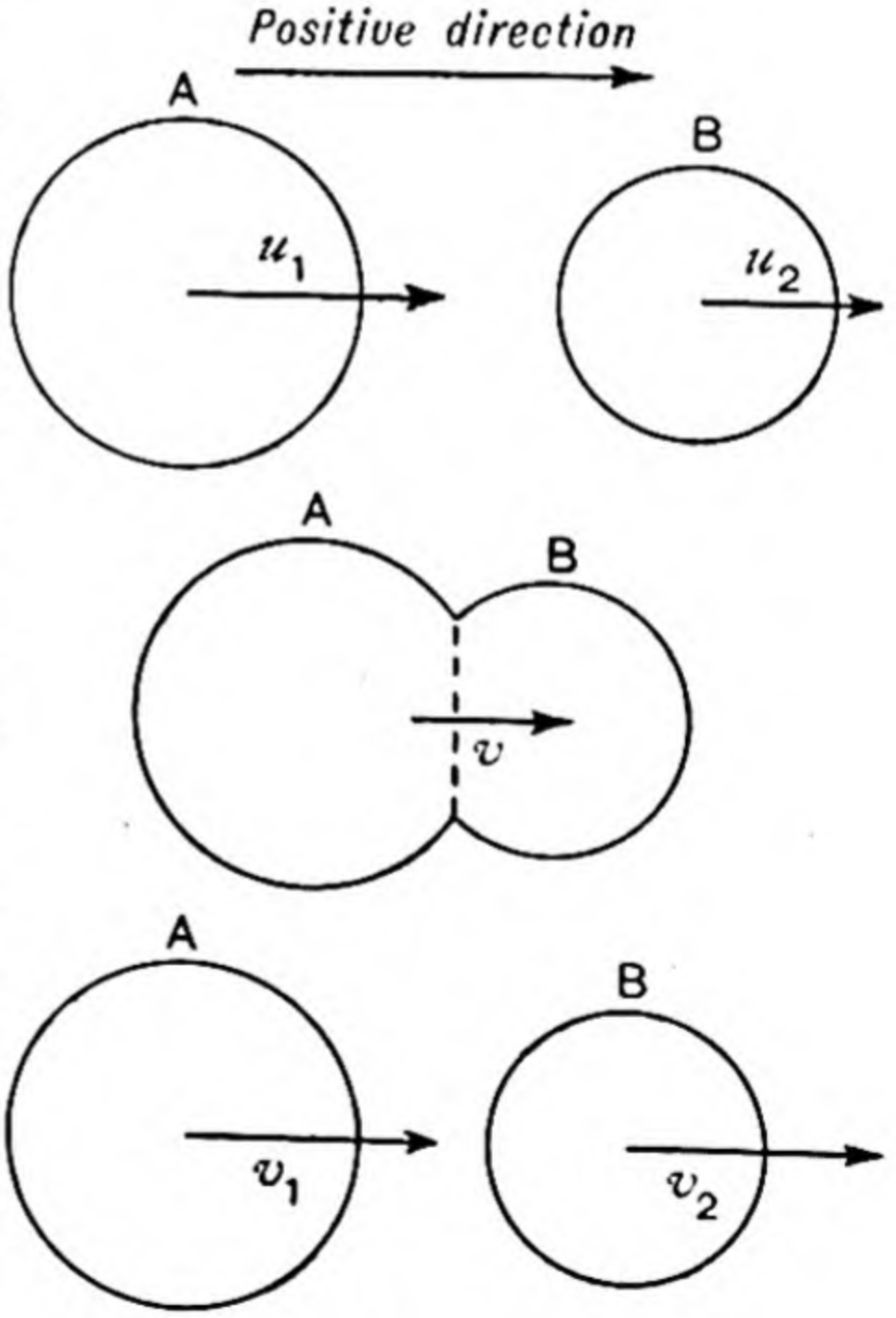
and $$v = 7{\cdot}68 \text{ ft./sec.}$$

Impact

Direct impact. When two bodies A and B moving in the same straight line collide, some of their combined kinetic energy is expended in deforming them. If the deformation is supposed perfectly elastic, so that the bodies afterwards regain their original shapes completely, this energy is stored in the deformed system as potential energy, and is recovered as the bodies return to their original shapes, being converted into kinetic energy of the separating bodies. If the deformation is supposed perfectly inelastic, so that no recovery at all occurs, all the energy lost in producing deformation is converted into heat, etc., the bodies coalesce, and travel as a single system with reduced kinetic energy.

Perfectly inelastic impact does happen, but most impacts are partly elastic; that is, they lie somewhere between the two extreme cases. Some of the kinetic energy expended in deformation is stored as potential energy during the impact, and is recoverable, while some is permanently expended in producing heat.

Consider the successive stages shown in the collision of the bodies A and B; call the masses m_1 and m_2, the initial velocities u_1 and u_2 (both shown directed to the right, the positive direction in this discussion; $u_1 > u_2$, *algebraically*, or there can be no overtaking or collision), and



Approach, compression, and separation of colliding bodies.

the final velocities v_1 and v_2 (also shown directed to the right; $v_2 > v_1$, or there can be no separation); the intermediate stage of maximum deformation is also shown; at this instant the relative velocity of approach has been reduced to zero, and they have not yet begun to separate, so they are instantaneously moving as a single body with a common velocity v. We shall develop the argument with all the velocities positively directed; using algebraic signs to denote direction, velocities to the left would be accounted negative.

(*a*) *Before impact*,

velocities are u_1 and u_2, $(u_1 > u_2)$

relative velocity of approach (that is, velocity of A relative to B) $= u_1 - u_2$,

total momentum $= (m_1u_1 + m_2u_2) \rightarrow$

total K.E. $= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$.

(*b*) *At instant of greatest compression*,

common velocity $= v$

relative velocity $=$ *zero*

total momentum $= (m_1 + m_2)v \rightarrow$

total K.E. $= \frac{1}{2}(m_1 + m_2)v^2$.

(If the collision is completely inelastic, and there is no separation, matters finish at this stage.)

(*c*) *After separation*,

velocities are v_1 and v_2, $(v_2 > v_1)$

relative velocity of separation (that is, velocity of A relative to B) $= v_1 - v_2$,

total momentum $= (m_1v_1 + m_2v_2) \rightarrow$

total K.E. $= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$.

By the Law of Conservation of Momentum,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v,$$

and

$$(m_1 + m_2)v = m_1v_1 + m_2v_2.$$

From the first of these equations,

$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2},$$

so the kinetic energy of the combined masses at the instant (b) is

$$\tfrac{1}{2}(m_1+m_2)\left(\frac{m_1u_1+m_2u_2}{m_1+m_2}\right)^2.$$

$$=\frac{1}{2}\cdot\frac{m_1^2u_1^2+m_2^2u_2^2+2m_1m_2u_1u_2}{m_1+m_2}.$$

The total K.E. at stage (a) can be written

$$\frac{1}{2}\cdot\frac{(m_1+m_2)}{(m_1+m_2)}(m_1u_1^2+m_2u_2^2)$$

$$=\frac{1}{2}\cdot\frac{m_1^2u_1^2+m_2^2u_2^2+m_1m_2u_1^2+m_1m_2u_2^2}{m_1+m_2},$$

so, subtracting the K.E. at stage (b) from that at stage (a) and rearranging,

$$\text{K.E. lost}=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1^2+u_2^2-2u_1u_2),$$

$$=\frac{1}{2}\cdot\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2.$$

In general, part of this lost kinetic energy is stored up as potential energy and returned as kinetic energy when the bodies separate ; part is dissipated irrecoverably. To what extent is restitution made at separation?

Reworking exactly the same algebra for the second of the momentum equations, it is found that the kinetic energy restored is

$$\frac{1}{2}\cdot\frac{m_1m_2}{m_1+m_2}(v_1-v_2)^2.$$

Whence,

$$\frac{\textit{K.E. restored at separation}}{\textit{K.E. lost during compression}}=\frac{(v_1-v_2)^2}{(u_1-u_2)^2}.$$

It is found experimentally that, within wide limits, this fraction is constant for a given pair of materials over a wide range of values of the individual velocities and masses ; that is, it is a property of the materials concerned, and we write,

$$\frac{(v_1-v_2)^2}{(u_1-u_2)^2}=e^2,$$

where e is a constant for the pair of materials composing the colliding surfaces, and is called the **coefficient of restitution** between them—

a very appropriate name, since e^2 measures the extent to which restitution is made on separation of that part of the kinetic energy abstracted during compression.

Taking square roots,

$$\frac{\pm (v_1 - v_2)}{(u_1 \div u_2)} = e.$$

It is usual to take the negative sign of the square root, since, as we have seen, collision and separation are only possible if $u_1 > u_2$ and $v_2 > v_1$ algebraically, so that $(v_1 - v_2)$ and $(u_1 - u_2)$ are essentially opposite in sign. Thus the coefficient of restitution is defined by the equation

$$e = \frac{-(v_1 - v_2)}{(u_1 - u_2)};$$

or, $(v_1 - v_2) = -e(u_1 - u_2)$, whence

relative velocity of separation $= -e \times$ *relative velocity of approach.*

This last equation is known as **Newton's experimental law of impact.**

Returning to the expression for the kinetic energy lost during compression, we see that the kinetic energy restored is

$$e^2 \cdot \frac{m_1 m_2}{2(m_1 + m_2)} (u_1 - u_2)^2,$$

and the kinetic energy irrecoverably transformed to heat, etc., is

$$(1 - e^2) \frac{m_1 m_2}{2(m_1 + m_2)} (u_1 - u_2)^2.$$

Thus, an experimental knowledge of the value of e is necessary before problems on impact can be completely solved. From the point of view of the Law of Conservation of Energy, this is equivalent to saying that we can apply the Law only if we know how much allowance to make for the mechanical energy that is permanently converted to other forms during the action.

Experimental measurement of e. One means of measuring the value of the coefficient of restitution between two materials is to employ the ballistic balance of p. 82, surfacing the bodies A and B with the materials concerned. Referring to the theory there given, it will be seen that if $x_1 \leftarrow$ and $x_2 \rightarrow$ are the initial displacements of the bodies, the relative velocity of approach is proportional to

$(x_1 + x_2) \rightarrow$; if $y_1 \leftarrow$ and $y_2 \rightarrow$ are the maximum displacements after collision, the relative velocity of separation is proportional to $-(y_1 + y_2) \rightarrow$,

$$\text{whence} \quad e = \frac{y_1 + y_2}{x_1 + x_2}.$$

Another method, useful for measuring e between, say, steel ball-bearings and a plate of glass or steel, is to drop a ball from a measured height h_1, and note the height of rebound h_2.

Since

$$v^2 = 2gh,$$

relative velocity of approach is $\sqrt{2gh_1}\downarrow$, and relative velocity of separation is $\sqrt{2gh_2}\uparrow$; whence

$$e = \frac{\sqrt{2gh_2}}{\sqrt{2gh_1}} = \sqrt{\frac{h_2}{h_1}}.$$

Simple problems on direct impact. In all problems on direct impact it is necessary to remember :

(i) momentum and velocity are both vectors ; it is thus necessary to distinguish between opposing directions by algebraic signs, and state clearly which direction is positive ;

(ii) relative velocity must be relative to something ; it is best therefore to state this as, e.g., " the velocity of A relative to B ", and to stick to this meaning throughout.

The following example shows the method applicable to all such problems.

EXAMPLE. Two smooth spheres, A and B, are approaching one another along the same line. The mass of A is 20 gm., and its velocity 30 cm. per sec. ; the mass of B is 30 gm., and its velocity 40 cm. per sec. The coefficient of restitution is 0·6. Find the velocities after the impact, and the kinetic energy lost as the result of the collision.

Suppose A comes from the left, B from the right, and take the left-to-right direction as positive.

Before collision ;

Velocity of $A = +30$ cm. per sec.

Velocity of $B = -40$ cm. per sec.

Velocity of A relative to $B = \{30 - (-40)\}$
$= 70$ cm. per sec. to the right.

Momentum of $A = 30 \times 20 = 600$ units $\rightarrow$

Momentum of $B = 30 \times 40 = 1200$ units $\leftarrow$

Total momentum $= -1200 + 600$ units $\rightarrow = -600$ units $\rightarrow$
(or, 600 units $\leftarrow$).

After collision :

Let v_1 and v_2 be the velocities of A and B, both supposed positive for the moment.

Velocity of A relative to $B = (v_1 - v_2)$ to the right.

Momentum of $A = 20v_1$ units $\rightarrow$.

Momentum of $B = 30v_2$ units $\rightarrow$.

Total momentum $20v_1 + 30v_2$ units $\rightarrow$.

By the Law of Conservation of Momentum,

$$20v_1 + 30v_2 = -600 \text{ units} \rightarrow .$$

By Newton's experimental impact law,

$$(v_1 - v_2) = -0{\cdot}6 \times 70 = -42 \text{ cm. per sec.}$$

Solving,

$$v_1 = -37{\cdot}2 \text{ cm. per sec.},$$

$$v_2 = 4{\cdot}8 \text{ cm. per sec.}$$

Hence, A rebounds to the left at 37·2 cm. per sec.,

and B rebounds to the right at 4·8 cm. per sec.

Initially,

the kinetic energy of A was $\frac{1}{2} \times 20 \times 30^2 = 9000$ ergs ;

" " " B " $\frac{1}{2} \times 30 \times 40^2 = 24{,}000$ ergs ;

so, total kinetic energy was 33,000 ergs.

After collision :

kinetic energy of A is $\frac{1}{2} \times 20 \times 37{\cdot}2^2 = 13{,}840$ ergs ;

" " B " $\frac{1}{2} \times 30 \times 4{\cdot}8^2 = 345$ ergs ;

total kinetic energy is $= 14{,}185$ ergs.

Hence, kinetic energy lost is $33{,}000 - 14{,}185 = 18{,}815$ ergs.

Oblique impact. When a smooth sphere collides obliquely with a smooth plane, there can be no change of momentum parallel to the surface. So, if u be the initial velocity at an angle α to the normal at the point of impact, and v the final velocity at an angle β to the

normal, as shown in the figure, then, as the component of the velocity parallel to the plane is unaltered,

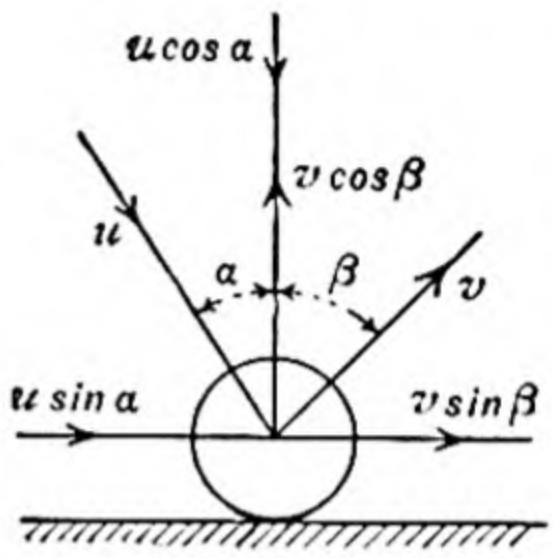
$$u \sin \alpha = v \sin \beta.$$

If e is the coefficient of restitution between the surfaces, then, for the velocity components normal to the plane

$$eu \cos \alpha = v \cos \beta,$$

whence $$v = u\sqrt{\sin^2 \alpha + e^2 \cos^2 \alpha},$$

and $$\tan \beta = \frac{1}{e} . \tan \alpha.$$



Oblique impact of sphere on plane.

For perfectly elastic impact $e = 1$, $v = u$ and $\beta = \alpha$; while for perfectly inelastic impact, $e = 0$, $v = u \sin \alpha$, and $\beta = 90°$; that is, the sphere does not rebound, but slides along the plane with velocity $u \sin \alpha$.

Oblique impact of two smooth spheres is treated in a similar way. Draw the line joining the two centres at the moment of impact (the "line of centres"), and resolve the initial velocities along and at right angles to the line of centres. As the spheres are smooth, there is no change of momentum in a direction at right angles to the line of centres; and the laws of impact as for two spheres impinging directly are applied to the momentum and velocity components along the line of centres.

Force exerted by a continuous fluid jet. Let a jet of fluid of cross-sectional area A sq. cm. of liquid of density ρ gm. per c.c., moving with velocity v cm. per sec., strike a solid wall and be arrested there without rebound.

The volume reaching the wall per second is vA c.c., its mass ρvA gm., and the momentum destroyed per second is $\rho vA \times v = \rho v^2 A$ units. This, by Newton's Second Law, gives the force in dynes on the wall, and the force per unit area or pressure is ρv^2 dynes per sq. cm.

By Newton's Third Law, a body emitting such a jet in the $\rightarrow$ direction experiences a force $\rho v^2 A$ dynes in the $\leftarrow$ direction.

If the direction of the jet make an angle θ with the wall, then the stream is not arrested but slides laterally along it. In each second the component of the momentum $\rho v^2 A \sin \theta$ perpendicular to the wall is destroyed, while the component $\rho v^2 A \cos \theta$ parallel to the

wall is retained by the jet. The force experienced by the wall is $\rho v^2 A \sin\theta$ dynes at right angles to its surface. Thus the resolution of the " force of the wind " by the sail of a yacht (p. 42) is justified.

Force exerted by a stream of elastic particles. Suppose a similar stream contains ν particles per c.c., each of mass m gm., moving with velocity v cm. per sec. normally towards a solid wall from which they rebound normally with reversed velocity v, making perfectly elastic collisions. The cross-sectional area is, as before, A sq. cm.

The change of momentum of each particle on impact, from $mv\rightarrow$ to $mv\leftarrow$, is $2mv$ units, and the number of particles striking the wall per second is the number included in a length v of the jet, that is, νAv. So the rate of change of momentum at the wall is $2m\nu Av^2$ units, whence the normal force on the wall is $2m\nu Av^2$ dynes, and the pressure exerted by the jet on the wall is $2m\nu v^2$ dynes per sq. cm. This calculation is examined in greater detail on p. 218, where the pressure exerted by a gas on the walls of its container is worked out in a similar manner.

Machines

A machine is a device enabling energy to be used for the performance of work in a convenient way. In the simple machines we shall consider, the energy is supplied to the machine by a force P called the **effort** moving through a distance x in its own direction, and the machine does work by overcoming a resistance W called the **load** through a distance y in the direction of action of W (in many cases simply raising a weight W through a vertical distance y). If W is being overcome at a uniform speed, so that no work is expended in giving kinetic energy to any part of the system, there is a definite relation between W and P; this may be different for different loads, but for the same machine a given load W always requires the same effort P to raise it uniformly.

The **mechanical advantage** of a machine is the ratio

$$\frac{\text{Load}}{\text{Effort}} = \frac{W}{P}.$$

The **velocity ratio** of a machine is the ratio

$$\frac{\text{distance moved by effort}}{\text{corresponding distance moved by load}} = \frac{x}{y}.$$

The **efficiency** of a machine is

$$\frac{\text{useful work done by machine on load}}{\text{total work done on machine by effort}}$$

$$= \frac{W \times y}{P \times x}$$

$$= \frac{W}{P} \div \frac{x}{y}$$

$$= \frac{\text{mechanical advantage}}{\text{velocity ratio}}.$$

In all machines there is friction to be overcome, and often moving parts have to be raised; so, applying the law of conservation of energy,

$$\begin{array}{ccccccc} \textit{work done} & & \textit{work done} & & \textit{work done} & & \textit{work done} \\ \textit{by effort} & = & \textit{by machine} & + & \textit{overcoming} & + & \textit{raising} \\ \textit{on machine} & & \textit{on load} & & \textit{friction} & & \textit{moving parts} \end{array}$$

The work done on the load is thus always less than the work done by the effort, so the efficiency must always be less than unity.

Useful information may often be obtained by imagining that a machine is perfectly efficient. This is equivalent to neglecting friction and any moving parts.

With this assumption,

$$\textit{work done by effort} = \textit{work done on load},$$

or,

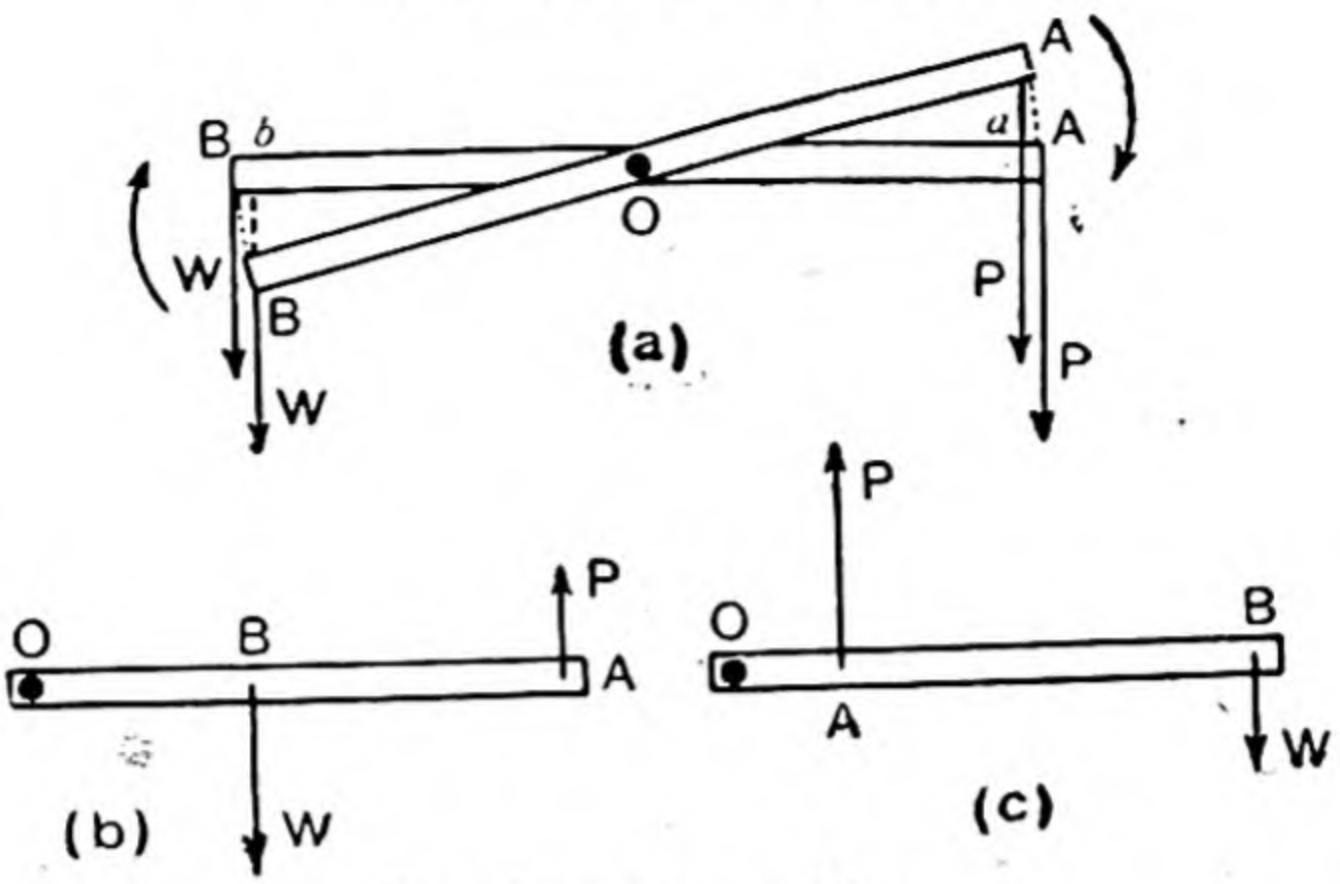
$$P \times x = W \times y,$$

a statement known as the **Principle of Work.**

Then,

$$x/y = W/P,$$

or the mechanical advantage is equal to the velocity ratio. This is often employed to calculate the velocity ratio, under the name of **theoretical mechanical advantage.** It is perhaps an unnecessary stretch of the imagination, since the velocity ratio can always be written down by simple inspection of the machine. But, if we include moving parts and friction as an effective additional load, then of course the total work done by the machine is equal to the work done by the effort on the machine, and it may be convenient to estimate the effect of friction by comparing the actual "pay-load" with the load that a perfectly efficient machine would raise.



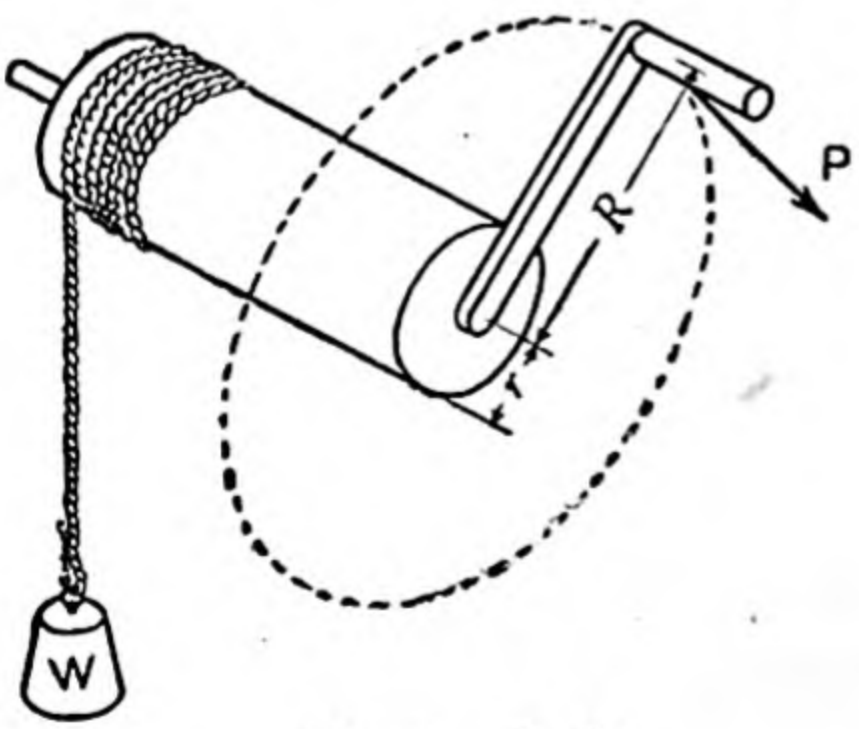
The three types of lever.

Levers. A lever is a light rod turning freely about a fixed pivot (or fulcrum) O. The effort P applied at A raises the load W applied at B; AO and BO are called the **effort arm** and **load arm.** The three possible arrangements are shown in Figs. a, b, and c. In (a), showing the lever displaced to raise W, the velocity ratio is Aa/Bb, which, by similar triangles is equal to AO/BO; so

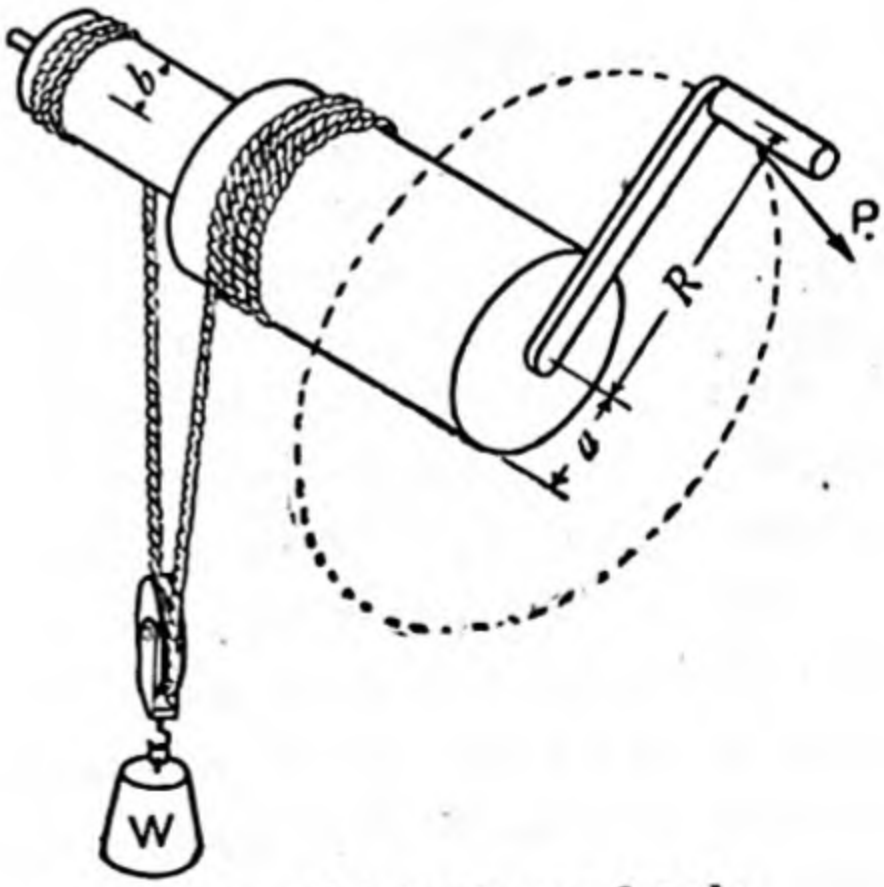
$$velocity\ ratio = effort\ arm/load\ arm.$$

Wheel devices. In the simple winch illustrated, considering one revolution, if the effort travels $2\pi R$, the load rises $2\pi r$, so

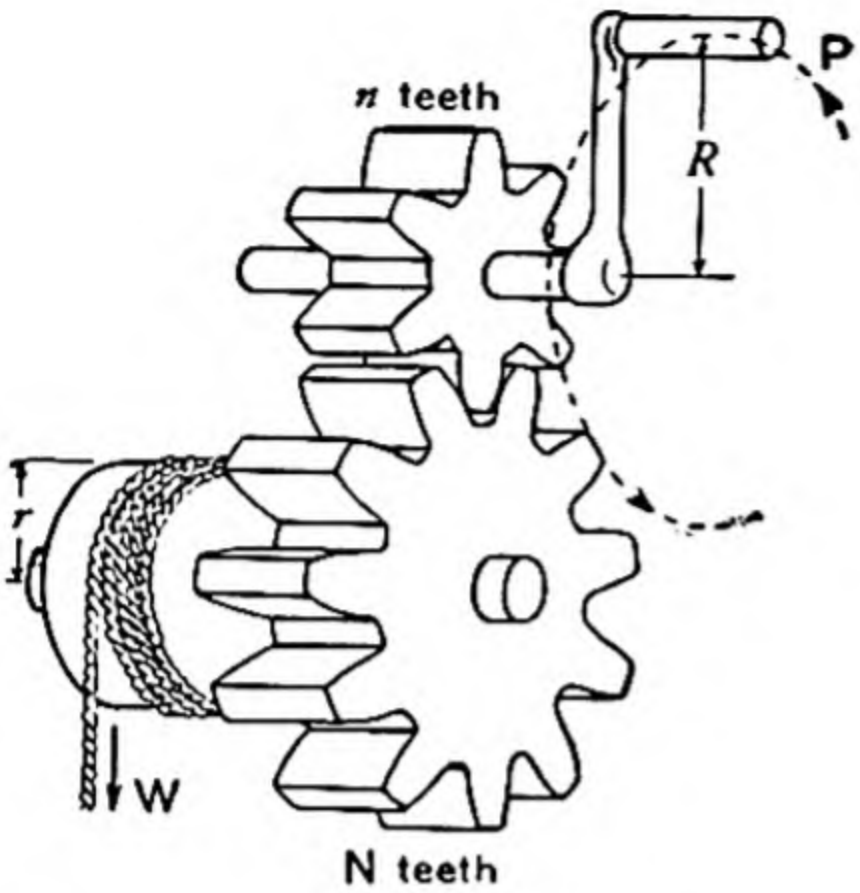
$$\text{velocity ratio} = R/r.$$



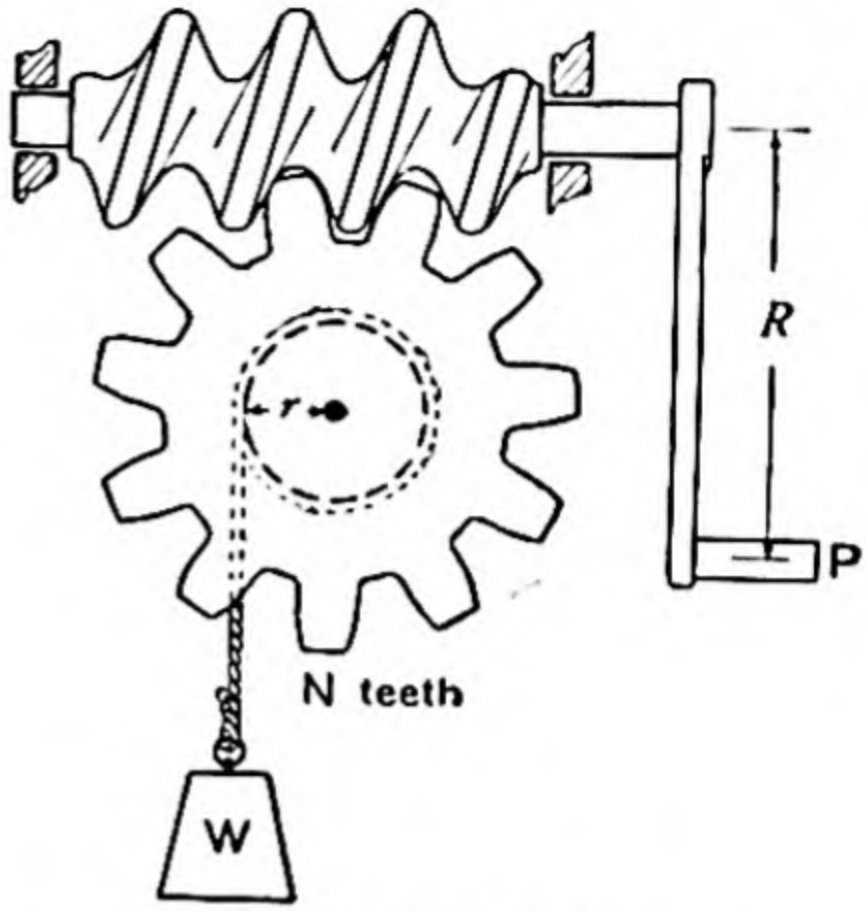
Simple winch.



Differential brake and axle.



(*a*) Gear train.



(*b*) Worm and pinion.

In the differential wheel and axle or winch, the rope is wound up on an axle of radius a and unwound from an axle of radius b. In one revolution, P travels $2\pi R$; a length $2\pi a$ of the load loop is wound up on to the larger axle and a length $2\pi b$ is unwound from the smaller, so that the shortening is $2\pi\ (a-b)$; as this shortening is shared between the two sides of the loop, W rises through half this distance, $\pi(a-b)$; so

$$\text{velocity ratio} = 2\pi R/\pi(a-b) = 2R/(a-b).$$

Gear wheels used to increase (or reduce) the velocity ratio (see diagrams) can be worked out by inspection. Thus in Fig. *a*, one revolution of the effort handle turns the axle through only n/N of a revolution, so the velocity ratio is N/n of that of an ungeared winch, or, $N/n \times R/r$. In Fig *b*, showing the worm-and-pinion gear, one turn of the handle turns the axle through only $1/N$ of a revolution, giving a velocity ratio N times that of an ungeared winch, or, NR/r.

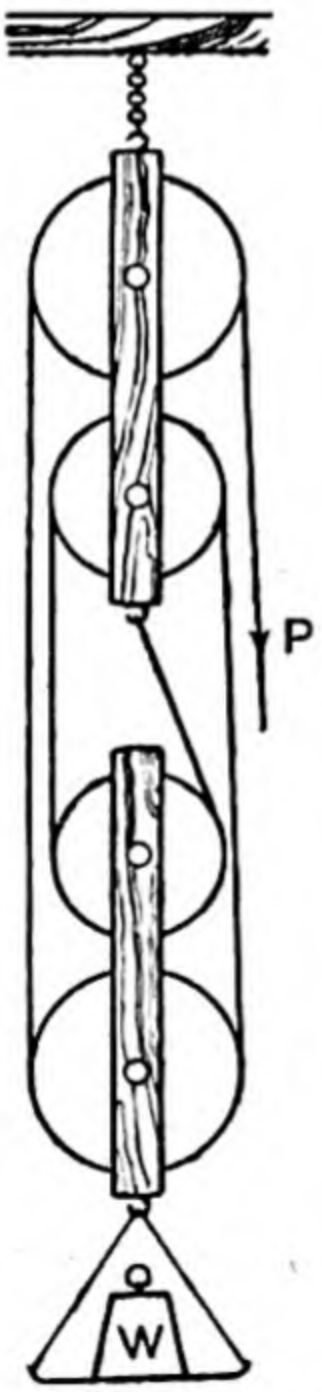
Pulley systems. A single pulley, used to change the direction of application of a force, is really a machine in which the velocity ratio is unity.

The common sheaved pulley system has two blocks within which the individual pulleys rotate independently. Usually the pulleys are side by side on a common axle; the arrangement of the figure, which is also used in practice, shows more clearly the course

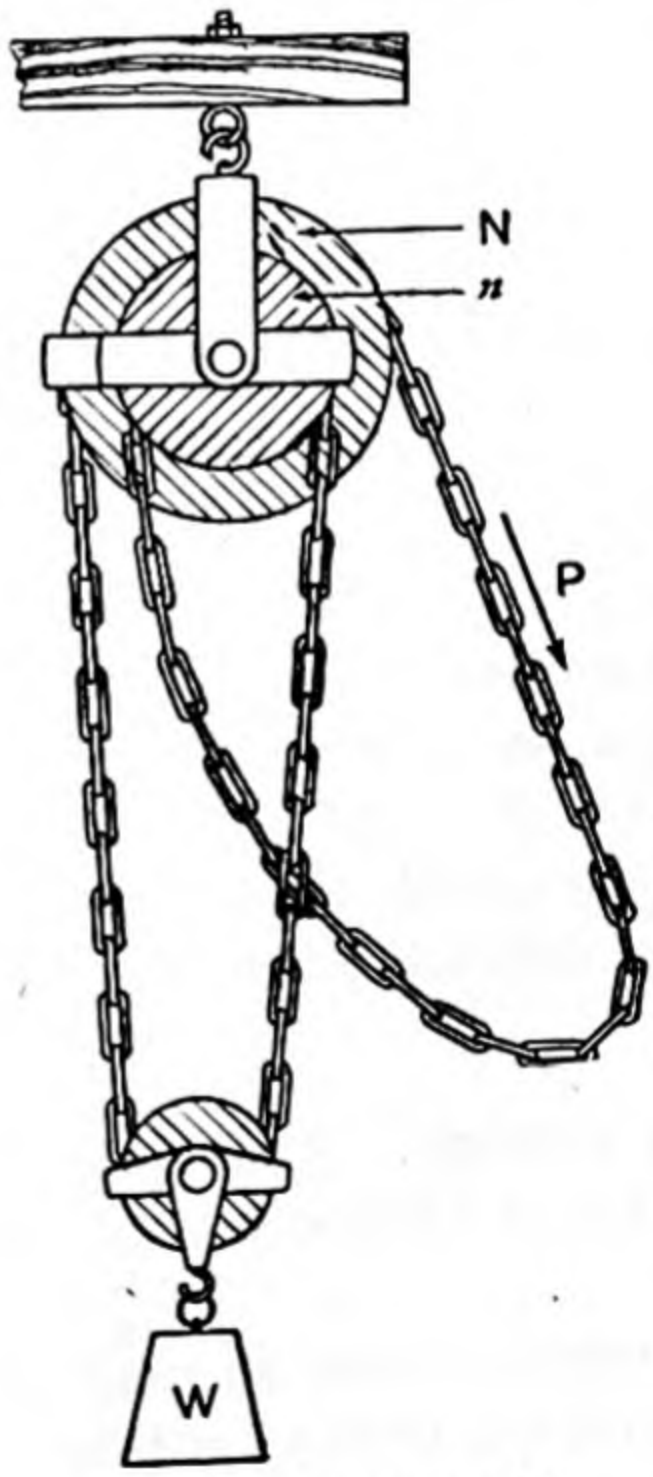
of the rope, which is one continuous cord fastened on one of the blocks. The whole system is suspended from a rigid beam.

The velocity ratio is found by counting the number of cords supporting the lower block. For, if W rises 1 ft., each of these shortens by 1 ft., and as the rope is taut the sum of these shortenings must equal the number of feet of rope pulled out by the effort P. The velocity ratio of the system shown in the figure is 4.

The Weston Differential has in the upper block a composite pulley consisting of two wheels of different radii cast side by side as a single unit, so that both revolve together. A continuous chain passes over the larger wheel, under the load pulley, over the smaller wheel, and out to the slack side of the loop. Both the wheels of the upper pulley are indented so that the links of the chain are securely held. Suppose there are N such recesses in the circumference of the larger wheel, and n in the smaller. Take one link of chain as unit length,



Sheaved pulley system, of velocity ratio 4,



Weston differential.

and consider one revolution of the composite wheel. The distance P moves is N links. The left-hand side of the chain rises by N links, the right-hand side falls by n links; the net shortening of the load loop is then $(N-n)$ links, and as each side shortens by half this amount, the load W rises $\frac{1}{2}(N-n)$ links.

Thus the velocity ratio is $N/\frac{1}{2}(N-n)$, or $2N/(N-n)$.

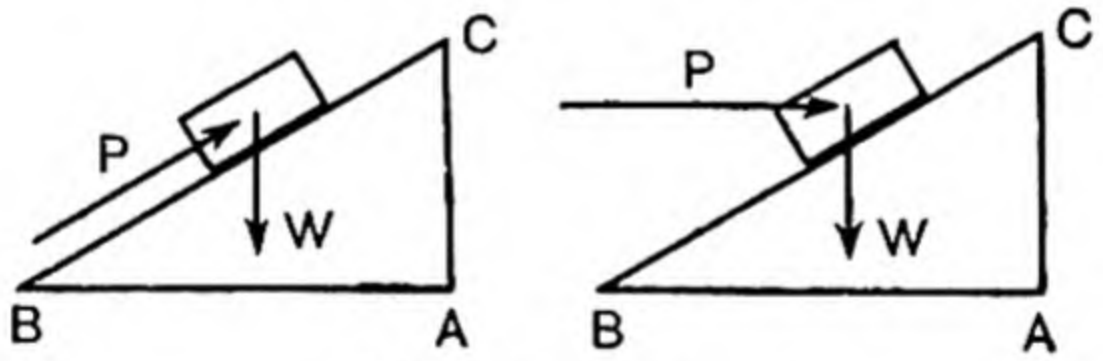
In practice, it is usually found that the efficiency of this machine is less than 0·5, and that it does not "overhaul", or run backwards when the effort is removed. The second of these observations is a consequence of the first; for, supposing that more than half the work done by the effort is used in overcoming friction, the maximum force that friction can exert must exceed the load, so that if the effort is removed and the load tends to fall, friction (which always opposes motion) will suffice to support the load.

Inclined Plane. By inspection of the diagram, it can be seen that, if P is applied parallel to the line of greatest slope,

$$\text{velocity ratio} = BC/AC;$$

while if P is applied horizontally,

$$\text{velocity ratio} = AB/AC.$$

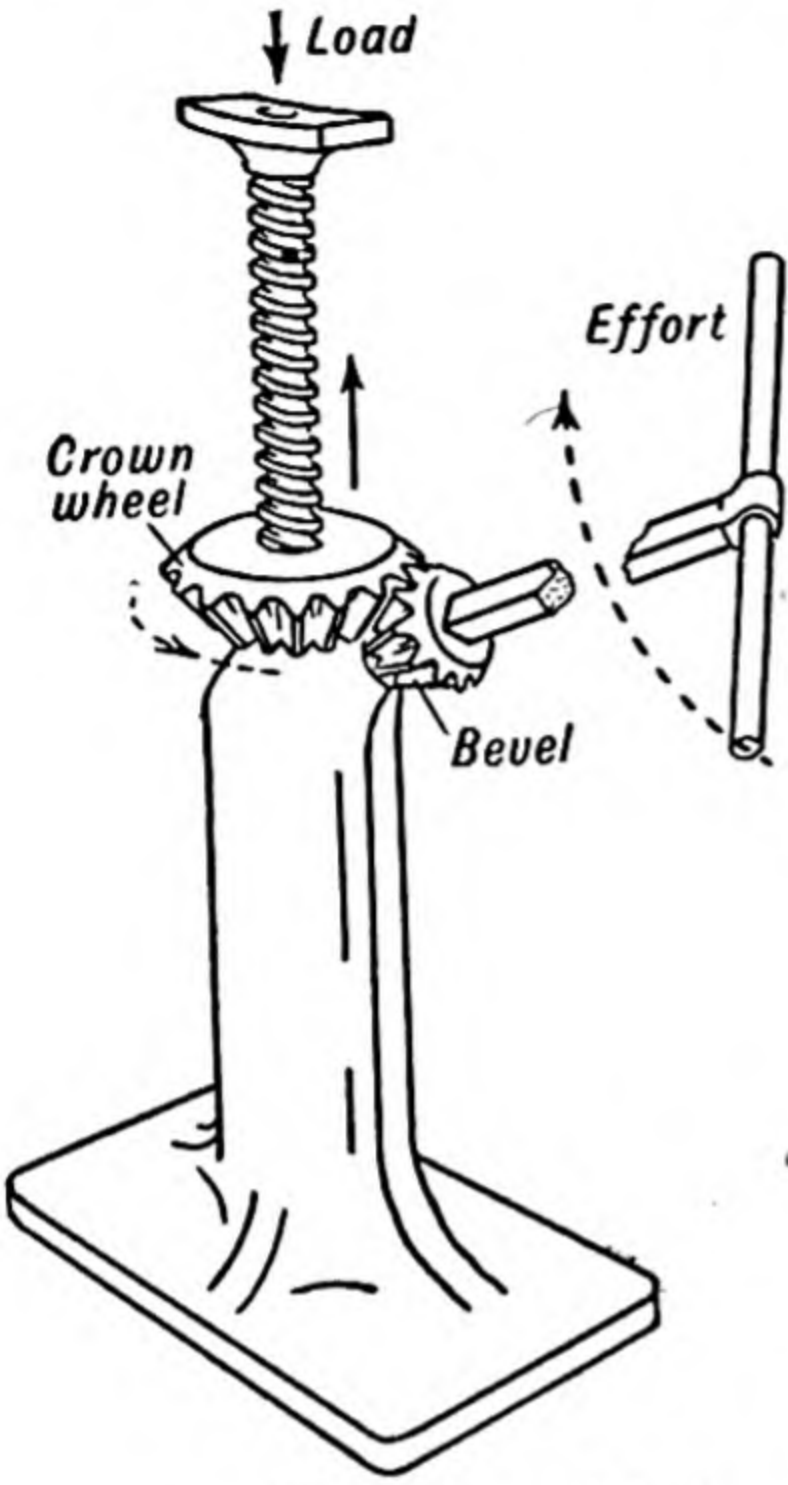


Inclined plane.

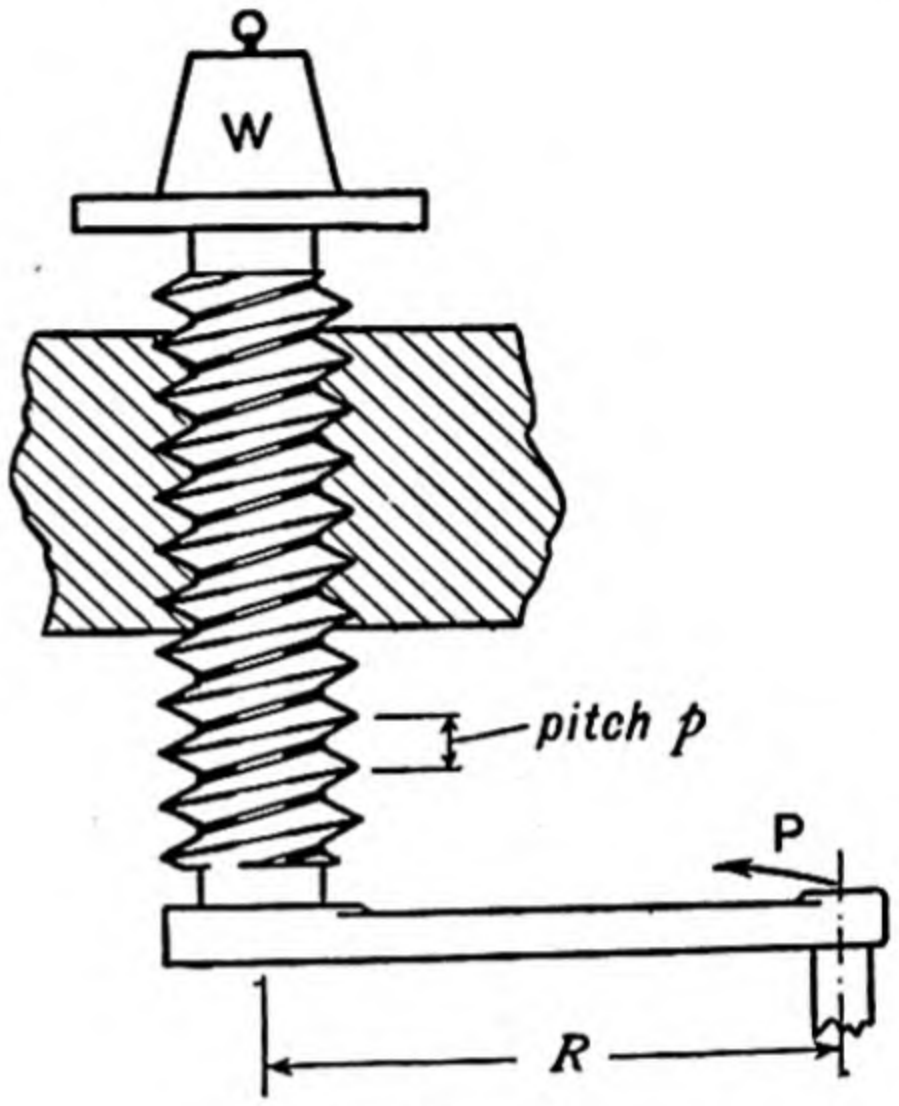
Screw. A screw, carrying a helical thread of constant pitch, working in a fixed nut, moves forward through one pitch for each revolution of the shaft. If the effort is applied at the end of a lever arm of length R (see p. 144), and the pitch of the thread is p, in one revolution the effort moves through a distance $2\pi R$, and the load, borne on the end of the screw, rises a distance p;

hence, $$\text{velocity ratio} = 2\pi R/p.$$

A simple form of motor-car jack is also illustrated on p. 144. Here the screw is pushed upwards by rotation of the nut. Between the lever arm to which the effort is applied and the nut there is a *crown-and-*



Car-jack simplified.



Principle of screw.

bevel gear. Suppose the length of the lever arm is 1 ft., the bevel wheel has 6 teeth, the crown wheel 30 teeth, and the pitch of the screw is 0·2 in. Consider one revolution of the effort lever; the effort travels a distance $2\pi \times 12$ in.; the crown wheel turns through $6/30, = 0{\cdot}2$, of a revolution; the load rises through 0·2 of a pitch, $= 0{\cdot}2 \times 0{\cdot}2 = 0{\cdot}04$ in.; so the velocity ratio is $2\pi \times 12/0{\cdot}04 = 600\pi$.

Behaviour of an actual machine. Suppose that the "pay-load" is W, and the moving parts to be raised are of weight w; the total load raised is $(W + w)$; as friction is supposed to be proportional to the pressure between surfaces in contact, write this as an additional load of $\alpha(W + w)$, where α is a constant for the machine.

The total resistance to be overcome by the effort is then

$$(1 + \alpha)(W + w).$$

As we have already taken into account the reasons why the efficiency is less than unity, we can now suppose that the machine, with this

computed load, is perfectly efficient, and say that the mechanical advantage is equal to the velocity ratio, V; hence,

$$\frac{(1+\alpha)(W+w)}{P} = V,$$

$$\text{or } P = \frac{(1+\alpha)}{V} \times (W+w),$$

$$\text{so, } P = \frac{(1+\alpha)}{V} \cdot W + \frac{(1+\alpha)}{V} \cdot w.$$

The relation between P and W should thus be linear, and the graph of P against W a straight line of which the gradient is $(1+\alpha)/V$, and the negative intercept on the W axis is w.

The efficiency is *mechanical advantage/velocity ratio*, or $\frac{W}{P \cdot V}$.

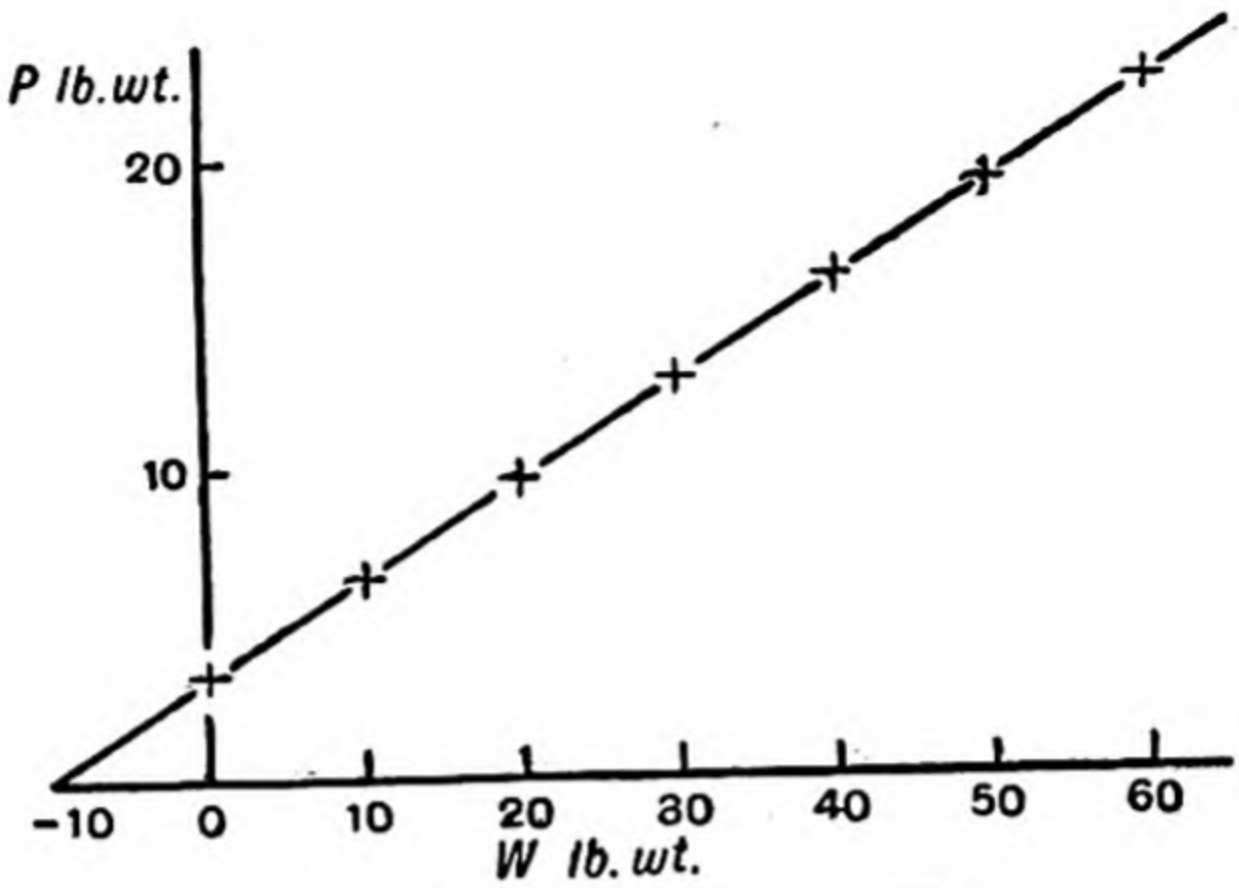
Rearranging the expression above gives

$$\text{efficiency} = \frac{W}{PV} = \frac{1}{(1+\alpha)} - \frac{w}{PV};$$

if P is small (and hence for small loads) the efficiency is considerably less than $1/(1+\alpha)$, but for large values of P it approaches the limiting value $1/(1+\alpha)$.

EXAMPLE. The first two columns of the table below give values of load and effort for a sheaved pulley system of velocity ratio 4. Find the weight of the moving parts, and express the effect of friction as a fraction of the total load; plot graphs showing the relation between (*a*) W and P, (*b*) mechanical advantage and W, (*c*) efficiency and W.

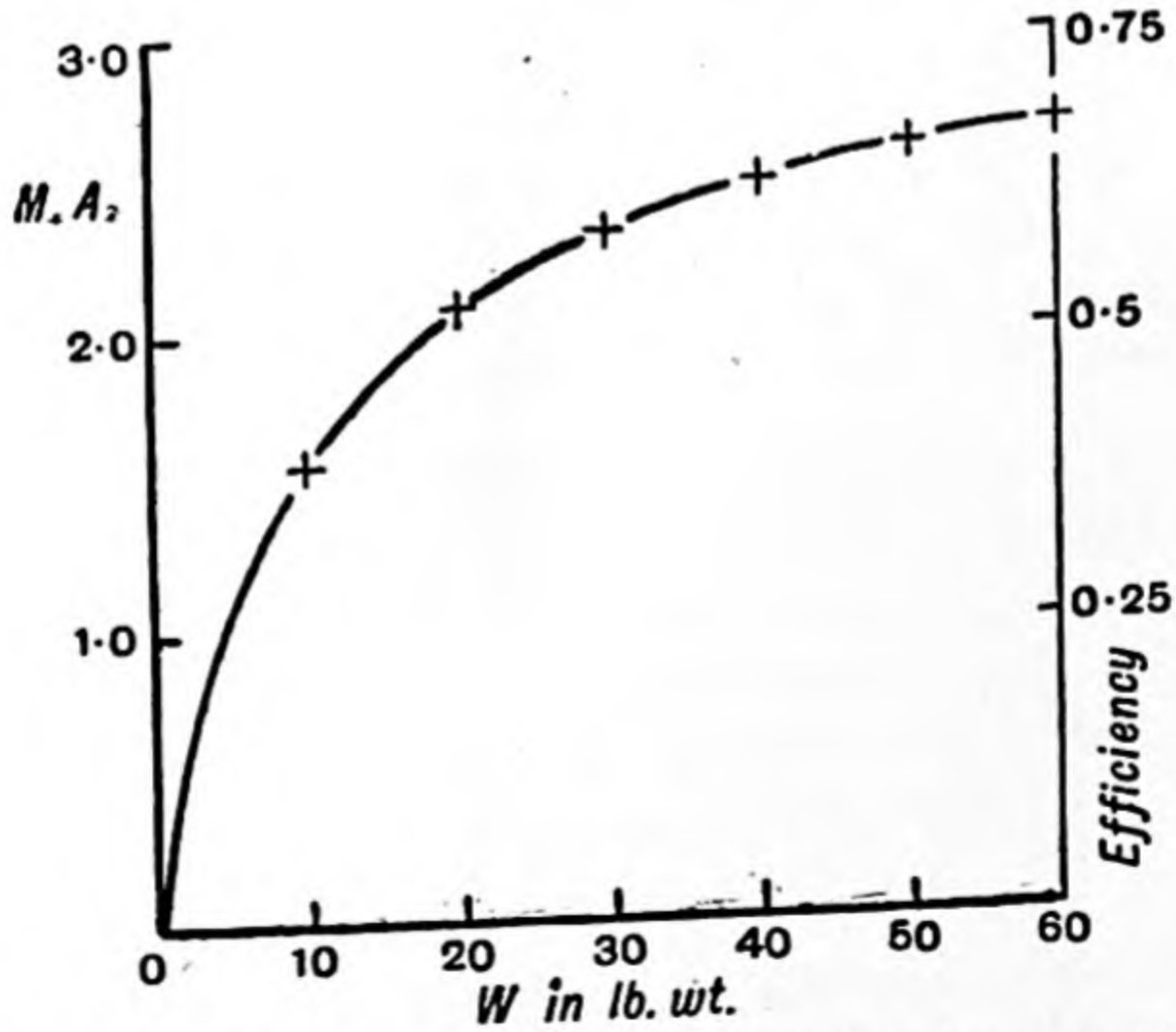
Load W lb. wt.	Effort P lb. wt.	Mechanical advantage	Efficiency
0	3·2	0	0
10	6·5	1·56	0·40
20	9·5	2·1	0·53
30	13	2·3	0·58
40	16	2·5	0·63
50	19·5	2·6	0·65
60	22·5	2·7	0·68



(*a*) Graph of P against W.

The graph of P against W is linear ; the negative intercept on the W axis represents 10 lb. wt., and the gradient is about 0·31 ; as $(1+\alpha)/4 = 0{\cdot}31$, $\alpha = 0{\cdot}24$, so the friction is 0·24 of the total load (Fig. *a*).

The graph of mechanical advantage and efficiency against load (Fig. *b*) shows a great increase of both with increasing loads for small values of W ; but this increase is less marked for greater loads.



(*b*) Graph of Mechanical Advantage and Efficiency against W.

EXAMPLES ON CHAPTER 5

1. Explain what is meant by *force*, *work* and *power*. Why is a force of 1 lb. wt. equal to a force of 32 poundals?

A car weighing 10 cwt. and working at 16 h.p. travels up a slope of gradient 1 in 10. Neglecting air and frictional resistances, at what steady speed will it travel? (O. & C.)

2. Explain the terms *force*, *work*, *momentum*.

A mass of 500 lb. falls from a height of 16 ft. upon an inelastic pile of mass 1000 lb. If the mean resistance of the ground to penetration by the pile is $1\frac{1}{2}$ tons weight, how far will the pile be driven at each blow. (O. & C.)

3. Explain the terms *work* and *power* and state the units in which they are measured.

State the principle of work as applied to a machine and indicate why it is possible by means of a machine to raise a heavy mass by the application of a small force.

A force of 20 lb. wt. is used on a wheel and axle to raise a mass of 300 lb. If the radii of the wheel and axle are 6 ft. and 3 in. respectively, calculate the efficiency of the machine. (O. & C.)

4. A boat of mass 1 ton, starting from rest and moving with uniform acceleration, acquires in 3 minutes a velocity of 30 m.p.h. There is an additional constant resistance to the motion of 30 lb. wt. per ton. Calculate (*a*) the acceleration, (*b*) the distance travelled in the 3 minutes, (*c*) the force required to produce this acceleration, (*d*) the total work done in the time. (O. & C.)

5. Give an example of a machine by which a large weight may be raised by the application of a small force, and explain how this is made possible.

A capstan is 1 ft. in diameter and has 4 bars at right angles to the axle. One man pushes against each bar and exerts a force of 50 lb. wt. so as just to raise an anchor weighing 1000 lb. How far along the bars from the centre of the capstan's axis must each man push? (O. & C.)

6. State Newton's second law of motion and explain how, from this law, we derive a method of measuring force.

A train of mass 200 tons travels at 60 miles per hour. The brakes are suddenly applied producing a retarding force equal to one-tenth the weight of the train. Find the distance the train runs on a level track before coming to rest. (O. & C.)

7. Explain why it is possible by means of a screw jack to raise a heavy car by applying a small force.

Define the terms *mechanical advantage*, *velocity ratio* and *efficiency* as applied to a machine.

A screw of pitch 1 in. is worked by a lever 7 ft. in length. If a force of 3 lb. wt. applied at the end of the lever is just sufficient to raise a weight of 1000 lb., calculate (*a*) the velocity ratio, (*b*) the efficiency of the machine, (*c*) the work done by the effort in raising the weight 3 in. [Take π as 22/7.] (O. & C.)

8. What do you understand by the term *energy*? Distinguish between potential and kinetic energy. In what units is energy measured on the cm. gm. sec. system?

A ball of mass 10 lb. is thrown vertically upward with a kinetic energy of 20,480 foot-poundals. At what height will its kinetic energy be half the initial value? What is the maximum height it will reach?
(O. & C.)

9. Explain how it is possible by the use of a machine to raise a heavy weight by the application of a small force, and illustrate your answer by reference to (*a*) a lever, and (*b*) a block and tackle system of pulleys.

A block and tackle has three pulleys in each block, and the weight of the lower block is negligible. If it is required to raise a mass of 120 lb. and the efficiency is 80%, calculate the velocity ratio, the effort which must be applied, and the mechanical advantage. (O. & C.)

10. What do you understand by *work* and *power*?

Define the units *erg*, *watt*, *horse-power*.

A boy weighing 110 lb. climbs a ladder 20 ft. high in 10 secs. Find the average horse-power he has exerted. (O.)

11. Explain the terms *mechanical advantage*, *velocity ratio*, and *efficiency* as applied to a machine.

Explain how it is that the efficiency of a machine is always less than 100 per cent., and why the efficiency of a pulley system is greater for moderately large loads than for small ones.

A man sits in a light sling attached to the lower block of a pulley system whose mechanical advantage is 3. What fraction of his own weight must he exert as an effort in order to lift himself? (O.)

12. What do you understand by the *efficiency* of a machine? Explain, with reference to some simple machine, why the value of the efficiency is always less than unity.

The following results were obtained in an experiment with a sheaved pulley system with two pulleys in each block:

Load (lb. wt.)	20	40	60	80	100
Effort (lb. wt.)	14·3	16·7	22·1	28·6	35·7

Work out the mechanical advantage and efficiency for each value of the load.

Plot a graph of efficiency (*Y*-axis) against load (*X*-axis), and comment on the shape of the graph. (O.)

13. Define the units *erg*, *watt*, *horse-power*.

The tension in the cable of a barrage balloon is 660 lb. wt. The balloon is brought down to ground-level from a height of 2,500 ft. in 5 minutes by a mechanically operated winch vertically below it. Find the work done in lowering the balloon, and also the horse-power at which the winch is working. (O.)

14. Define *coefficient of restitution* and describe an experiment to determine its value in the case of a steel sphere falling on a rigid steel plate.

A sphere A moving in a straight line with a velocity v makes a " head-on " collision with a sphere B initially at rest ; after impact B acquires a velocity $1{\cdot}6v$. Assuming the spheres to be perfectly elastic, what is the ratio of the mass of A to that of B? What percentage of A's energy is transferred to B as a result of the impact? (N.)

15. State the law of conservation of linear momentum and illustrate it by *two* different examples.

Describe an experimental verification of the truth of this law.

A target weighing 299 lb. is suspended by two long parallel cords so that it can swing upwards parallel to itself, the cords keeping in the same vertical plane. A cannon shell of mass 1 lb. is fired into the target in a horizontal direction through its centre of gravity and becomes embedded in the target which swings upwards rising through a distance of 1 ft. Calculate the velocity of the cannon shell. (N.)

16. State and discuss Newton's third law of motion.

A wooden beam of mass 25 kilos is suspended by two vertical cords, each 1 metre long, one at each end of the beam. A bullet of mass 200 gm. is fired into the beam along the line of its axis, and produces a horizontal deflection of 5 cm. Find (*a*) the velocity of the bullet at impact, and (*b*) the loss of energy. (C. S.)

17. Two steel balls, each of mass 50 gm., are suspended by cords of equal length 50 cm., so that when at rest they just touch. One of the balls is pulled aside by 25 cm. and let go. If the impact is perfectly elastic, find the velocity of both balls after impact. If the impact is made perfectly inelastic by a touch of plasticine, find the velocity of both balls after impact and the loss of kinetic energy. [$g = 980$ cm. per sec. per sec.] (C. S.)

18. A gun of mass M, set at elevation θ, is free to recoil on horizontal rails ; a shell of mass m is fired with velocity V relative to the barrel. What is the velocity of recoil of the gun, assuming that the rails and the barrel are in the same vertical plane? (C. S.)

19. State the laws of impact and explain the basis on which they rest.

A ball of mass 50 gm. is projected with velocity 20 cm./sec. so as to strike directly a ball of mass 100 gm. at rest. If the coefficient of restitution for the impact is 0·5, find the velocities after impact and the loss of energy due to impact. (C. S.)

20. State the laws of sliding friction, and explain how they may be tested experimentally.

A cube rests on a rough horizontal plane. A tension parallel to the plane is applied to it by a thread attached to the upper face. Show that the cube will slide or topple according as the coefficient of friction is less or greater than 0·5. (C. S.)

21. State the laws of direct impact and indicate the evidence on which they are based.

A smooth sphere of mass half a kilogram, moving with a speed of half a metre per second, collides directly with a similar sphere of mass one kilogram moving in the opposite direction with speed one metre per second. The kinetic energy after collision is half what it was before. Find the velocity of each sphere after impact, and the coefficient of restitution. (C. S.)

22. Name and define the absolute unit of energy in (*a*) the F.P.S. system, (*b*) the C.G.S. system.

Show that while a body is moving freely under the action of gravity its mechanical energy is constant.

A car of mass 1/2 ton is held stationary by its brakes on an incline rising 1 ft. in 30 ft. measured along the slope. When the brakes are released it travels downhill without the engine running. After it has travelled 480 ft. its speed is 28 ft. per sec. What is the change in the total mechanical energy? Give the result in ft. lb. wt. (N.)

23. Describe some form of screw press or screw jack and deduce an expression for its velocity ratio.

How would you find the efficiency of the machine you describe for one particular load?

A screw jack used for lifting a car has an efficiency of 20 per cent. The operator applies an equal effort with each hand at the ends of a handle of total length 10 in., and has to make 5 complete turns to raise the load 1 in. If the load on the jack is 3 cwt., what is the force exerted by each hand? (N.)

24. State the principle of conservation of momentum, and show how it follows from Newton's laws of motion.

A 300 gm. mass hangs from the end of a light string so as to form a plumb-line of length 100 cm. A body of mass 50 gm., travelling horizontally, collides with the bob of the plumb-line, and adheres to it. The system is deflected through an angle of 30° ; calculate the velocity of impact of the body. (O. & C.)

25. Explain the meaning of (*a*) work, (*b*) power, (*c*) efficiency of a machine.

A friction band passes round the flywheel of an engine ; one end of the band supports an 84 lb. weight and the other is attached to a spring balance. The diameter of the flywheel is 2 ft. It is found that, when the flywheel is making 1,320 revolutions per minute, the spring balance reads 24 lb. What horse-power is the engine developing? If the total power supplied by the combustion of the fuel is 3×10^6 ft. lb. per minute, what is the overall efficiency of the engine? [1 h.p. = 550 ft. lb. per second.] (O. & C.)

26. Deduce and give examples of the principle of the conservation of momentum.

A bullet is fired into a heavy block which can swing as a pendulum,

and it remains embedded in the block. If the mass of the pendulum is M, its length l, the mass of the bullet m, and if the lower end of the pendulum moves through a horizontal distance d, find expressions for the velocity of the bullet and the fractional loss of kinetic energy at impact. Assume that the pendulum can be regarded as " simple ", and that d is small compared with l.

27. Define *foot-poundal*, *horse-power*, *coefficient of friction*.

Find the shortest distance in which a motor-car, travelling along a level road at 30 miles per hour, can be brought to rest, if the coefficient of friction between the wheels and the road is 7/8. Find also the average rate of working (in horse-power) necessary to stop the car in this distance. The mass of the car is 3,360 lb. (O. & C.)

28. Define the terms " work " and " horse-power " and state the units in which they are measured.

An escalator is 300 ft. long at a slope of 45° to the horizontal and carries 200 people, of an average weight of 10 stone each, per minute at a speed of 3 miles per hour. Find the horse-power necessary to do this (*a*) if the passengers remain stationary relative to the staircase, (*b*) if they walk up at 2 miles per hour. (O. S.)

29. State Newton's second and third laws of motion, and from them deduce the principle of the conservation of momentum.

An aircraft has 8 machine-guns directed straight ahead. Each gun fires bullets of mass 1 oz. at the rate of 1,200 per minute with muzzle velocity (measured when the gun is stationary) of 2,400 ft. per sec. By how much (in ft. per sec.) is the speed of the aircraft reduced in 5 sec. when all the guns are firing, if the total mass of the machine and pilot is 6,000 lb.? Find also the backward force (in lb. wt.) exerted on the aircraft by the firing. (N.)

30. State the laws of sliding friction, and distinguish between static and kinetic friction.

A body A rests on a horizontal table, the coefficients of static and kinetic friction between the body and the table being μ_1 and μ_2, respectively ($\mu_1 > \mu_2$). A light string, attached to A, passes over the edge of the table and supports a body B, hanging freely. If the system is in limiting equilibrium, and is then set in motion, show that its acceleration is $\frac{(\mu_1 - \mu_2)g}{\mu_1 + 1}$. If B reaches the ground after A has travelled a distance s, so that the string becomes slack, how much farther will A be able to travel along the table before it comes to rest? (O. & C.)

31. Discuss and illustrate Newton's third law of motion and show how it leads to the principle of the Conservation of Momentum. What other relationship is required for solving problems of collision? Find an expression for the loss of energy in imperfectly elastic collision. (C. S.)

32. State the principle of the parallelogram of forces and discuss critically how you would verify it experimentally.

Deduce from the principle the rule for resolving forces.

Two particles of mass m_1 and m_2 are connected in a light framework in a horizontal plane so that they cannot move relative to each other. The framework can turn about a vertical axis O from which their distances are r_1 and r_2 respectively. A force F in a horizontal plane is applied to m_1 in a direction making an angle θ with r_1. Using the energy principle, show that the angular acceleration is proportional to the moment of F about O and obtain the constant of proportionality. You may assume the expression for the kinetic energy of a particle but not that for a rigid body. (C. S.)

33. Explain what is meant by the term mass. How may the ratio of the masses of two bodies be determined without weighing? Point out any assumptions involved in your method. (C. S.)

34. Explain what is meant by a work diagram, and give *two* examples of its use.

A certain mass of gas occupies 1,200 cm.^{3} at a pressure of 10^6 dynes cm.$^{-2}$. The volume is reduced to 100 cm.^{3}, the temperature remaining constant. Plot a graph of pressure against volume, and use it to calculate the work done on the gas during the change. (O.)

35. State the law of conservation of momentum, and explain how it is concerned in the jet propulsion of aircraft. Discuss whether jet propulsion is feasible beyond the limits of the atmosphere.

Describe how you would verify the law of conservation of momentum experimentally in the laboratory. (O.)

36. A thin-walled cylindrical vessel containing a non-viscous liquid has a small circular hole at the bottom. Show from considerations of energy or otherwise that the volume of liquid flowing through the hole per second is proportional to the square root of the height of liquid above it and to the square of the radius of the hole.

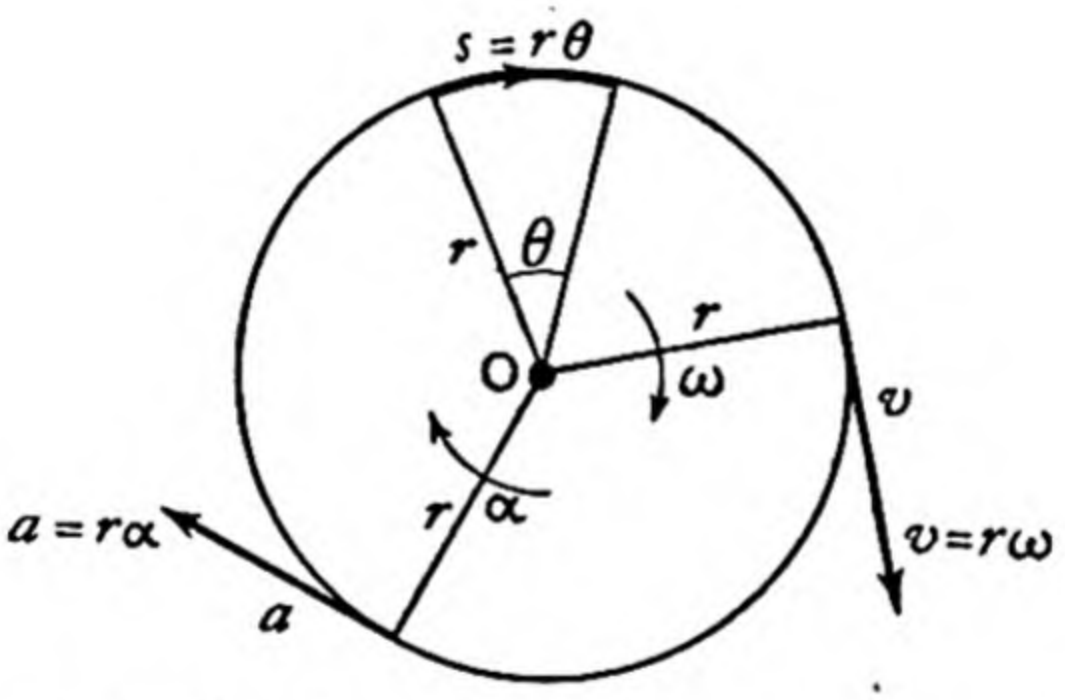
Describe briefly how you would verify this result experimentally. (O. & C.)

CHAPTER 6

CIRCULAR MOTION

THE motion of a particle describing a circular path can be considered in two ways (see diagram), either in terms of the linear motion of the particle along its path, or as the rotation about the centre of the circle of the radius vector drawn from the centre to the particle. Thus, the particle, in travelling once round the circle goes a distance $2\pi r$, where r is the radius; the radius vector makes one complete revolution and sweeps out an angle of 2π radians. Linear distance s and angular displacement θ radians are connected by the relation

$$s = r\theta.$$



If at any instant the linear speed is v cm./sec., and the radius vector is sweeping round with an angular velocity ω radians per second, then (as v is rate of change of s, while ω is rate of change of θ)

$$v = r\omega.$$

Again, if at any instant v is changing at a cm. per sec. per sec., while ω changes with an angular acceleration of α radians per sec. per sec.,

$$a = r\alpha.$$

For a particle rotating with uniform speed and angular velocity, the distance travelled, and the angle swept out, after t sec. are

$$s = vt \quad \text{and} \quad \theta = \omega t.$$

If the motion is uniform, it is often convenient to use the number of revolutions per second, n, or the time of one complete revolution, T sec. (which is $1/n$ sec.). Then, as there are 2π radians in one revolution,

$$\omega = 2\pi n = 2\pi/T \ ;$$

and

$$\theta = 2\pi nt = 2\pi t/T,$$

while

$$v = r\omega = 2\pi nr$$

$$= \frac{2\pi r}{T} \text{ cm. per sec.}$$

If the rate of change of speed, a, is uniform the radius sweeps round with uniform angular acceleration ; and if the initial speed and angular velocity are u cm. per sec., and ω_0 radians per sec., the kinematic equations of pp. 59 and 60 apply.

Hence :

$v = u + at$ becomes $\omega = \omega_0 + \alpha t$,

$s = ut + \frac{1}{2}at^2$ becomes $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$,

and $v^2 = u^2 + 2as$ becomes $\omega^2 = \omega_0{}^2 + 2\alpha\theta$.

Maintenance of uniform circular motion. It was shown (p. 54) that a particle moving with uniform speed v cm. per sec. in a circle of radius r cm. has an acceleration directed towards the centre of

$$\frac{v^2}{r} \text{ cm. per sec. per sec.}$$

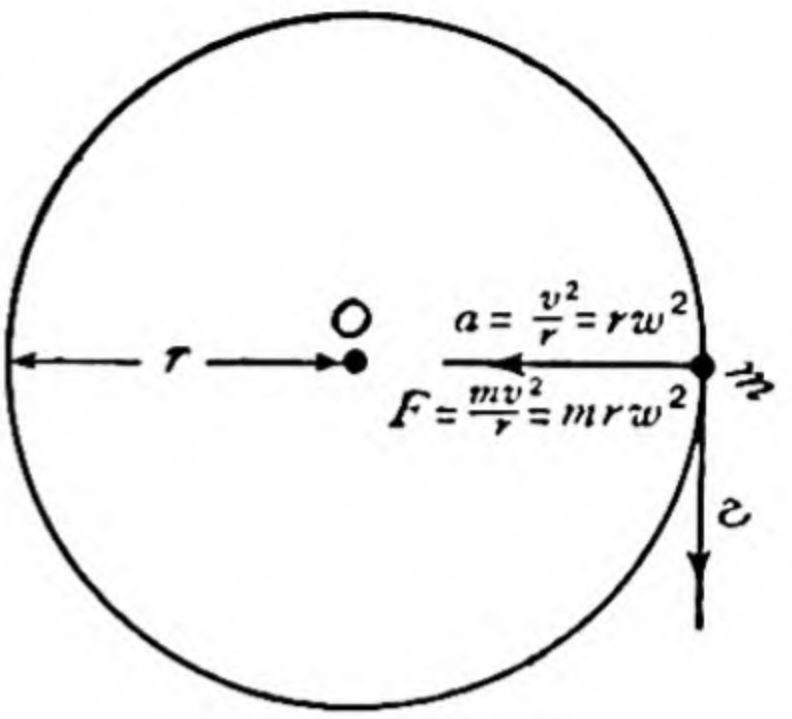
As $v = r\omega$, this can be written

$$r\omega^2 \text{ cm. per sec. per sec.}$$

Using $F = ma$, it can be seen that the force directed towards the centre of the circle which will provide this acceleration in a body of mass m gm. is

$$\frac{mv^2}{r} \text{ dynes or } mr\omega^2 \text{ dynes.}$$

Note that this is the *external force*, acting on the body from without, which is *necessary to maintain the motion*, and can be regarded as the cause of the motion. It is directed *always* towards the centre of the path.



On applying such a force by hand, for example, to keep a body whirling round on a string, one consciously experiences the equal and opposite reaction, outwards from the centre. People speak of this as "centrifugal force", and often infer that it indicates that the body itself exerts a force tending to make it fly radially outwards, or perhaps in some mysterious way conjures up an outwardly directed external force for this purpose. It is hard to combat this opinion by argument; the only test is that of experiment. What happens if the string breaks? The body does *not* fly radially outwards, but continues in a straight line along the tangent to its path at the instant of severance. For what has happened is that *the circular motion has ceased because the force towards the centre that maintained it is no longer acting*. Action and reaction are equal and opposite, and act on *different* bodies; that the reaction on your hand is outwards is the clearest possible evidence that the force acting on the rotating body is inwards.

This point is so important that it is worth while to consider more fully what justification there can ever be for the use of the term "centrifugal force". In rectilinear motion, when the equation $F = ma$ is used, the product *mass* × *acceleration* can be regarded as a kinetic reaction provided by the inertia of the body in opposition to the accelerating force (p. 76). But inertia, in resisting a forward acceleration, cannot be regarded as an *external force* tending to move the body backwards. If *mass* × *acceleration* is a kinetic reaction, then the value of this for the body describing a circle is certainly mv^2/r; but this is simply another way of stating that the body possesses inertia, and *does require an inward force to give it an inward acceleration*; it does not mean that the body tends of itself to move centrifugally, or that it is capable of exerting an external force upon itself.

Although the idea of kinetic reaction is reasonable, it is still not particularly helpful at this stage. For in all dynamical problems on the motion of a body, we require to know the external

force which acts upon the body from without and causes the motion. Here the external force is the force towards the centre, mv^2/r.

If, however, we have two external forces instead of one—that is, mv^2/r towards the centre and mv^2/r away from the centre, *both deliberately applied to the body from without*, the problem presents a quite different aspect. For there is now no resultant external force to maintain circular motion, which thus ceases. The body is arrested (or moving *uniformly in a straight line*, under zero resultant force), and we are in the realm of statics, not of dynamics. This device is frequently used in applied mathematics, since the solution of a statical problem follows straightforward lines (p. 104). But it is one thing to arrest the motion so that we may resolve and take moments legitimately, and quite another thing to believe that the feat of the imagination by which this is done means that the motion is maintained by the very force we have had supposedly to introduce in order to stop it.

R. W. Pohl, in his *Physical Principles of Mechanics and Acoustics* (Blackie and Son) discusses the notion of "centrifugal force" very clearly. His argument amounts to this: If you are a passenger in a vehicle rounding a corner, you do experience every sign of an external force directed radially outwards upon you; but you are in a very privileged position as an accelerated observer, carrying your own accelerated "frame of reference" with you. You likewise experience every sign of a force acting on you to the rear when the car accelerates in a straight line, and forwards when it decelerates, for the same reason—a moving observer carries his "frame of reference" with him and is unaware of its motion. But these subjective experiences must not be allowed to prejudice the outlook of a fixed external observer of the motion. The passenger has no *dynamical* awareness of any motion at all and only experiences the force needed to maintain his illusion of *rest*. The fixed external observer is aware both of the accelerated motion, and of the external force which is applied to maintain the acceleration.

To illustrate this further, consider the man of mass m gm. in the lift accelerating upwards (p. 78).

The external observer says, "This man is accelerating upwards at

a cm. sec.$^{-2}$ with reference to the external surroundings which I believe to be fixed ; this is because he is experiencing a resultant *upthrust* of *ma* dynes."

The man in the lift says, " All I know is that I have suddenly become *ma* dynes heavier ; I can, it is true, hear noises of moving machinery, but so far as I can see I have not moved with respect to the lift, which *I* call fixed. Surely I am at rest, and experiencing a *downthrust* of *ma* dynes." (More convincing still is the very real sensation of one's insides being forced *upwards* when the lift is accelerating *downwards!*)

Both statements are true, but few of us spend much time in accelerating lifts, and all of us agree to regard the surroundings as fixed, so the verdict of the external observer is accepted. **In applying the laws of motion to problems we usually adopt the point of view of the fixed external observer.**

Similarly, the man moving in a circular path with respect to the surroundings may suppose *himself* fixed, and to him it appears that there really is an outwardly directed force $mr\omega^2$ acting upon him. But external observers of the motion say the *surroundings* are fixed, and that the man experiences an inwardly-directed force $mr\omega^2$ which maintains him in motion.

In applying the laws of motion to problems on circular motion we usually adopt the point of view of the fixed external observer, who can not observe " centrifugal force ".

One further point arises here. The earth itself rotates about its axis, so that an imaginary observer outside the earth would say that we live in an accelerated frame of reference, though we suppose our accelerated surroundings to be fixed. It follows that *to us* the earth's rotation really does give rise to a genuine centrifugal force. However, for the sake of consistency of treatment, the effect of the earth's rotation on the apparent weight of a body is dealt with on p. 382 from the point of view of an imaginary fixed external observer, for he alone is in a position to observe the rotation of which we are merely informed by the astronomer. Centrifugal force is not the only consequence of our accelerated frame of reference, for other forces known as Coriolis forces arise when we attempt to move about in it. These cannot be explained here, but their effects are observed in the circulation of air in cyclones

and in the spin with which the last few litres of bath-water swirl down the plug-hole of a bath.

The centrifuge. Migration away from the axis of rotation is observed when a vessel containing solid particles suspended in a liquid is whirled rapidly round. Apparatus designed for this purpose is called a centrifuge ; the process is known as centrifuging, and if we can explain how it happens we should be well on the way to understanding the matter of the previous paragraphs.

From the point of view of the external observer, the explanation is as follows :

Consider an element of liquid of volume v c.c. and density ρ gm. per c.c., at distance r cm. from the axis of rotation. The force exerted on this by the surrounding liquid must just keep it in motion, and so must be $v\rho r\omega^2$ dynes inwards.

If now this element of liquid is replaced by the same volume of solid of density ρ' gm. per c.c., the surrounding liquid exerts the same inward force as before (cf. p. 183, for here Archimedes' Principle may be adapted to read, " When a body is centrifuged in a fluid, it experiences an inthrust equal to $mr\omega^2$ for the fluid displaced "). But this, if ρ' is greater than ρ, is inadequate to furnish the $v\rho' r\omega^2$ dynes needed to keep the solid in motion, so the denser particles eventually settle on the only solid wall capable of providing the necessary force towards the centre. The difference (*inward force needed* minus *inwards force furnished*) at distance r cm. from the axis is $v(\rho' - \rho)r\omega^2$ dynes.

This is not an outwardly directed force, but a " missing inward force ", and the particle merely fails to describe a circular path unless it is furnished. Frictional or other forces may, of course, supply it without the need for actual contact with the outer wall.

From the point of view of an internal observer. In a centrifuge we are concerned with the relative motions of the particles within the moving system with respect to one another, rather than with any circles an outside observer notices. Taking up a point of observation within the centrifuge cell itself, and looking around, we see all is apparently at rest except for this one solid particle. We find we must exert upon it a force $v(\rho - \rho')r\omega^2$ dynes inwards to make it join us at " rest "—that is, to partake of our unrealised motion ;

here we agree with the external observer. But we, applying Newton's First and Third Laws of Motion, and having our own idea as to what constitutes " rest ", immediately conclude that it must therefore be experiencing a force of $v(\rho - \rho')r\omega^2$ dynes directed outwards.

So centrifugal force is a reality to an observer within the rotating system, provided he takes his reference points from which to discuss motion also within the rotating system.

Thus, a pilot in an aircraft experiences all the visceral and cerebral sensations corresponding to centrifugal force when he turns at high speed ; for he, within the aircraft, forms the reference frame with respect to which small abdominal and other displacements are reckoned by him.

Again, in discussing sedimentation within a centrifuge cell, as it is relative motion within the system that has to be described, we speak of a centrifugal force, an outwardly directed centrifugal field and an outwards acceleration.

In these cases it is usual to describe the acceleration in terms of that of gravity, g. In an aircraft accelerations of $5g$ to $10g$ may be encountered. In modern ultracentrifuges, centrifugal fields giving accelerations of the order of $10^6 g$ have been obtained.

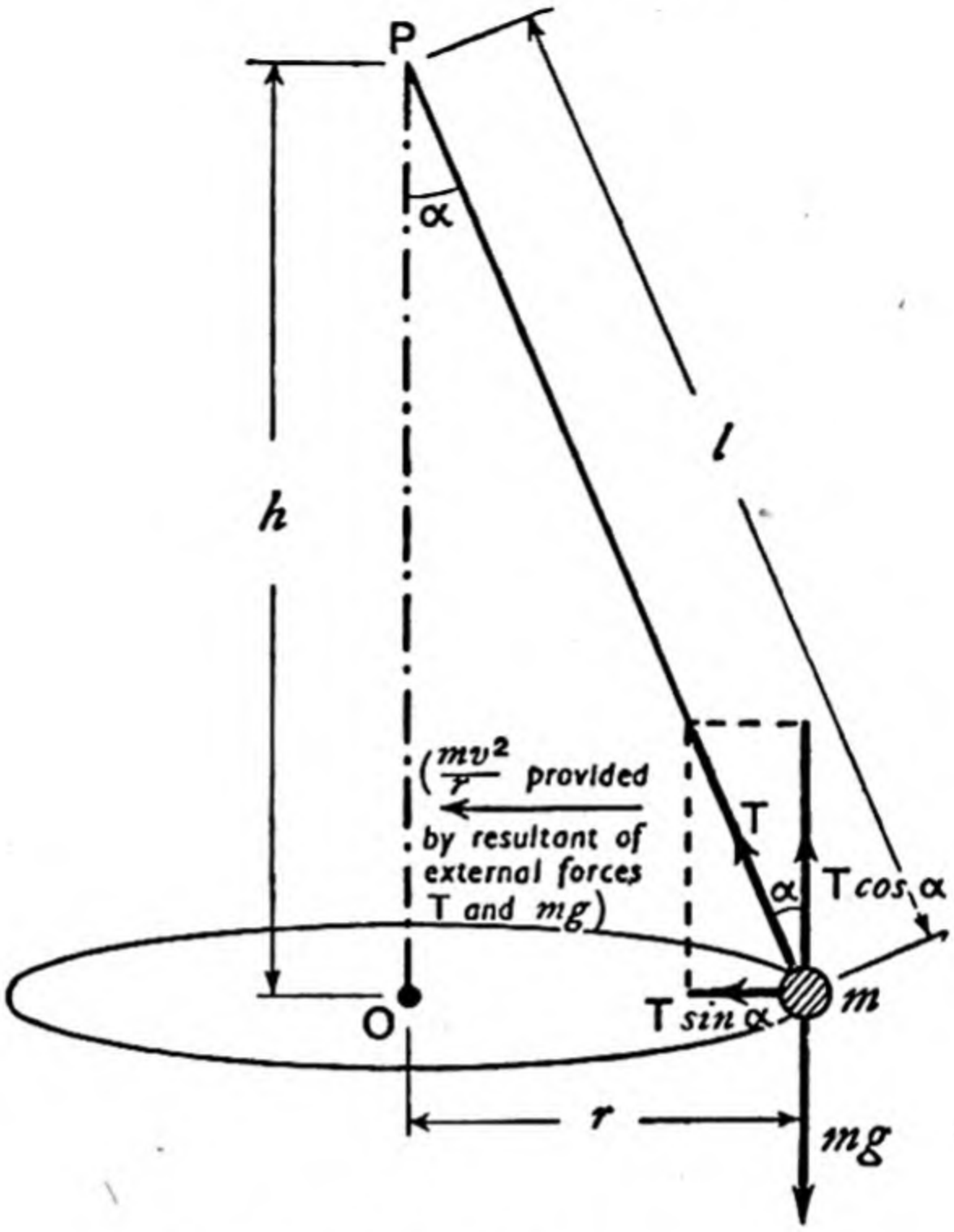
Problems on circular motion. In the following problems, bearing in mind the important point that the force maintaining motion must be directed *towards* the centre, we shall use the sign $\equiv$ (" identically equal ") when we equate the resultant inwardly-directed external force to $\frac{mv^2}{r}$, and the sign $=$ (" equal and opposite ") to connect two forces that balance statically. This distinction will be necessary only when the forces are first written down, and not in any subsequent algebra.

Conical pendulum. A body of mass m gm. describes a horizontal circle of radius r cm. at the end of a string of length l cm., sweeping out a cone of semi-vertical angle α. Find expressions for the angular velocity and the period.

Let T be the tension in the string (see diagram, p. 160), then

$$T \cos \alpha = mg \text{ dynes},$$

$$T \sin \alpha \equiv mr\omega^2 \equiv \frac{mv^2}{r} \text{ dynes}.$$



Whence, $\tan \alpha = r\omega^2/g = v^2/rg.$

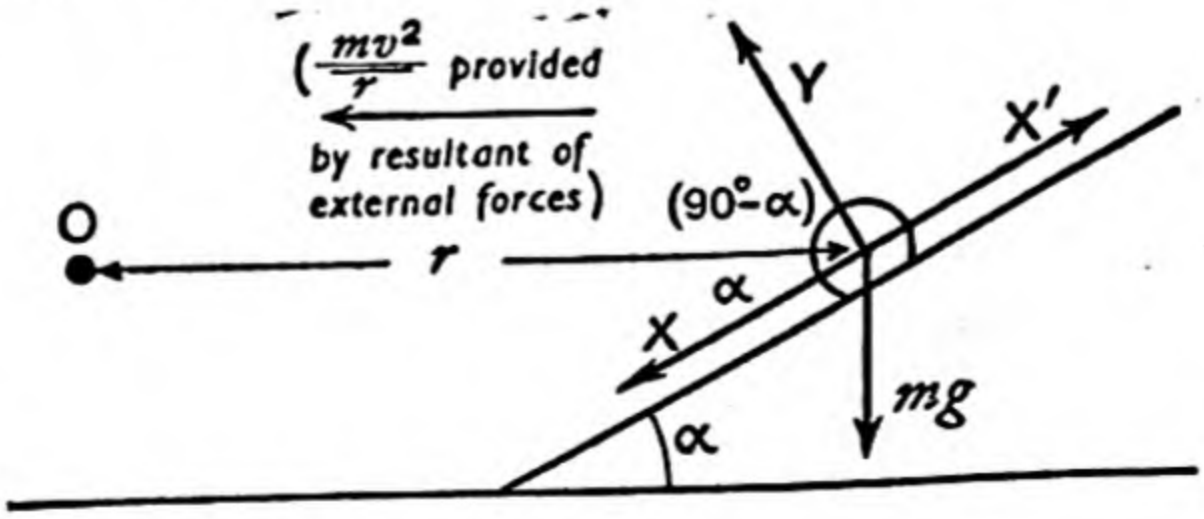
As $\tan \alpha = r/h$,

$$r/h = r\omega^2/g, \text{ whence } \omega = g/h,$$

and $$\text{the period} = 2\pi/\omega = 2\pi\sqrt{\frac{h}{g}}$$

$$= 2\pi\sqrt{\frac{l \cos \alpha}{g}} \text{ sec.}$$

Vehicle rounding a banked track. In the diagram, a vehicle of mass m is rounding a curve of radius r with velocity v. The track is



banked at an angle α. The reactions of the track parallel and perpendicular to its surface, are X and Y. That parallel to the surface may be either inwardly directed as X, or outwardly as X', and in the most favourable case is zero, for then the motion is possible on a *smooth* track.

Resolving horizontally and vertically, and working in absolute units

$$Y \sin \alpha + X \cos \alpha \equiv \frac{mv^2}{r}, \quad \text{or} \quad Y \sin \alpha - X' \cos \alpha \equiv \frac{mv^2}{r}$$

$$Y \cos \alpha - X \sin \alpha = mg, \quad \text{or} \quad Y \cos \alpha + X' \sin \alpha = mg.$$

If $X = 0$, $\tan \alpha = v^2/rg$ when X' also vanishes.

For this, the optimum angle of banking, the only force acting on the vehicle is that normal to the track. Friction, or rail guidance, is not called upon for X. An exactly similar calculation follows for the banking of an aircraft while turning, since here there is no frictional force X to rely on, and the reaction of the air on the plane, Y, must both support the weight and provide mv^2/r.

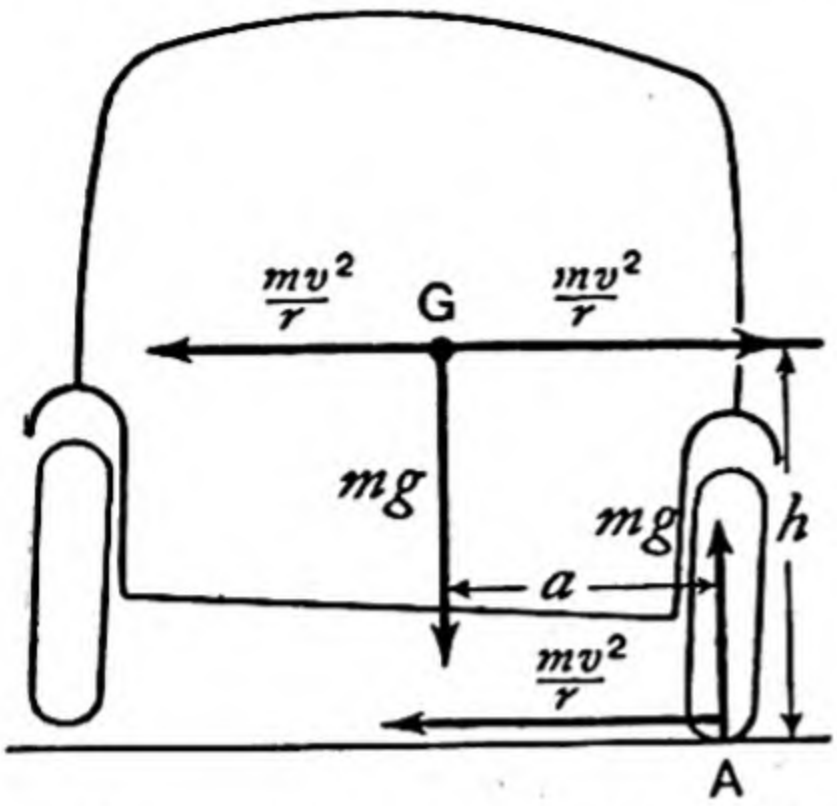
Vehicle cornering on a rough horizontal road. Here, if friction is limiting, its value is μmg absolute units ; any centrally-directed force smaller than this can be furnished.

Hence, $\frac{mv^2}{r}$ must be less than μmg,

or, v must be less than $\sqrt{\mu gr}$.

Vehicle about to overturn while cornering. So far, vehicles have been considered as " particles " ; we have ignored their dimensions, and also the fact that with an extended body the actual point of application of an external force is an important matter. Suppose the vehicle is of width $2a$, with its centre of gravity G at height h above the road (see p. 162). It is about to overturn, pivoting on the outer wheels A ; the vertical upthrust of the road is applied at A and so is the frictional force (not limiting) towards the centre.

Replace the force to the centre $\left(\frac{mv^2}{r} \text{ acting at } A\right)$ by a force $\frac{mv^2}{r}$ acting at G, and a couple of moment $\frac{mv^2}{r} . h$, as shown in the figure. We now have the force required to maintain the motion applied at G where it is needed, and a couple tending to overturn



Dangerous corner; the centre of the curve is to the left of the figure.

the vehicle outwards. The restoring moment is that of the couple mga, due to the weight at G and the vertical reaction at A.

If $\frac{mv^2}{r} . h$ is greater than mga, then the vehicle will start to overturn outwards; and as the arm of the overturning couple increases as G rises, and that of the restoring couple correspondingly decreases, overturning is assured.

Discussion on similar lines can be applied to normal cornering, when all the wheels are on the ground, by considering that the vertical upthrust of the ground is then divided between the two sets of wheels, say X at A and $(mg - X)$ on the other side.

Rotation of a Rigid Body

A rigid body can be regarded as an assemblage of particles, which preserve the same relative positions during any operation, so that the body as a whole is undistorted. When such a body rotates about any fixed axis, the individual particles describe circular paths about this axis as centre, each with its own r, each with its own v at any instant, but all with the same ω at any instant. To set such a body rotating, work must be done on it; if a couple C dyne cm. turns it through an angle θ radians, the work done on the body is $C\theta$ ergs. This work is recoverable from the body when its rotation is arrested. Thus, though the body has no translational motion as a whole, and so no translational kinetic energy, it does possess **kinetic energy of rotation.** This can be calculated by finding the sum of the individual kinetic energies of translation of the particles.

A particle of mass m gm. at distance r cm. from the axis of revolution,

describing its own circular path with speed v cm. per sec., has kinetic energy $\frac{1}{2}mv^2$ ergs; or, since $v = r\omega$, this is $\frac{1}{2}mr^2\omega^2$ ergs.

The total kinetic energy is thus the sum of the quantities $\frac{1}{2}mr^2\omega^2$ for all the particles, which, as $\frac{1}{2}\omega^2$ is the same for each, is

$$\tfrac{1}{2}\omega^2 \times (\text{the sum of all the } mr^2\text{'s for all the particles}),$$

or,

$$\tfrac{1}{2}\omega^2 \,.\, \Sigma mr^2.$$

The quantity Σmr^2, which can be calculated in simple cases and always measured experimentally by suitable means, is called the **moment of inertia** of the body about the axis concerned, and usually given the symbol I.

The moment of inertia, I, of a rigid body about a given axis is the sum of the products mr^2 taken for all the particles composing the body, where m is the mass of a particle, and r its distance from the axis concerned.

The kinetic energy of the body is thus

$$\tfrac{1}{2}I\omega^2 \text{ ergs.}$$

Comparing this with the expression $\frac{1}{2}Mv^2$ ergs for the translational kinetic energy of a body of mass M gm. moving with velocity v cm. per sec., we see that ω replaces v, while I replaces M. It appears that I takes the part in angular rotation of M in linear translation —hence the name "moment of inertia", which simply means "rotational inertia" or "rotational counterpart of mass".

The equation

$$\text{K.E.} = \tfrac{1}{2}\ \textit{moment of inertia} \times (\textit{angular velocity})^2$$

is the rotational counterpart of

$$\text{K.E.} = \tfrac{1}{2}\ \text{mass} \times (\text{velocity})^2.$$

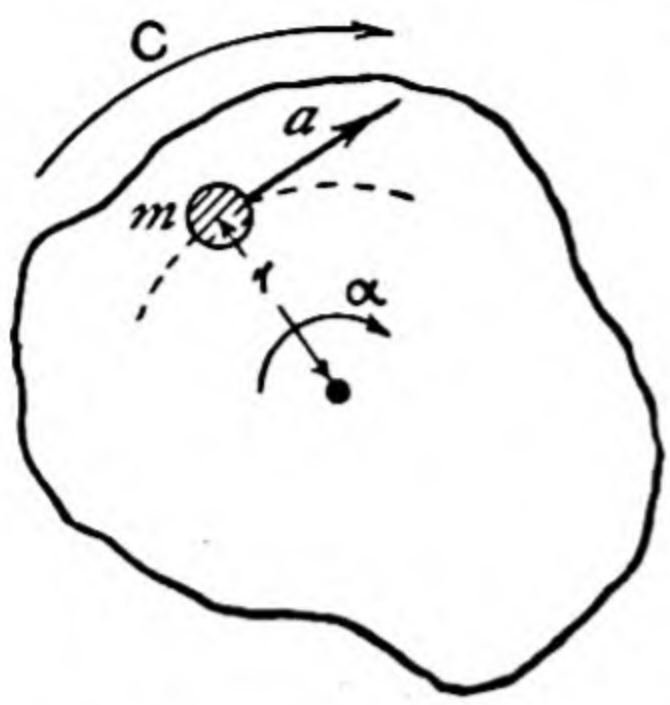
Now let a couple C dyne cm. give the body as a whole an angular acceleration α radians per sec. per sec. about the axis.

A particle of mass m gm. at distance r cm. from the axis (see p. 164) experiences a linear acceleration a cm. per sec. per sec., and thus must be acted on by a force ma dynes tangential to its path.

The moment of this about the axis is mar dyne cm.; as $a = r\alpha$, this is $mr^2\alpha$ dyne cm.

The sum of all these moments must equal C; so

$$C = \Sigma mr^2\alpha = \alpha\Sigma mr^2 \text{ dyne cm.};$$



and since, $\Sigma mr^2 = I$,

$$C = I\alpha \text{ dyne cm.}$$

The equation

couple = moment of inertia × angular acceleration

is the counterpart of

force = mass × acceleration.

Again, suppose a steady couple C dyne cm. acting for t sec. increases the angular velocity from ω_0 to ω.

Since $\omega = \omega_0 + \alpha t$ $\qquad \alpha t = \omega - \omega_0$.

Hence, $\qquad Ct = I\alpha t = I\omega - I\omega_0$.

The product of moment of inertia and angular velocity is called the **angular momentum** of the body about the axis and the product $C \times t$ is the impulse of the couple.

The equation $\qquad Ct = I\omega - I\omega_o$, or

impulse of couple = change of angular momentum

is the counterpart of

impulse of force = change of momentum.

The law of conservation of momentum applies also to angular momentum; the total angular momentum of a system about a chosen axis is unaltered by interactions between members of the system.

Throughout the above it has been stressed that I is the moment of inertia of the body *about the given axis of rotation.* If the same body rotates about a different axis, r for every particle will be different, so Σmr^2 will be different, and the moment of inertia has a quite different value. *The axis concerned must always be specified when a value for I is quoted*, though if the term " moment of inertia " is used unqualified it usually means that the axis concerned is an obviously indicated one passing through the centre of gravity of the body.

To sum up, all the formulae of straight-line particle dynamics can be converted to apply to the rotation of a rigid body about a fixed

axis, by substituting moment of inertia I for M, and the appropriate *angular* quantities for displacement, velocity, and acceleration.

Rotation Equations for Rigid Body.	Translational Counterparts for Particle.
K.E. $= \frac{1}{2} \cdot I\omega^2$.	$K.E. = \frac{1}{2}Mv^2$.
Angular momentum $= I\omega$.	$Momentum = Mv$.
"Second Law" equation $C = Ia$.	"*Second Law*" *equation* $F = Ma$.
Energy equation $C\theta = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$.	*Energy equation* $Fs = \frac{1}{2}Mv^2 - \frac{1}{2}Mu^2$.
Momentum equation $Ct = I\omega - I\omega_0$.	*Momentum equation* $Ft = Mv - Mu$

The value of I for a given body about any axis depends on two factors: the mass M of the whole body, and its distribution about the axis. These two factors can be separated by writing

$$I = MK^2,$$

where K is called the **radius of gyration** of the body about that axis.

If the axis of rotation passes through the centre of gravity of the body, we often write I_G for the moment of inertia, and k for the radius of gyration, so

$$I_G = Mk^2.$$

The dimensions of moment of inertia are those of

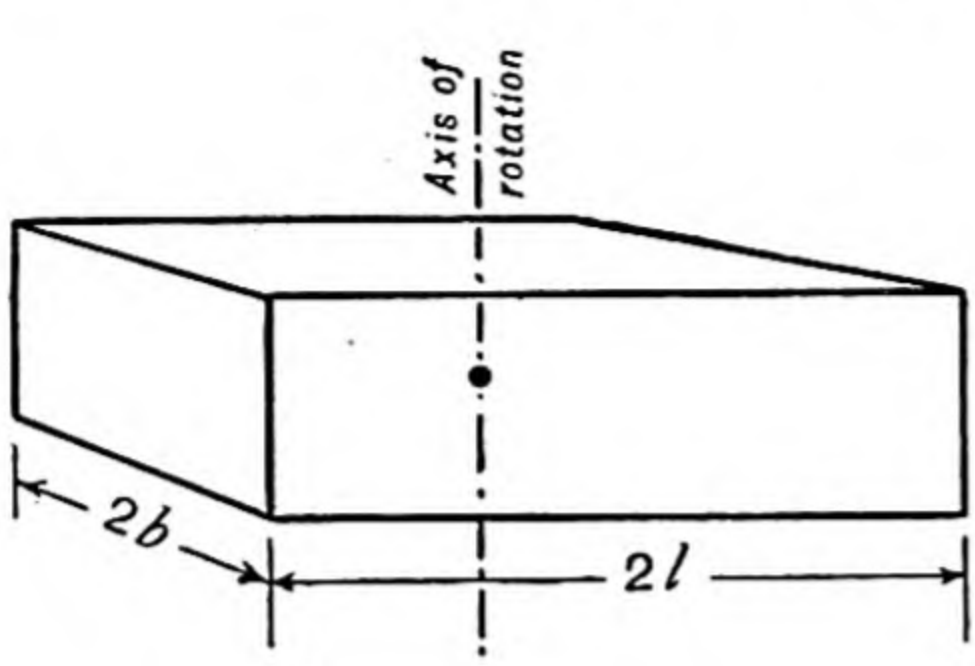
$$(mass) \times (length)^2, \quad \text{or} \quad ML^2,$$

and the units on the C.G.S. system are gm. cm.^{2}.

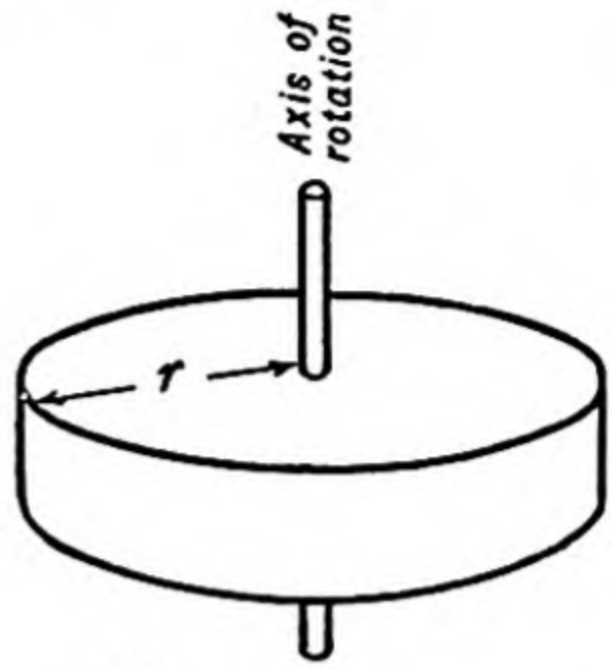
Values of I_G in simple cases. The summation Σmr^2 can be done fairly easily by the integral calculus for bodies of simple shape made of uniform material. Details of the method are not needed for our purposes; all that is required is to memorise three or four results for uniform symmetrical figures rotating about an axis of symmetry through the centre of gravity.

A thin circular hoop, of mass M and radius r, about an axis through the centre perpendicular to its plane,

$$I_G = Mr^2.$$



(a)



(b)

The same formula holds for a thin cylindrical shell about its geometric axis.

A long thin rod, mass M, length 2l cm. about an axis through its mid-point, at right angles to its length.

$$I_G = \frac{Ml^2}{3}; \quad k = \frac{l}{\sqrt{3}}.$$

A uniform rectangular bar, of length 2l and breadth 2b, about an axis through its centre of gravity perpendicular to the lb face, as shown in Fig. (a),

$$I_G = \frac{M(l^2+b^2)}{3}; \quad k = \frac{\sqrt{(l^2+b^2)}}{\sqrt{3}}.$$

A uniform disc of radius r and mass M, about an axis through the centre and perpendicular to its plane (Fig. (b)),

$$I_G = \frac{Mr^2}{2}; \quad k = \frac{r}{\sqrt{2}}.$$

The thickness or depth of the disc is not involved, and the same formula holds for the moment of inertia of a cylinder about its geometric axis, for it is simply a very thick disc.

A uniform sphere of mass M and radius r, about any axis passing through its centre,

$$I_G = \frac{2}{5}Mr^2; \quad k = r \cdot \sqrt{\frac{2}{5}}.$$

The above should be known ; it is best to try to memorise them singly as they stand. These results are useful because, as will be seen below, once the moment of inertia about an axis through the centre of gravity is known, that about any parallel axis anywhere is easily

written down, and also the moment of inertia of any body that can be split up into simple symmetrical parts can be found by summing up the individual moments of inertia of the parts about the chosen axis.

Theorem of Parallel Axes. A body of mass M and radius h rolling on a horizontal surface (see diagram) can be regarded in two ways:

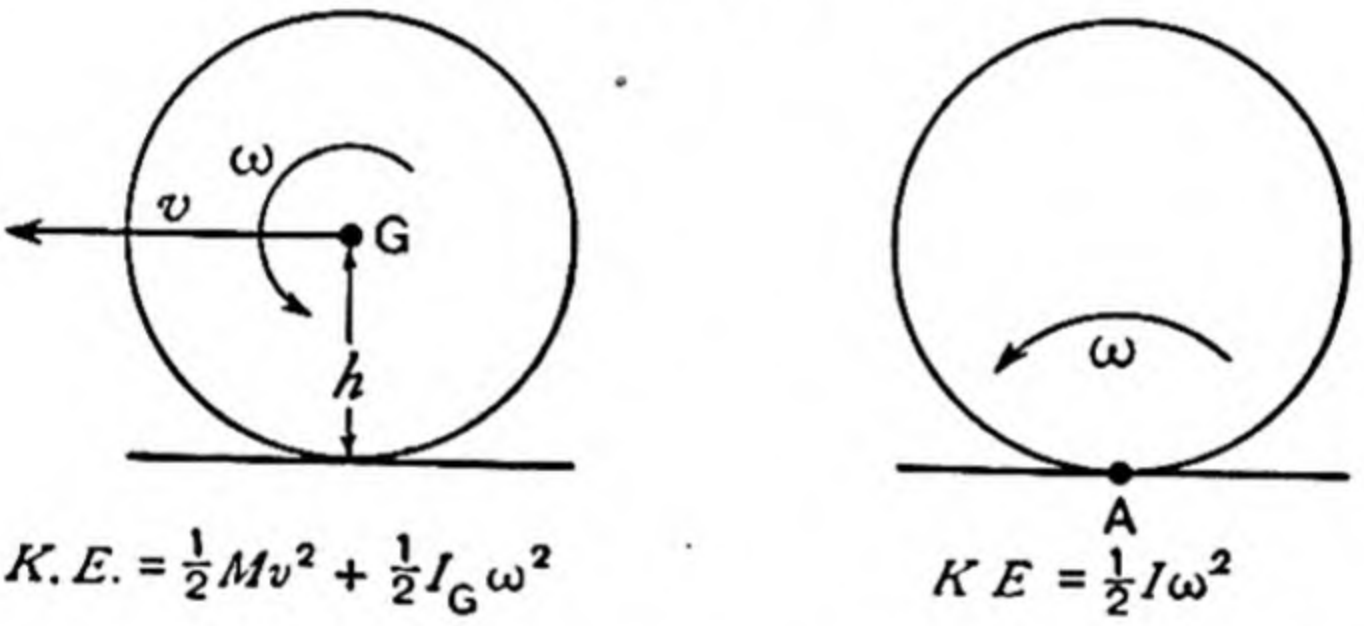
(a) We can say that it has two independent motions,

(i) rotation with angular velocity ω about the centre of gravity, and

(ii) superposed on this, translation of the centre of gravity with linear velocity v, where $v = h\omega$.

The kinetic energy of rotation is $\frac{1}{2}I_G\omega^2$, and that of translation $\frac{1}{2}Mv^2$; so the total energy is

$$\tfrac{1}{2}I_G\omega^2 + \tfrac{1}{2}Mv^2 = \tfrac{1}{2}I_G\omega^2 + \tfrac{1}{2}Mh^2\omega^2.$$



$K.E. = \frac{1}{2}Mv^2 + \frac{1}{2}I_G\omega^2$ $\quad$ $K\,E = \frac{1}{2}I\omega^2$

(b) We can fix attention on the point of contact with the surface, A, which is, for an instant only, a fixed axis about which the body as a whole is rotating with angular velocity ω. This must be so, because the centre of gravity is moving at right angles to it, at distance h with speed v^2, and angular velocity v/h (which is ω).

From this point of view the motion is entirely rotational, but the moment of inertia concerned is its value I about the point of contact A.

Thus, the kinetic energy is $\frac{1}{2}I\omega^2$.

The two expressions for the total kinetic energy must be the same, so

$$\tfrac{1}{2}I\omega^2 = \tfrac{1}{2}I_G\omega^2 + \tfrac{1}{2}Mh^2\omega^2,$$

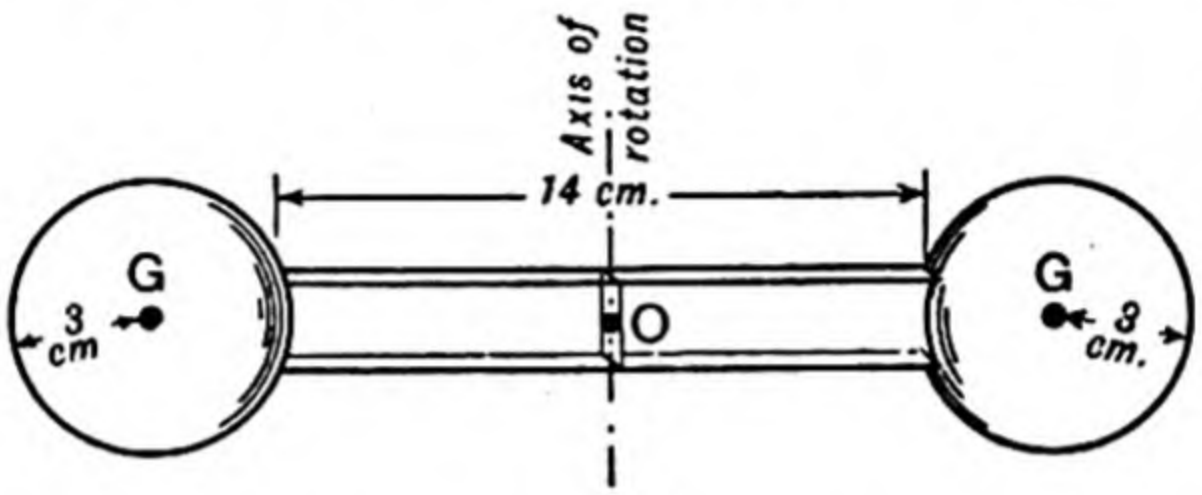
or,
$$I = I_G + Mh^2.$$

This is a perfectly general result, though obtained by considering a

particular case that enabled an energy method to be used. It is known as the **Theorem of Parallel Axes :**

The moment of inertia, I, of a body about any axis equals the moment of inertia, I_G, about a parallel axis through the centre of gravity, plus the product Mh^2, where M is the mass of the body and h the distance between the two parallel axes.

EXAMPLE. To illustrate the application of the theorem of parallel axes, and the summation by which the moment of inertia of a body made up of simple shapes can be found, consider the following :



A dumb-bell is made up of a rectangular rod of length 14 cm., breadth 2 cm., and mass 200 gm., and at each end solid spheres of radius 3 cm. and mass 300 gm. Find the moment of inertia of the whole about an axis O through the centre of gravity, as shown in the diagram.

Consider the system split up into the rod and the two separate spheres.

For the rod, the moment of inertia about O, is, using $M(l^2+b^2)/3$,

$$\frac{100\times(7^2+2^2)}{3}=1767 \text{ gm. cm.}^2.$$

For one sphere,

$$I_G\left(=\tfrac{2}{5}Mr^2\right)=\tfrac{2}{5}\times 300\times 3^2=1080 \text{ gm. cm.}^2.$$

Hence, using the theorem of parallel axes,

$$I_0=(I_G+Mh^2)=1080+300\times 10^2,$$
$$=31{,}080 \text{ gm. cm.}^2.$$

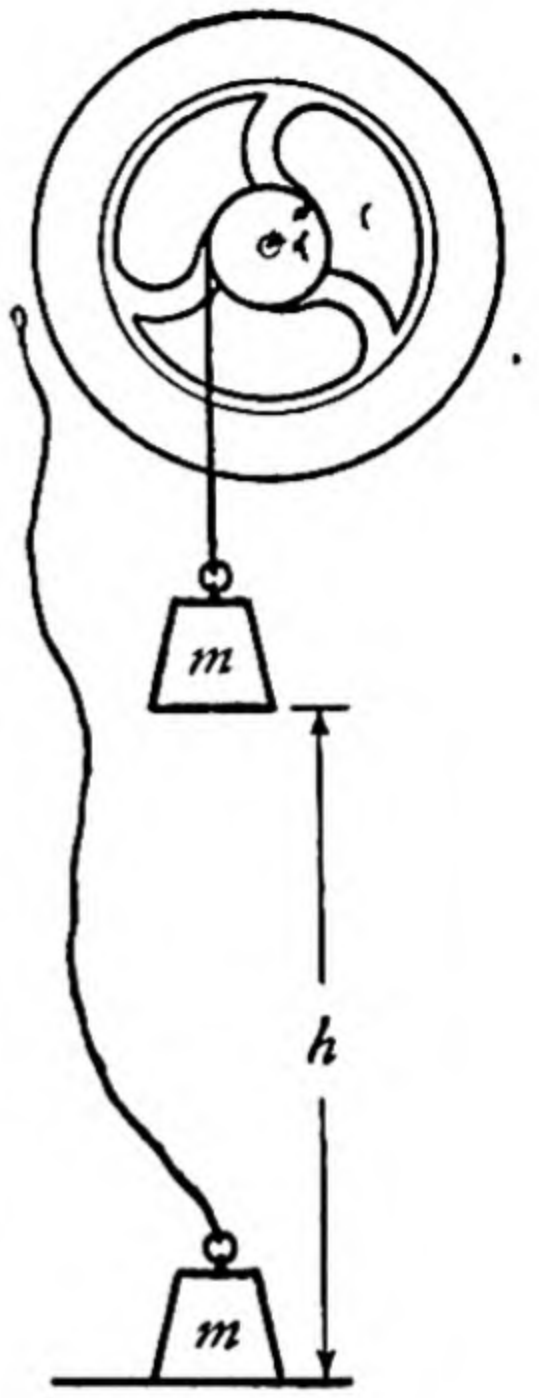
For the other sphere, similarly, $I_G=1080$, and $I_0=31{,}080$ gm. cm.^{2}.

The moment of inertia of the whole is obtained by summing these " partial sums of mr^2 " about O, thus

$$I=1767+31{,}080+31{,}080$$
$$=63{,}927 \text{ gm. cm.}^2.$$

Moment of inertia of a flywheel about a fixed axis through the centre of gravity. The flywheel is mounted on an axle of radius

r cm., which rotates in fixed bearings (see diagram). These may conveniently be ball-bearings, or should be reasonably well lubricated; in this experiment friction cannot be ignored and is corrected for, but it should be reduced as much as possible.



A light strong thread, wrapped round the axle, holds a mass m gm., at height h cm. above the ground; the thread is looped on a small pin on the axle, and its length is so adjusted that it disengages itself when the mass m hits the ground.

The time t sec. for m to fall the vertical distance h cm. is determined. The number of revolutions, n_1, made by the wheel during the fall is counted, and also the number, n_2, made after m has detached itself and *before* the wheel comes to rest.

The mass m falls with uniform acceleration, giving the wheel uniform angular acceleration; let v cm. per sec. be the linear velocity of m, and ω radians per sec. the angular velocity of the wheel, at the instant the mass strikes the ground.

Then, since the acceleration has been uniform, the final velocity

$$v = 2 \times (\textit{average velocity}) = 2h/t \text{ cm. per sec.},$$

and

$$\omega = v/r = 2h/rt \text{ radians per sec.}$$

The potential energy lost by m is mgh ergs; this has

(*a*) furnished kinetic energy of translation to m itself, $\frac{1}{2}mv^2$ ergs;

(*b*) provided kinetic energy of rotation to the wheel, $\frac{1}{2}I_G\omega^2$ ergs, and

(*c*) performed a certain amount of work against friction, $n_1 w$ ergs,

where w is the work done per revolution in overcoming friction.

Thus,

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I_G\omega^2 + n_1 w. \text{ ergs.}$$

Now, when m drops off, the wheel is left with kinetic energy $\frac{1}{2}I_G\omega^2$ ergs, which is entirely used in overcoming friction during the n_2

turns described before it comes to rest. Assuming that the work done against friction per turn is again w ergs,

$$\tfrac{1}{2}I_G\omega^2 = n_2 w.$$

Substituting for w in the previous equation,

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I_G\omega^2(1 + n_1/n_2)\ ;$$

and from this, by substituting $v = 2h/t, \quad \omega = 2h/rt$,

I_G can be calculated, in gm. cm.^{2}.

Acceleration of roller on inclined plane. A body sliding down a smooth inclined plane would have a uniform acceleration of $g \sin \alpha$, where α is the inclination of the plane to the horizontal. The acceleration of a roller, while still uniform, is always less than $g \sin \alpha$.

Suppose a body (hoop, solid cylinder, or sphere) of mass M gm., radius r cm., and moment of inertia about its centre of gravity $I_G = Mk^2$, rolls down a slope inclined at an angle α, of length l and vertical height h (see diagram).

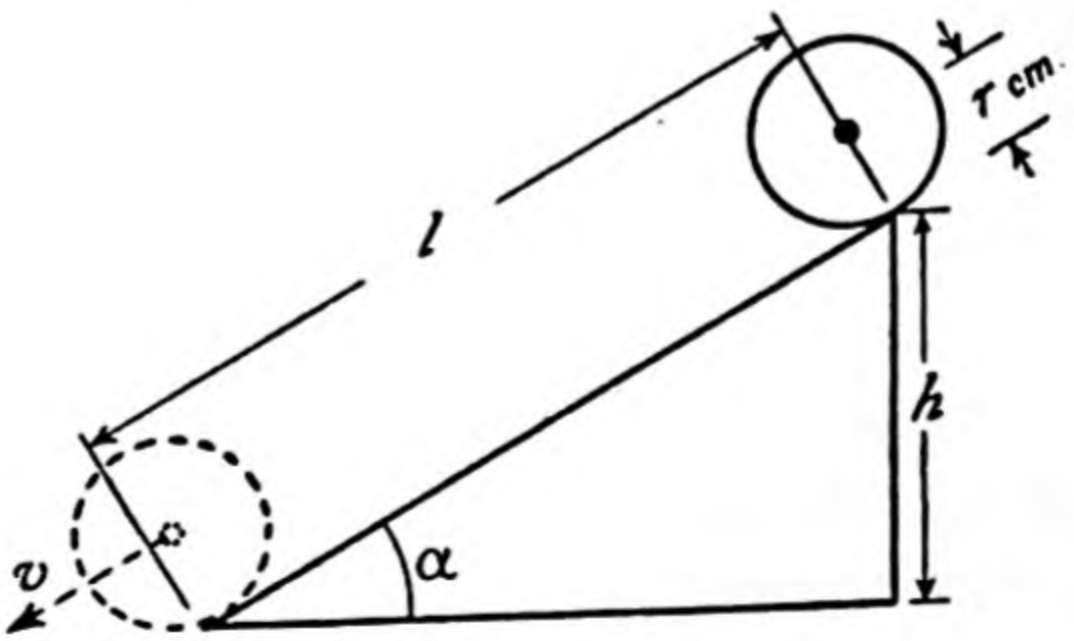
The motion can be regarded as two independent occurrences :

(a) linear motion of the centre of gravity, giving a final velocity v cm. per sec. at the end of the slope ; and, superposed on this,

(b) rotation about the centre of gravity, giving a final angular velocity about the centre of gravity of ω radians per sec., where $\omega = v/r$.

The kinetic energy of translation finally acquired is $\tfrac{1}{2}Mv^2$ ergs, and that of rotation $\tfrac{1}{2}I_G\omega^2$ ergs. Both are provided by the potential energy lost by the mass in falling through a vertical distance h cm., which is Mgh ergs, so

$$Mgh = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_G\omega^2,$$



or, as $\omega = v/r$,

$$Mgh = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_G \cdot v^2/r^2,$$

and, putting Mk^2 for I_G

$$Mgh = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}Mk^2v^2/r^2;$$

whence, cancelling M,

$$2gh = v^2(1 + k^2/r^2),$$

and, rearranging,

$$v^2 = \frac{2gh}{(1 + k^2/r^2)} = \frac{2g \sin \alpha \cdot l}{(1 + k^2/r^2)},$$ where α here stands for the inclination of the slope.

Now, if a cm. per sec.^{2} is the uniform acceleration down the slope,

$$v^2 = 2al,$$

so $$a = \frac{g \sin \alpha}{(1 + k^2/r^2)} \text{ cm. per sec. per sec.}$$

Thus,

for a hoop ($k^2 = r^2$), the acceleration is $\frac{1}{2}g \sin \alpha$ cm. per sec. per sec. ;

for a solid cylinder ($k^2 = \frac{1}{2}r^2$), the acceleration is $\frac{2}{3}g \sin \alpha$ cm. per sec. per sec. ;

for a sphere ($k^2 = \frac{2}{5}r^2$), the acceleration is $\frac{5}{7}g \sin \alpha$ cm. per sec. per sec.

For a body of radius of gyration k cm., rolling on rails on an axle of radius r cm. instead of on its own periphery, the equation

$$a = \frac{g \sin \alpha}{(1 + k^2/r^2)} \text{ cm. per sec. per sec.}$$

is generally applicable.

EXAMPLES ON CHAPTER 6

1. Define the moment of inertia of a body about a given axis.

Describe how the moment of inertia of a flywheel can be determined experimentally.

A horizontal disc rotating freely about a vertical axis makes 100 r.p.m. A small piece of wax of mass 10 gm. falls vertically on to the disc and adheres to it at a distance of 9 cm. from the axis. If the number of revolutions per minute is thereby reduced to 90, calculate the moment of inertia of the disc. (N.)

2. Describe and explain an experiment to find the moment of inertia of a wooden cylinder about its axis, the diameter of the cylinder being about 2 in. (N.)

3. Derive from first principles an expression for the acceleration of a particle describing a circular path with constant speed. Explain why it is desirable to bank the road at corners, and calculate the optimum angle of bank for a curve of radius r to be traversed at a speed v.

A train is travelling round a circular arc of radius 500 ft. which is not banked. The gauge of the railway is 4 ft. 9 in. and the hoight of the centre of gravity is 3 ft. 6 in. above the track. If there is no pressure on the inner rail, what is the speed of the train?

4. Explain the terms *moment of inertia* and *angular momentum*. Derive the relation between them for a rigid body rotating uniformly about a fixed axis.

Describe an experimental determination of a moment of inertia of *either* a flywheel *or* a rod. (N.)

5. The mass of the earth is increasing at the rate of 1 part in 5×10^{19} per day by the accretion of meteors falling normally on the earth's surface. Find the corresponding rate of change of the period of rotation of the earth supposing that the density of the earth is uniform.

[The moment of inertia of a thin spherical shell of mass m and radius r about a diameter is $\frac{2}{3}mr^2$.] (C. S.)

6. Enumerate the external forces which are acting on a bicycle and its rider when going round a curve on a level road with uniform speed, and show how the acceleration is produced. Illustrate your answer with a diagram.

Calculate the angle which the bicycle and its rider must make with the vertical when going round a curve of 22 ft. radius at 10 miles an hour. What is the least value of the coefficient of friction between the tyres and the road which will prevent slipping. If the bicycle and rider together have a mass of 200 lb., what frictional force must the ground exert on the wheels? (N.)

7. Show that in order to maintain a particle of mass m moving in a circular path of radius a, with angular velocity ω, a force of magnitude $ma\omega^2$ must be applied directed towards the centre of the circular path.

One end of a light flexible string of length 100 cm. is attached to a fixed point and supports at the other end a small bob of mass 100 gm. The whole forms a conical pendulum, the bob describing a circle in a horizontal plane. Show that $n = \frac{30}{\pi}\sqrt{\left(\frac{g}{100\cos\theta}\right)}$, where n is the number of revolutions per minute and θ is the angle the string makes with the vertical.

If the string breaks under a tension of 300 gm. weight, find the maximum angle the string can make with the vertical. (O. & C.)

8. Explain the meaning and the usefulness of the quantity " moment of inertia ", and deduce expressions for (a) the angular acceleration of a body of moment of inertia I about a given axis in terms of a couple G

applied to the body about that axis, (*b*) the kinetic energy of rotation when the body is rotating with angular velocity ω about that axis.

A steel cylinder of radius 3 cm. and mass 2,000 gm. is supported horizontally in frictionless bearings on a light shaft passing through its axis of symmetry. A long light cord with one end fixed to the cylinder is wrapped round it a large number of times, and a weight of 100 gm. fastened to its free end. Find (*a*) the linear acceleration of the weight, (*b*) the tension in the cord, (*c*) the angular acceleration of the cylinder. (The moment of inertia of a cylinder of mass M and radius a about its axis is $\frac{1}{2}Ma^2$.) (O. & C.)

9. How long will it take a penny, starting from rest, to roll a distance of 1 metre down an inclined plane which makes an angle of 30° with the horizontal? (Acceleration due to gravity = 981 cm./sec.²) (C. S.)

10. Deduce an expression for the force required to cause a particle to move in a circle with uniform speed.

Two particles resting on a rough horizontal disc are connected by a piece of string which is drawn tight and lies along a radius. The disc is now set spinning and is speeded up until the particles begin to slip. Show that, if the coefficient of friction between each particle and the disc is the same, the speed at which they slip is the same as that at which a similar particle would slip if placed at their centre of gravity. (C. S.)

11. A steel rod of length 50 cm. and area of cross-section 1 sq. cm. has masses of 2,000 gm. attached to its ends. Neglecting the weight of the rod, find the greatest speed at which the system can be rotated without breaking if the tenacity of the material is 10^{10} dynes per sq. cm. (C. S.)

12. Discuss the application of the principles of the conservation of energy and the conservation of momentum to mechanical problems.

A mass is attached by means of a light string 3 metres long to a point on the surface of a fixed vertical cylinder whose radius is 10 cm. The mass is set revolving about the cylinder, with the string horizontal, at the rate of 10 revolutions/sec. Find the angular velocity of the mass when the string has wound itself exactly twice round the cylinder, neglecting the vertical motion of the mass. (C. S.)

13. A cricket ball of mass 0·3 lb. and diameter 0·25 ft. is bowled at 30 ft. sec.$^{-1}$, spinning about an axis through its centre with an angular velocity of 80 radians sec.$^{-1}$. Calculate (*a*) its rotational energy, and (*b*) its total kinetic energy. (The moment of inertia of a sphere about an axis through its centre is 0·4 mr^2, where m is the mass and r the radius.) (O.)

14. State the laws of *sliding friction*, and describe an experiment by which you could measure the coefficient of friction between two surfaces.

A horizontal section of road whose radius of curvature is 200 feet, the coefficient of friction between whose surface and the tyres of vehicles on it is $\frac{1}{3}$, is banked at such an angle that vehicles at rest just remain

on it. What is the maximum speed at which vehicles can take the curve? (O.)

15. Show that a body describing a circular path with uniform speed experiences an acceleration towards the centre of the circle, and find an expression for this acceleration.

Find the speed of an aircraft describing a horizontal circle of radius 258 yds. if the acceleration towards the centre of the curve is $6g$. (O.)

16. Define *moment of inertia*, and find an expression for the kinetic energy of a rigid body rotating about a fixed axis.

A sphere, starting from rest, rolls (without slipping) down a rough plane inclined to the horizontal at an angle of 30°, and is found to travel a distance of 1,350 cm. in the first 3 seconds of its motion. Assuming that F, the frictional resistance to the motion, is independent of the speed, calculate the ratio of F to the *weight* of the sphere. (For a sphere of mass m and radius r, the moment of inertia about a diameter is $\frac{2}{5}mr^2$.) (O. & C.)

17. Show that when a body of mass m is describing a circular path of radius r with a velocity v, a force must act upon it of magnitude mv^2/r directed towards the centre of the circle.

A vehicle travelling on a horizontal surface goes round a curve of radius 300 feet at 60 miles per hour. If the distance between a pair of wheels is 5 feet, and there is no skidding, find the maximum height of its centre of mass to avoid overturning. (O. & C.)

18. Define the term " moment of inertia " of a body, and show that the same equations apply to angular motion as to linear motion if "force" is replaced by " moment of couple ", " mass " by " moment of inertia ", and " linear displacement " by " angular displacement ".

Show that $I = I_0 + Mh^2$, where I is the moment of inertia of a given body about any axis of rotation, I_0 is the moment of inertia about a parallel axis through the centre of mass, M the mass of the body, and h the perpendicular distance from the centre of mass to the axis of rotation. (O. & C.)

19. Explain why a lorry going round a corner at too great a speed is apt to overturn.

A symmetrically loaded lorry weighs 5 tons and the height of its centre of gravity is 5 ft. above the ground in a vertical plane midway between the wheels. The breadth of the wheel base may be taken to be 6 ft. 3 in. If there is no sideslip, what is the maximum speed at which the lorry can take a curve of mean radius 6 yards without beginning to overturn?

Determine the minimum tangential force and the minimum coefficient of friction to prevent sideslip. (O. & C.)

20. Explain what is meant by the moment of inertia of a body and describe some experimental method of comparing moments of inertia.

A flywheel with a horizontal axle, 2 cm. in diameter, has a cord wound round the axle to the end of which is attached a 5-kilogram weight. The weight is allowed to fall, causing the wheel to rotate. The weight falls from rest a distance of 40 cm. in the first 10 sec. What is the moment of inertia of the flywheel? (O. & C.)

21. A body moves in a circle with uniform speed. Find the magnitude and direction of the acceleration of the body at any instant.

An aeroplane flies in a horizontal circle of radius 2 miles at a speed of 330 miles per hour. What is the direction of the resultant force on the seaplane? (O. & C.)

22. Define " coefficient of friction " and " normal acceleration ".

A gramophone disc is set revolving in a horizontal plane and reaches a steady state of motion of two revolutions per second. It is found that a small coin placed on the disc will remain there if its centre is not more than 5 cm. from the axis of rotation. Explain this, and calculate the coefficient of friction between the coin and the disc. (O. & C.)

23. Masses of 95 gm. and 105 gm., hanging freely, are connected by a light string which passes over a pulley of mass 20 gm. When released, the system moves with an acceleration of 46·7 cm. per sec.^{2}. Calculate a value of g if the mass of the pulley is (a) neglected, (b) taken into account. Regard the pulley as a simple disc of moment of inertia $Mr^2/2$ and assume that no kinetic energy is lost in friction. (N.)

CHAPTER 7

PRESSURE IN FLUIDS AT REST

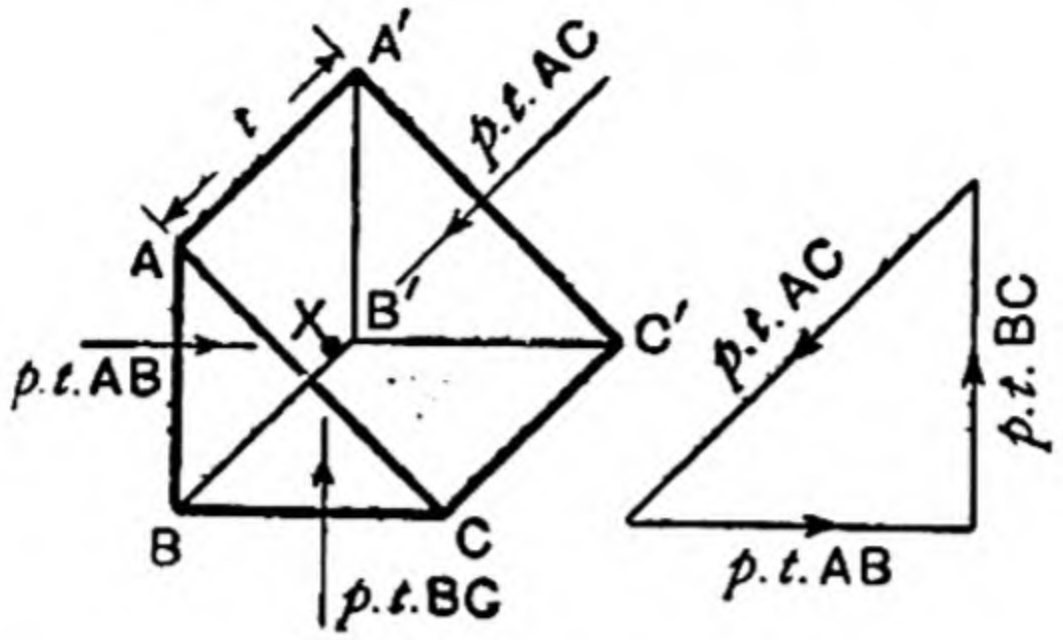
A liquid has a definite density, and a given mass of any liquid always occupies a definite volume, but has no definite shape. A gas completely fills its container, and a given mass of gas has no definite volume or shape. Both are **fluids.**

External forces are usually applied to fluids through the intervention of a boundary which has an appreciable area of contact with the fluid. In a fluid at rest, there can be no component of the external force parallel to the boundary, for this would have the effect of causing motion parallel to the boundary, whereas the fluid is at rest. The force at any element of the boundary must be directed normally to the element. In all problems the **pressure,** or **normal force per unit area** is the important quantity, rather than the force itself. It must be emphasized that **pressure is normal force per unit area.**

In the C.G.S. system, pressure is measured in dynes per sq. cm. The atmosphere exerts a very considerable pressure, about 10^6 dynes per sq. cm. at sea-level usually. The name given to a pressure of 10^6 dynes per sq. cm. is one **bar**; one millibar is 10^3 dynes per sq. cm. Other usual units are the gm. wt. per sq. cm., the kgm. wt. per sq. cm., and the common British System units are lb. wt. per sq. in., tons wt. per sq. ft., etc. The dimensions of pressure are those of *force÷area*, or $ML^{-1}T^{-2}$.

By the *pressure at a point* in a fluid, we mean the normal force per unit area, *taken over a small element of area in the neighbourhood of the point.*

Pressure at a point in a fluid at rest is the same in all directions. This must be so, for otherwise a small closed element of volume drawn round the point would experience a resultant force, and the fluid could not be at rest. For example, if X is the point, describe round it a right triangular prism, with sides ABC, $A'B'C'$, and width $AA' = t$ (see diagram). This is at rest, so the forces over ABC



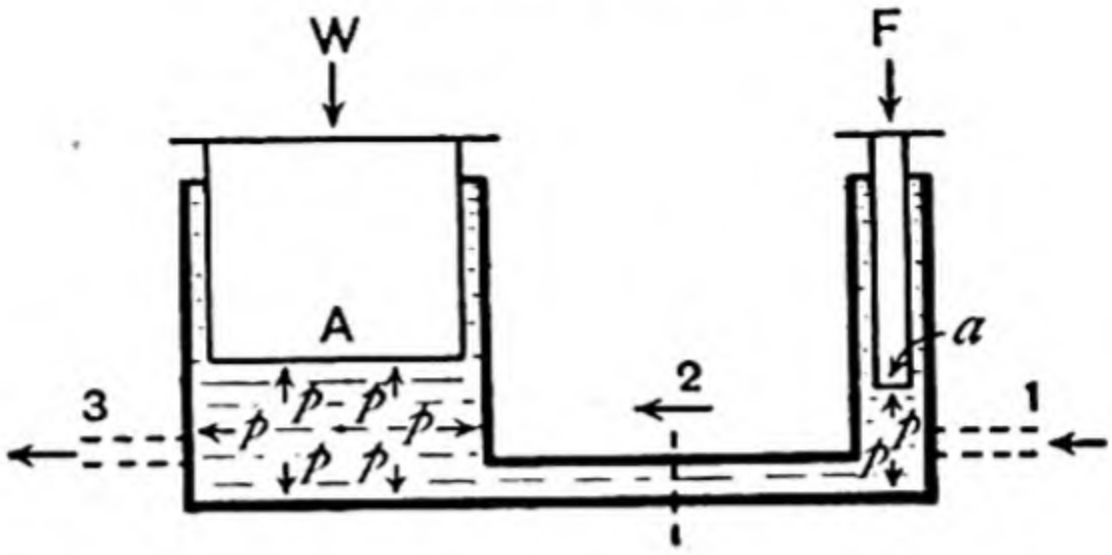
and $A'B'C'$ must be equal and opposite; suppose for the moment that the pressure over the rectangular faces is actually the same, p. Then the force over the face $AA'BB'$ is $p \times t \times AB$, acting at right angles to the face and through the centre of it; similarly for the faces AC and BC, the forces are $p \times t \times AC$ and $p \times t \times BC$, at right angles to the faces and through their centres. These three forces all meet in one point, and as they can be represented by lines are proportional in length to the sides of ABC, and as each is directed at right angles to the appropriate side, they can be drawn as the three sides of a closed triangle. Thus they are in equilibrium, justifying the assumption that the pressure was the same on all faces.

Pressure at all points at the same horizontal level in a fluid at rest is the same. This must be so, for any horizontal pressure gradient would cause the fluid to flow, and it does not.

A fluid transmits an externally applied pressure increment equally in all directions. This is a consequence of the fact that the pressure at any given point acts equally in all directions; any externally applied pressure increment must thus be transmitted undiminished from the boundary at which the external pressure is applied to the more remote parts of the fluid at all levels.

Simple experiments to verify the foregoing statements, or to illustrate them, are easily devised.

Hydraulic press. Two cylinders communicating by a horizontal tube are filled with liquid. One carries a plunger of area a, to which an effort F is applied; the other has a larger plunger of area A which bears a load W.



The effort F exerts a pressure $p=F/a$ on the fluid, which is transmitted equally in all directions, and acts equally all over the boundary. The load plunger experiences a force $F/a \times A$, which just overcomes the load W, so

$$\frac{F}{a} \cdot A = W, \quad \text{or} \quad \frac{W}{F} = \frac{A}{a}.$$

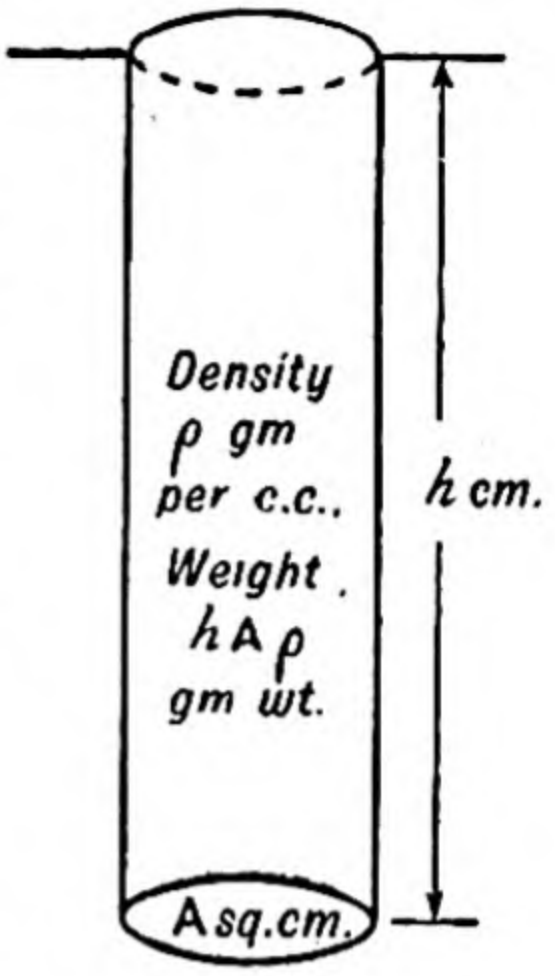
That is, the mechanical advantage of the arrangement equals *area of load plunger/area of effort plunger*.

If F moves down a distance x and W rises a distance y, then if the liquid is incompressible $ax = Ay$, for each represents the volume of liquid transferred from one cylinder to the other. Then $Fx = Wy$, so that in the absence of friction the arrangement would be perfectly efficient. But if the liquid is compressible, $ax > Ay$, whence $Fx > Wy$, and some work is expended in compressing the liquid instead of raising the load. Thus the fact that the compressibility of liquids is very small as well as their property of transmitting pressure equally in all directions, is an essential reason why most hydraulic mechanisms operate.

In an actual machine, a valve 1 enables fresh liquid to be drawn in when the effort rises for a second stroke ; a non-return valve 2 allows transfer of liquid from right to left only ; and a valve or tap 3 allows the load to fall and the liquid to return to a reservoir when required. The figure really represents a hydraulic hoist rather than a Bramah hydraulic press ; for a fixed upper plate should be provided for the plunger to press its load against.

Pressure due to a vertical column of liquid. Consider a horizontal area A sq. cm. at depth h cm. below the free surface of a liquid at rest (see diagram). Suppose for a moment that there is no external pressure on the free surface, so that we calculate the pressure *due to*

the fluid column ; and suppose that the density of the fluid is uniform, of value ρ gm. per c.c.



Standing above the area A is a vertical column of fluid of volume Ah c.c. The mass of this is $Ah\rho$ gm., and its weight $Ah\rho$ gm. wt. The thrust exerted on the area A is $Ah\rho$ gm. wt., and the pressure $p = Ah\rho/A = h\rho$ gm. wt. per sq. cm. Or, as there are g dynes in one gm. wt.,

$$p = hg\rho \text{ dynes per sq. cm.}$$

This, of course, gives the pressure at any point in any direction at the same horizontal level (or vertical depth) as the chosen area.

Thus, the pressure due to a column of liquid depends only on the vertical depth below the free surface and the density of the fluid : for a liquid of uniform density,

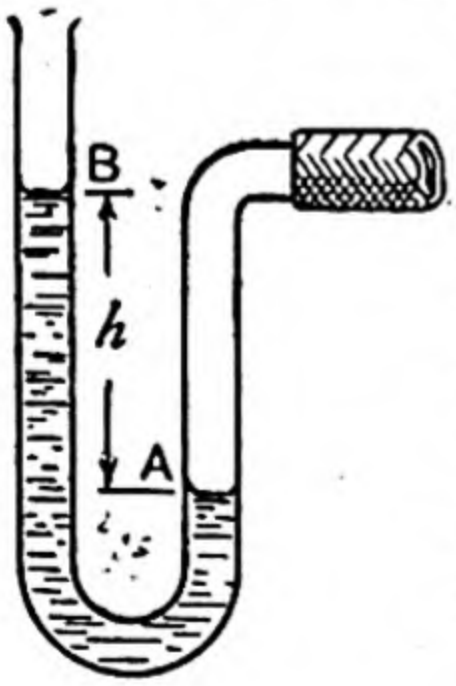
$$\text{pressure} = \text{vertical depth} \times \text{density} \quad \text{gm. wt./sq. cm.,}$$

or,

$$\text{pressure} = \text{vertical depth} \times \text{density} \times g \quad \text{dynes/sq. cm.}$$

Usually, the free surface of a liquid is exposed to atmospheric or other external pressure ; if the external pressure on the free surface is P gm. wt. per sq. cm., then the *total* pressure at depth h cm. is $(P + h\rho)$ gm. wt. per sq. cm., or $(P + h\rho)g$ dynes per sq. cm.

Liquid pressure gauges. The diagram illustrates a simple manometer or pressure gauge, of the type used to measure the pressure of the laboratory gas supply. As the pressure at the same level in the same liquid is always the same, the pressure at level A is the same in both limbs. Thus the pressure of the gas supply is balanced by the pressure of a vertical column AB of liquid, plus the external atmospheric pressure on the surface B. The limbs need not be of equal diameter, nor need they actually be set vertically, provided that the *vertical difference* of height between A and B is measured. If this is h cm., and the density of the liquid ρ gm. per c.c., the pressure of the gas

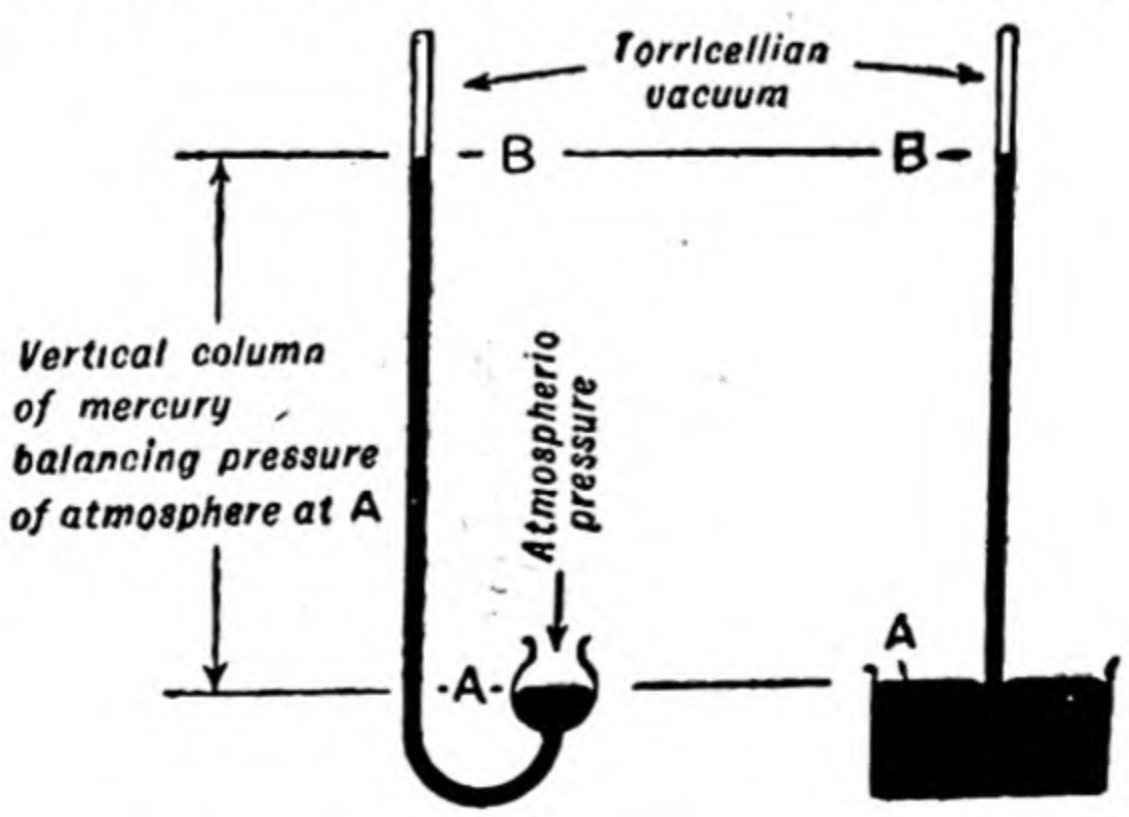


in the right-hand limb is that of the atmosphere plus $h\rho$ gm. wt. per sq. cm. (or plus $h\rho g$ dynes per sq. cm.).

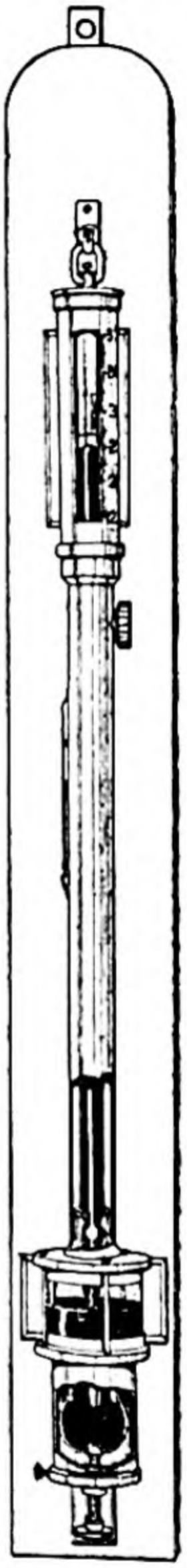
Mercury barometer. A device similar to the gauge described above is used to measure the pressure of the atmosphere itself. The top of the one limb (about 1 m. long) is sealed up and the limb filled with mercury; the mercury descends a short distance from the sealed end, and the space above the mercury is, apart from a little mercury vapour, a vacuum (known as a Torricellian vacuum after its discoverer), so that no appreciable pressure is exerted on B (see diagram below); the atmosphere presses on A; and the pressure of the vertical column AB must thus balance that of the atmosphere.

For many purposes, it is not necessary to convert the observed vertical height (or "head") into the usual pressure units; since the pressure of each centimetre of a liquid of density ρ gm. per c.c. is ρ gm. wt. per sq. cm., or ρg dynes per sq. cm., it suffices to state a reading as "so many centimetres of water" or "so many centimetres of mercury".

A simple mercury barometer can be made as follows: a straight glass tube, sealed at one end, of moderately wide bore and about one metre long, is carefully cleaned and dried. It is filled to within about a centimetre of the open end, then the thumb is placed over this end and the tube inverted slowly several times to remove air bubbles that may be clinging to the side. It is then completely filled, and, with the thumb placed over the end, inverted in a vessel of mercury; the thumb is removed, and the mercury drops leaving



a space at the top (the Torricellian vacuum). The upright tube is held in a firm support. The vertical height AB in the diagram is that of the column of mercury supported by the pressure of the atmosphere on the free surface of the mercury in the vessel. This is usually round about 76 cm., when the pressure of the atmosphere is



76 cm. of mercury ;
or $76 \times 13{\cdot}6$ gm. wt. per sq. cm. ;
or $76 \times 13{\cdot}6 \times 981$ dynes per sq. cm. ;
or $\dfrac{76 \times 13{\cdot}6 \times 981}{1000}$ millibars.

It is difficult to make accurate measurements with this simple barometer ; one of the main reasons is that a fixed scale cannot be used, since changes in the height of the mercury column cause corresponding but smaller changes in the level of the free surface in the reservoir.

The Fortin barometer. In this instrument, the barometer tube is enclosed for most of its length by a tube of brass or steel, on which is engraved a fixed scale. Only the region of possible readings on this scale is actually divided, and the zero of the scale is an ivory point. The base of the mercury reservoir can be raised or lowered by a screw which moves a false bottom consisting of a washleather bag. Moving over the scale is a vernier, with a non-parallax edge, so that readings can be taken to the nearest 1/20 mm., or the nearest 1/500 in.

In use, the reservoir is first adjusted until the mercury level just touches the ivory point. The vernier is then brought down so that the edge is level with the top of the mercury meniscus, and the reading taken. A thermometer by the side of the tube is also read, for correction has to be made for the expansion of the scale (which is only correct at the temperature at which it was graduated), and also for the change in density of the mercury with temperature (since 1 cm. of mercury only denotes a definite pressure in gm. wt. per sq. cm. if the density can be specified). For an accurate reading

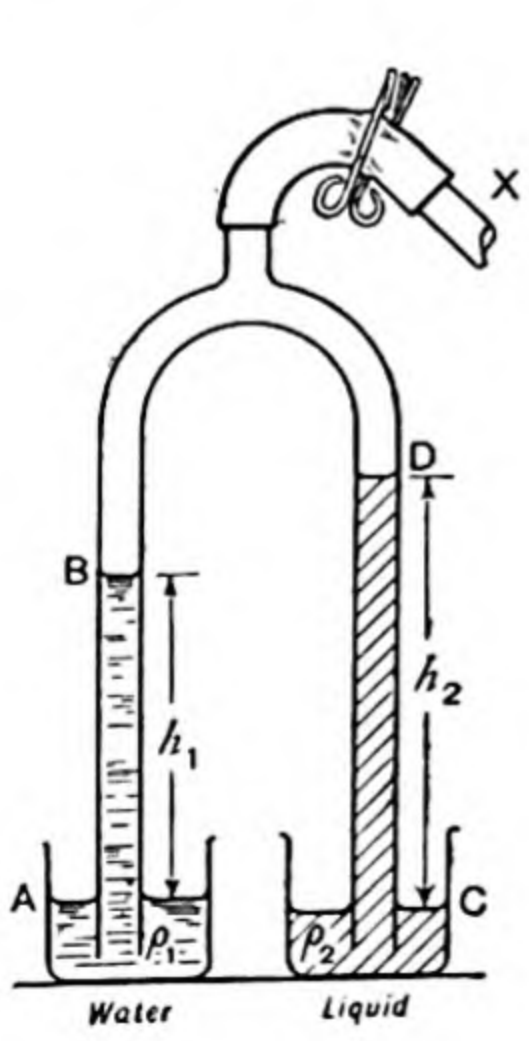
in dynes per square centimetre, the true local value of g must be used. A further correction is needed if the result is wanted for comparison with other observations elsewhere, such as the construction of meteorological charts. As the density of mercury is 13·6 gm. per c.c., and that of air about 0·00126 gm. per c.c. under ordinary conditions, a column of 1 mm. of mercury gives a pressure equivalent to a column of 1100 cm. of air, so that a considerable allowance for the height of the instrument above the chosen datum line is needed. The barometer can be read to 1/20 mm., which is equivalent to only 55 cm. of air, so all the accurate adjustment and careful use of the vernier are wasted if this last correction is not carefully applied. But, as stated above, this is only necessary when "sea-level pressures" are being collated for meteorological purposes.

Pressure method for specific gravity of liquid : Hare's Apparatus. The two separate limbs of an inverted U-tube dip into containers of water and the liquid under investigation (see diagram). By sucking at the outlet X, the pressure in the tube is reduced, and the two columns AB and CD both exert the same pressure—which is equal to the difference between that of the atmosphere outside and the reduced pressure in the tube. The vertical heights h_1 cm. of water and h_2 cm. of liquid are measured ; if ρ_1 gm. per c.c. be the density of the water, and ρ_2 gm. per c.c. that of the liquid, then

$$\text{pressure of liquid} = \text{pressure of water},$$

$$h_2\rho_2 = h_1\rho_1,$$

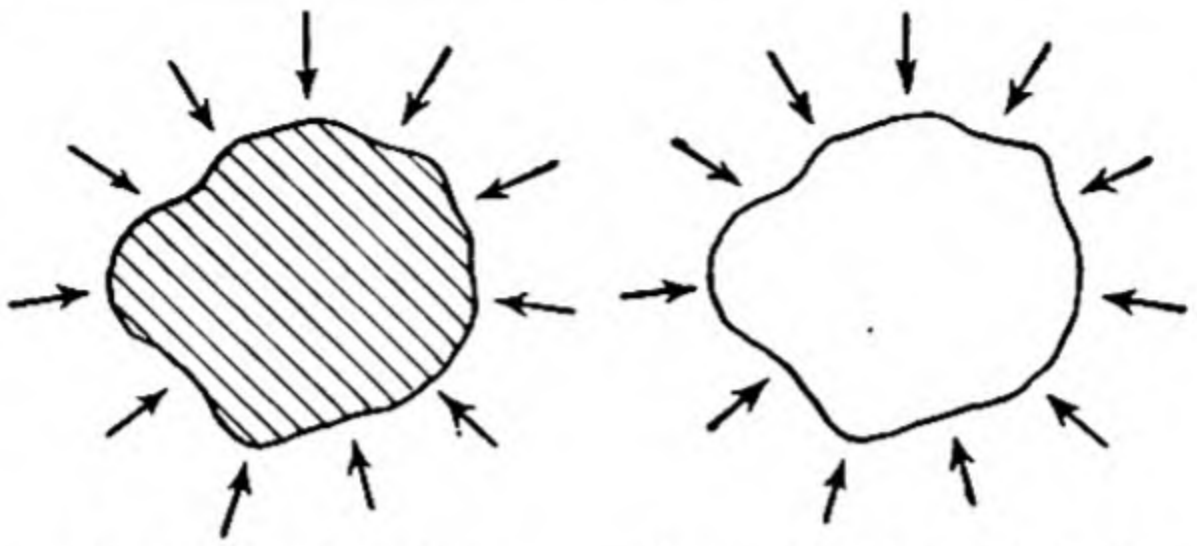
$$\therefore \frac{\rho_2}{\rho_1} = \frac{h_1}{h_2},$$



so the specific gravity of the liquid, which is

$$\frac{\text{density of liquid}}{\text{density of water}}, \text{ is equal to } \frac{h_1}{h_2}.$$

Archimedes' Principle. Because there is a vertical pressure gradient in a fluid at rest, a solid immersed therein is subjected to a greater pressure from below than above ; on the other hand, as there is no lateral pressure gradient, the pressure is uniform on all sides at the same horizontal level. It follows that a solid im-



Forces at boundary of solid

Same forces acting on same boundary enclosing fluid, just support weight of enclosed fluid

mersed in a fluid must experience a force acting vertically upwards (an upthrust), and no horizontal force.

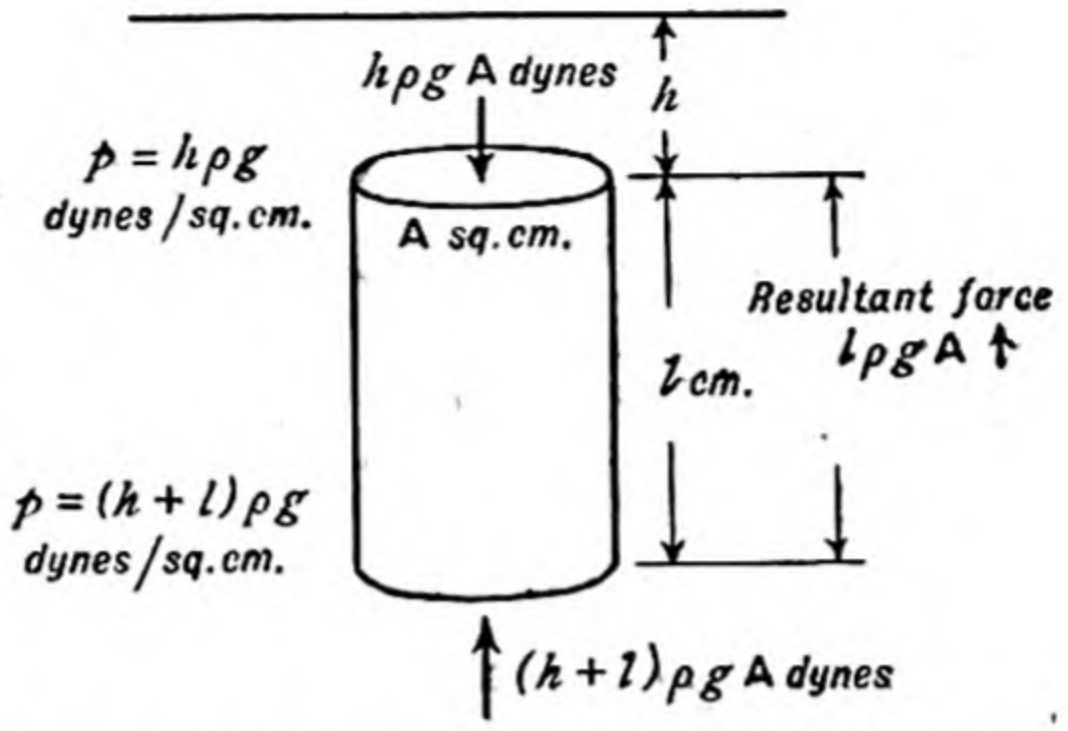
How great will this upthrust be? Consider a body of any shape immersed in a fluid (see diagram); the pressure acts from all sides on its boundary, and the upthrust is the resultant of a very large number of independent forces, which for each element of area is equal to *pressure* × *area of element* normal to the element. This resultant is determined solely by the extent and situation of each element of the boundary, the contents of which cannot influence matters. Thus if the solid is removed, and its place taken by a mass of the fluid, the same external forces acting round the same boundary now act on this fluid; but this fluid is, of course, just supported at rest. Hence, the combined forces on the boundary are just able to support the weight of the fluid that would fill the space within the boundary, or *the upthrust on the solid is exactly equal to the weight of the fluid displaced by the solid.* This important conclusion is the Principle of Archimedes, which is usually stated:

When a solid body is wholly or partially immersed in a fluid, it experiences an upthrust equal to the weight of the mass of that fluid displaced. This upthrust acts vertically through the centre of gravity of the displaced fluid.

Note: (*a*) that the term "fluid" covers both liquid and gas;

(*b*) that an upthrust is a *force*, and thus can only equal a *weight* (not an "amount" or "volume");

(*c*) the phrase "that fluid" means the fluid in which the body is immersed; if the body is immersed in brine, the upthrust



equals the weight of the brine it displaces ; if the body is immersed in alcohol, the upthrust equals the weight of the alcohol it displaces ; only *if it happens to be immersed in water* does the upthrust equal the weight of the water it displaces.

To illustrate further how the upthrust arises, consider a solid cylinder of length l cm., area of cross-section A sq. cm., immersed with its axis vertical and its upper face at a depth h cm. below the surface of a liquid of density ρ gm. per c.c. (see diagram).

The pressure on the upper face is $h\rho g$ dynes/sq. cm., and the downward thrust is $h\rho gA$ dynes ; the pressure on the lower face is $(h+l)\rho g$ dynes/sq. cm., and the upward force is $(h+l)\rho gA$ dynes ; there is no resultant horizontal thrust, so the resultant thrust of the liquid on the body is the difference between the two vertical forces on the horizontal faces, that is,

$$(h+l)\rho gA - h\rho gA = l\rho gA \text{ dynes.}$$

But lA c.c. is the volume of the solid body, $lA\rho$ gm. is the mass of liquid it displaces, and $lA\rho g$ dynes is the weight of the liquid displaced. Thus, when a body is immersed in a fluid (in this case, of course, a liquid), it experiences an upthrust equal to the weight of that fluid displaced.

To demonstrate the principle, a solid body of measured volume V c.c. (such as an aluminium cube) is weighed in air and then totally immersed in a liquid of measured density ρ gm. per c.c. The difference between the two weighings gives the upthrust, which is found to equal the weight of $V\rho$ gm. ; that is, the weight of fluid displaced. The vessel of liquid is supported clear of the balance pan, on a bridge or some other firm independent structure during weighings.

Use of Archimedes' Principle to determine specific gravities. As specific gravity of a substance may be defined as

$$\text{S.G.} = \frac{\text{mass of a certain volume of that substance}}{\text{mass of an equal volume of water}},$$

and as masses are proportional to weights, we can write

$$\text{S.G.} = \frac{\text{weight of a certain volume of substance}}{\text{weight of an equal volume of water}}.$$

If now the body is weighed, first in air, and then totally immersed in water, *in this case, the upthrust on the body is equal to the weight of an equal volume* of *water* ; whence

$$\text{S.G.} = \frac{\text{weight of body composed of that substance}}{\text{upthrust on that body when totally immersed } \textit{in water}}.$$

In applying this method experimentally, care must be taken to see that the vessel of water is supported, by means of a suitable bridge or other means, clear of the beam of the balance ; it is also important to see that the immersed body is clear of the surface of the liquid, and not touching the sides or bottom of the vessel.

A substance less dense than water can be totally immersed by using a suitable sinker, the upthrust on which is allowed for.

A substance soluble in water may be weighed completely immersed in a liquid of known specific gravity in which it is insoluble. The upthrust then gives the weight of the volume of that liquid displaced, from which the weight of the same volume of water is calculated.

EXAMPLE. A piece of rock salt weighs 108·2 gm. in air, and 48·2 gm. in saturated brine, of specific gravity 1·2. Find the specific gravity of rock salt.

Upthrust in brine $= 108{\cdot}2 - 48{\cdot}2 = 60{\cdot}0$ gm. wt. ;

Weight of brine displaced is, by Archimedes' Principle, therefore

$60{\cdot}0$ gm. wt. ;

weight of an equal volume of water is then

$60{\cdot}0/1{\cdot}2 = 50{\cdot}0$ gm. wt. ;

$$\therefore \text{ specific gravity of rock salt} = \frac{\text{weight of salt}}{\text{weight of equal volume of water}} = \frac{108{\cdot}2}{50} = 2{\cdot}16.$$

Flotation. For a body to float, whether on the surface of a liquid or totally immersed in a liquid or a gas, the upthrust on it must be able to just support its weight ; thus,

upthrust on floater = weight of floater.

But, by Archimedes' Principle, the upthrust is equal to the weight of the fluid displaced ; whence

upthrust on floater = weight of fluid displaced by floater.

From these two expressions, it can be seen that

weight of floater = weight of fluid displaced by floater, or

a floating body always displaces its own weight of the fluid that supports it.

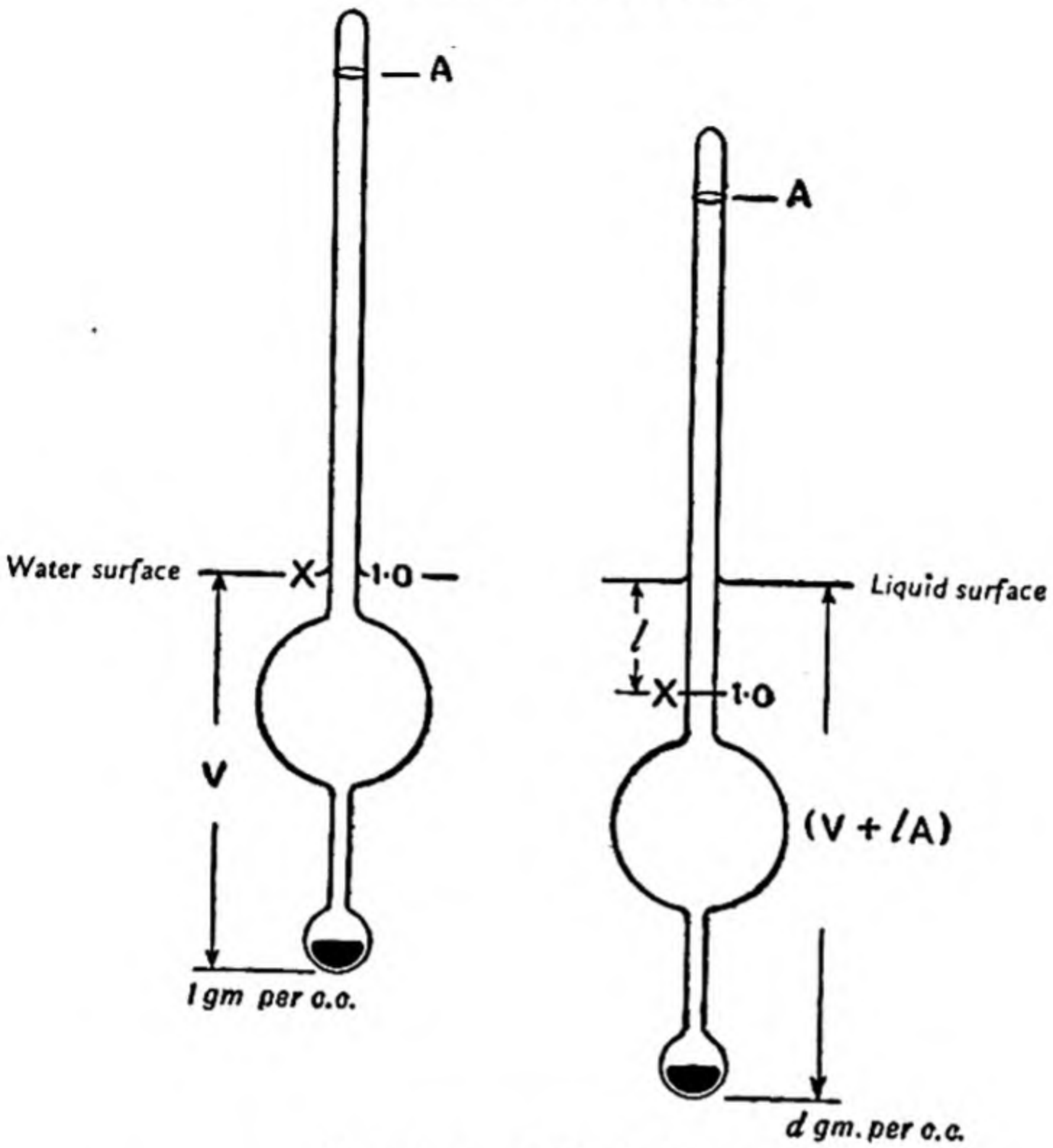
A body floating on the free surface of a liquid is in stable equilibrium for small vertical displacements ; thus, if it is further depressed, more liquid is displaced, and the upthrust, now greater than the weight, restores it to the original position.

For small rotational displacements, it is stable if the upthrust through the centre of gravity of the displaced liquid (the centre of buoyancy, B) gives, with the weight, a restoring couple, as p. 92. For a ship, this is expressed by saying that the point called the **metacentre**, at which the vertical line through B intersects the plane of symmetry of the vessel, must always be above its centre of gravity.

Constant-weight hydrometer. A floating body of mass M gm. and weight M gm. wt. experiences an upthrust of M gm. wt. whatever be the liquid in which it floats, and must thus always displace M gm. of the liquid supporting it. If it floats in a liquid of density d_1 gm. per c.c., it must sink until it displaces a volume V_1 c.c., where $M = V_1 d_1$; in a liquid of density d_2 gm. per c.c., it will displace V_2 c.c., where $M = V_2 d_2$. Thus $V_1 d_1 = V_2 d_2$, or, $\frac{d_2}{d_1} = \frac{V_1}{V_2}$. If the liquid indicated by the suffix 1 is water, then $\frac{d_2}{d_1}$ is the specific gravity of the other liquid ; so

$$\textit{Specific gravity of liquid} = \frac{\textit{volume of body submerged in water}}{\textit{volume of body submerged in liquid}}.$$

This principle is used in the simple hydrometer. The instrument has a buoyancy chamber, below which is a weighted " keel ", so that it floats upright. The part normally protruding from the liquid is made fairly narrow so that small changes in the submerged



Constant-weight hydrometer

volume are easily evident, and it bears a scale graduated directly in specific gravities.

If V c.c. be the volume up to the mark X (see diagram), where the 1·0 graduation is made at the level to which the hydrometer sinks in pure water, A sq. cm. the area of cross-section of the stem, and l cm. the distance above X of the mark to which the hydrometer sinks in a liquid of density d gm. per c.c., or specific gravity d, then when it is in water,

$$M = V \times 1 ;$$

when in the liquid,

$$M = (V + lA) \times d.$$

Hence

$$V = (V + lA) \times d,$$

and

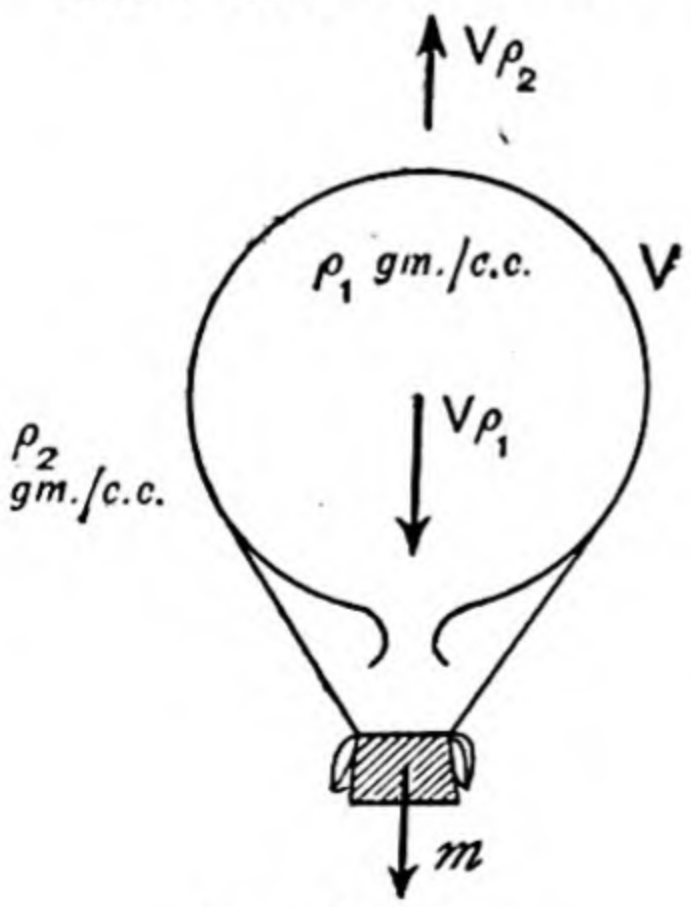
$$l = \frac{V(1-d)}{Ad}.$$

This illustrates the method of graduating such a hydrometer. Examples are best worked out from first principles, without relying on any formula.

Air buoyancy. The Archimedes' upthrust of the air on the objects around us is by no means negligible. A human being has a volume of about 60 litres, displacing 60 litres of air ; the density of air is about 1·25 gm. per litre ; so everyone experiences an upthrust of about 75 gm. wt., or nearly 3 oz. wt. If about 100 c.c. of water is weighed on a balance, counterpoising it with about 12 c.c. of brass, the upthrust on the water is about 0·12 gm., and that on the brass about 0·014 gm., the difference of about 0·11 gm. being a simple error in the weighing. This shows that, as a rule, correction for air buoyancy should be made whenever accuracy of the order of one part in a thousand is required in a weighing (see p. 190). As every litre of air displaced means 1·25 gm. wt. upthrust, very considerable upthrusts are experienced by bodies of large enough volume ; this is the principle of the balloon and other lighter-than-air craft.

Balloons. (*a*) *Open balloons :* These are fully distended at ground-level, and remain so in flight, pressure being equalised by an open vent at the bottom of the envelope, so that gas escapes as the balloon rises. The gas, hydrogen, coal gas (and sometimes helium) is less dense than air under the same conditions.

Let V c.c. be the volume of the envelope, ρ_1 gm. per c.c. the density of the gas inside, ρ_2 gm. per c.c. that of the air outside, and m gm. the mass of the envelope and fittings.



Open balloon

The mass of gas and container is $(V\rho_1 + m)$ gm., and its weight

$$(V\rho_1 + m) \text{ gm. wt.}$$

The mass of the air displaced is $V\rho_2$ gm., and the upthrust it exerts is $V\rho_2$ gm. wt.

The resultant upward force is

$$V\rho_2 - (V\rho_1 + m) = [V(\rho_2 - \rho_1) - m] \text{ gm. wt.}$$

As the balloon rises, both the external and internal pressures diminish

together, and as both the gas and the air obey the gas laws, the *ratio* ρ_2/ρ_1 remains unchanged. But the *difference* $(\rho_2 - \rho_1)$, which is the important factor determining the lift, gets steadily less and less, and the upward force steadily decreases, until it becomes zero, and the balloon reaches its " ceiling " when

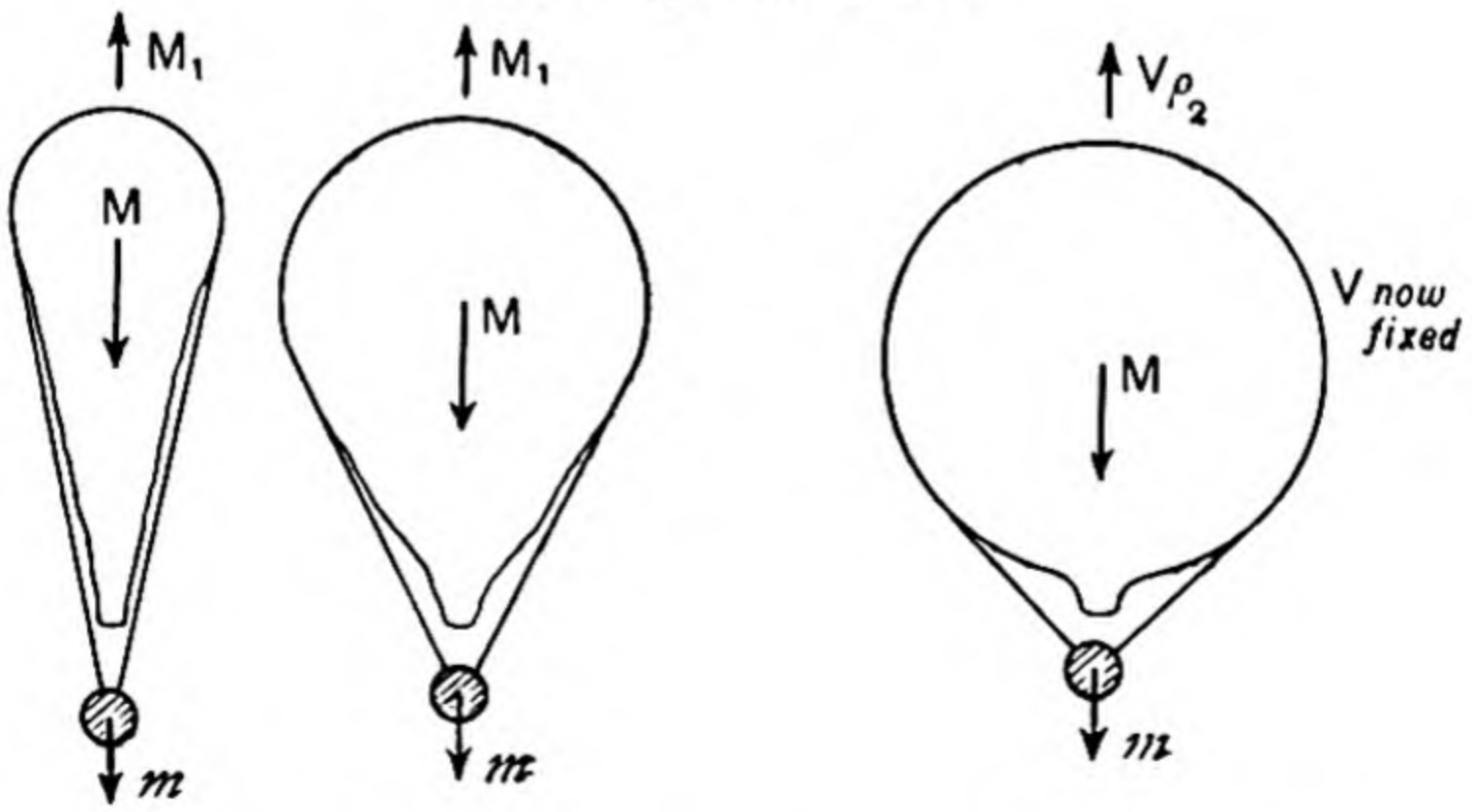
$$V(\rho_2 - \rho_1) - m = 0.$$

The early balloonists controlled height by adjustment of the factors $(\rho_2 - \rho_1)$ and m ; to rise, they reduced m by throwing out ballast, while to descend gas was released from a valve at the *top* of the envelope, its place being taken by air from below giving a denser mixture.

(*b*) *Closed balloons* : The large balloons intended to reach great heights, as used in the 1930 period for stratosphere exploration, were closed. A fixed mass M gm. of gas partially filling the envelope at ground-level, expands as the balloon rises and the external pressure decreases ; that is, V increases as the balloon rises.

Here, $V\rho_1$ is fixed, being M, and if the gas laws are followed by the surrounding air, $V\rho_2$ must also be constant, say M_1 gm., and the resultant upthrust is $[M_1 - (M - m)]$ gm. wt. at all levels until the envelope is fully distended. Thus the balloon rises with constant lift until this height is reached, after which as the volume can no longer increase, but remains V c.c., $V\rho_2$ steadily decreases with increasing altitude, the ceiling being reached when

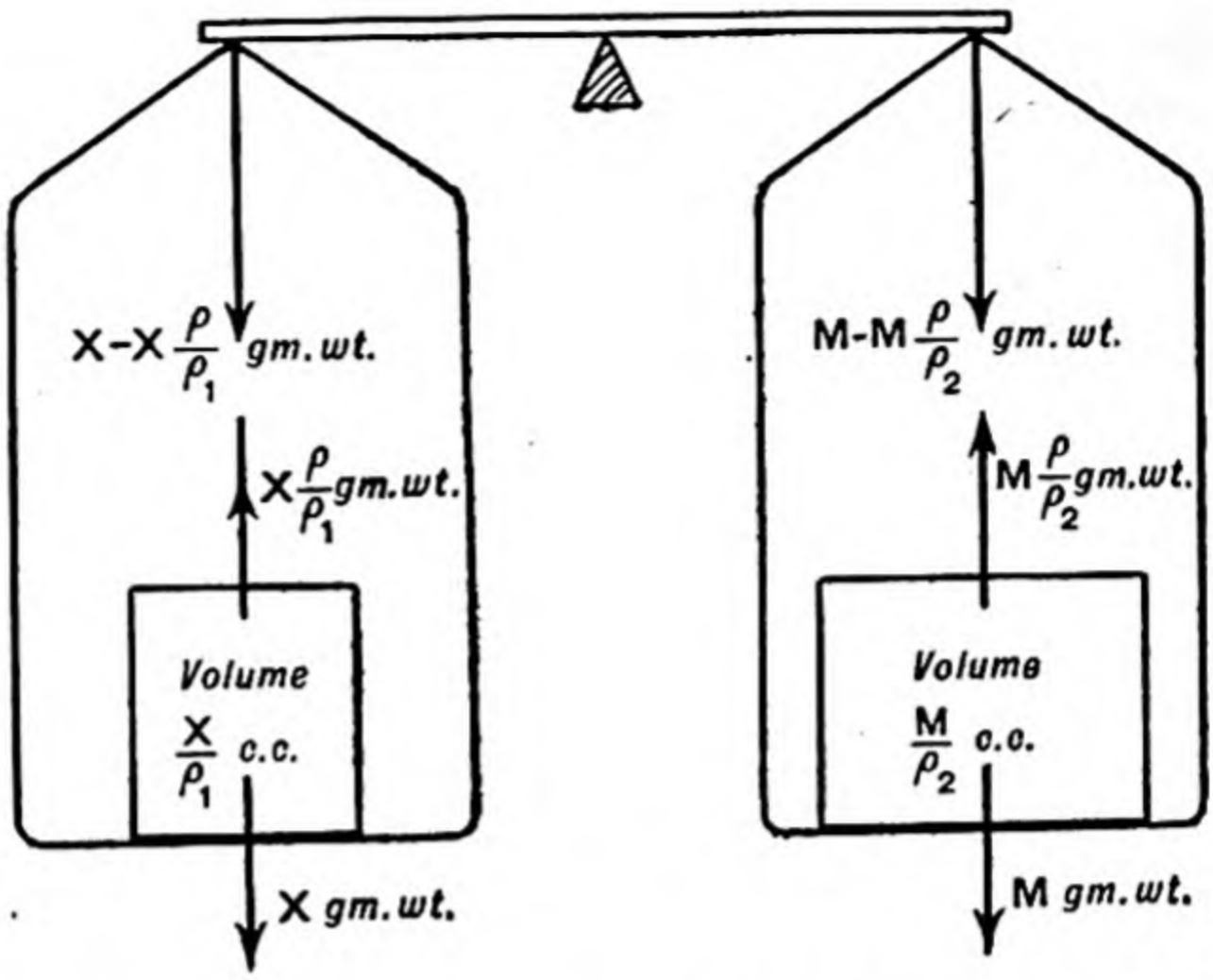
$$V\rho_2 - (M + m) = 0.$$



Constant upthrust

Constant volume

Closed balloon ascending



Buoyancy correction in weighing. Let X gm. be the true mass of a body of density ρ_1 gm. per c.c., counterpoised by M gm. of "weights" of density ρ_2 gm. per c.c. in air of density ρ gm. per c.c. (see diagram).

The volumes of the body and the counterpoise are

$$X/\rho_1 \quad \text{and} \quad M/\rho_2 \text{ c.c.},$$

the masses of air displaced,

$$(X/\rho_1)\,\rho \quad \text{and} \quad (M/\rho_2)\,\rho \text{ gm.},$$

and the upthrusts,

$$(X/\rho_1)\,\rho \quad \text{and} \quad (M/\rho_2)\,\rho \text{ gm. wt.}$$

The forces on the two sides of the balance arms are then

$$(X - X\rho/\rho_1) \quad \text{and} \quad (M - M\rho/\rho_2) \text{ gm. wt.},$$

so

$$X(1 - \rho/\rho_1) = M(1 - \rho/\rho_2).$$

Whence

$$X = M\,\frac{(1 - \rho/\rho_2)}{(1 - \rho/\rho_1)}$$

gives the true mass of the body in grams.

EXAMPLES ON CHAPTER 7

1. Define *pressure at a point* in a liquid, and show that, for a liquid at rest, the pressure is the same in all directions.

Explain why the force exerted by a liquid at rest against a flat surface in contact with it must be perpendicular to the surface.

In a hydraulic press, the diameter of the larger plunger is 30 cm. and the diameter of the pump plunger is 5 cm. Calculate the force exerted by the ram when a force of 2 kgm. wt. is applied to the smaller plunger. (O. & C.)

2. Define *pressure at a point* in a liquid and derive a formula relating it to the density of the liquid and the depth of the point below the surface.

Calculate the total thrust on a square plate of side 10 cm. immersed horizontally 15 cm. below the surface of a liquid of density 0·8 gm. per c.c. (O. & C.)

3. Define *pressure at a point* in a fluid. In what units is it measured?

Find the thrust exerted on the bottom of a vessel of uniform cross-section 50 sq. cm. by water which fills it to a depth of 20 cm. Find the additional thrust on the bottom of the vessel when a cubical block of wood of volume 50 c.c. and specific gravity 0·5 floats freely on the surface. (O. & C.)

4. What do you understand by *the pressure at a point* in a fluid?

Describe the *Bramah press*, and explain the principles of its action.

The diameters of the two plungers of a simple hydraulic press are 6 inches and 1 inch respectively; the effort is applied to the longer arm of a lever whose other arm moves the smaller plunger, the ratio of the lengths of the lever arms being 10 : 1. What force is exerted by the larger plunger when the effort applied to the lever is 20 lb. wt.? (O.)

5. What do you understand by the statement that the normal atmospheric pressure is 760 mm. of mercury?

Calculate the value of this pressure in gm. wt. per sq. cm.

Find the value of the atmospheric pressure on the top of a mountain 1,000 metres high at a time when normal atmospheric pressure is registered at a near-by station at sea-level.

Also calculate how much the level of a mercury barometer will fall when it is taken from sea-level to the mountain-top.

(Take the density of mercury as 13·6 gm./c.c., and the average density of the air as 0·0012 gm./c.c.) (O.)

6. Describe the Fortin barometer, and explain how you would proceed to read the value of the atmospheric pressure accurately from it.

An accurate barometer enables the atmospheric pressure to be read to the nearest 1/500 in. of mercury. What is the smallest change in altitude that can be detected by this barometer?

(Density of air, 0·00129 gm./c.c.; of mercury, 13·6 gm./c.c.) (O.)

7. Distinguish between *density* and *specific gravity*.

A hollow copper sphere of radius 1 cm. (specific gravity of copper 8·9) floats in water with half its volume immersed. Neglecting the mass of the air enclosed, find the volume of the cavity. (O. & C.)

8. Define *pressure at a point* in a liquid, and show how the resultant vertical thrust on a horizontal surface immersed in it can be found.

An ivory sphere (radius 1 cm.) is suspended from a balance arm by a light thread and is completely immersed in a beaker of turpentine (S.G. = 0·87). Its apparent weight in turpentine is 4·3 gm. Calculate (*a*) specific gravity of ivory, (*b*) weight of sphere in air, (*c*) tension in thread when sphere is in turpentine. (O. & C.)

9. Define the term *specific gravity*.

How many grams of salt (S.G. = 2·2) are present in 100 c.c. of brine of specific gravity 1·04? The volume of the solution may be assumed to equal the sum of the volumes of the salt and the water.

A body (S.G. = 0·94) is fastened by a thread to the bottom of a vessel containing brine of specific gravity 1·04, so that the body is completely immersed in the liquid. Given that the tension in the thread is 2 gm. wt., find the volume of the body. (O. & C.)

10. State the Principle of Archimedes.

How can it be applied to the determination of the specific gravity of a metal?

A cork of specific gravity 0·25 and mass 10 gm. floats on a liquid of specific gravity 0·9. Calculate (*a*) the fraction of the cork immersed in the liquid, (*b*) the minimum vertical force which must be applied to the cork to keep it totally immersed. (O. & C.)

11. Define *specific gravity*. Describe how you would measure the specific gravity of a liquid such as alcohol.

A cube of cork, with sides of length 5 in., floats in water with the top face horizontal and 1½ in. above the surface of the water. If it floats in a similar manner in a liquid of density 0·8 gm. per c.c., what distance will the top face be above the surface of the liquid? (O. & C.)

12. State the principle of Archimedes, and describe an experiment by which it could be verified.

A cork cylinder of cross-sectional area 20 sq. cm. and length 30 cm., is covered at one end with a layer of brass 2 cm. thick. If the whole floats in water with the metal below the surface and the axis of the cylinder vertical, what length of cork will project above the surface?

(Density of cork 0·25 gm./c.c. ; density of brass 8 gm./c.c.) (O. & C.)

13. Define *specific gravity*. Describe a common hydrometer and explain how it works.

A cubical block of density 1·5 gm. per c.c. and weighing 96 gm. in air is suspended by a string so that it is completely submerged in a liquid of specific gravity 0·75. What will be the tension in the string?

If the upper surface of the block is coated with cork of density 0·5 gm. per c.c. so that it floats with the upper surface of the cork in the surface of the liquid, what thickness of cork will be required? (O. & C.)

14. A simple form of the *Cartesian Diver*, is made of an inverted test-tube containing a suitable amount of air, immersed in a long jar of water which is tightly covered by a piece of sheet rubber *A*. The

tube is made to sink and rise at will by depressing and releasing *A*. Explain as fully as you can why this happens, and state each of the physical principles involved. (O.)

15. State Archimedes' Principle, and show how it follows as a consequence that a floating body displaces its own weight of fluid.

Give a good diagram of the common hydrometer, such as would be used for liquids of specific gravities between 0·8 and 1·0. Explain how it is used and why it works. (O.)

16. The volume of a diver encased in his diving suit is 4 cu. ft., and his total mass, including the suit, is 210 lb. Find the mass of lead which must be attached to his feet so that he may just sink in sea water.

(Density of fresh water, 62·5 lb. per cu. ft. ; specific gravity of sea water, 1·03 ; specific gravity of lead, 11·4). (O.)

17. How would you show experimentally that when a body is suspended in a liquid the upthrust on the body is equal to the downthrust of the body on the liquid?

An object of mass 900 kgm. and mean specific gravity 0·90 is held completely immersed in the sea by a cable which anchors it to the sea-bed. Calculate the upthrust on the object. (Sp. gr. of sea-water = 1·025.) If a current in the sea causes the cable to make an angle of 30° with the vertical, find the tension in the cable. (N.)

18. Distinguish between *fluid pressure* and *fluid thrust*.

Describe and explain the action of some form of hydraulic press.

The two cylinders of such a press have diameters of 3 in. and 3 ft. respectively. What is the velocity ratio of the machine? If a force of 10 lb. wt. on the smaller piston causes the larger one to exert a force of 1,350 lb. wt., what is the efficiency? (O. & C.)

19. Describe how the density of hydrogen at N.T.P. can be determined accurately.

A balloon of capacity 1000 cubic metres is filled with hydrogen and the weight of the balloon and accessories is 150 kilograms. If the balloon is moored by a steel wire of diameter 0·6 cm., what is its ceiling?

(Density of air, 1·3 ; of hydrogen, 0·09 gram per litre. Density of steel, 7·8 grams per c.c.) (O. S.)

20. A piece of brass of specific gravity 8·4 is attached by a light thread to a lump of wax of specific gravity 0·92, and the system is found to float totally immersed in salt solution of specific gravity 1·15. If the mass of the wax is 10 gm., what is (*a*) the mass of the brass, (*b*) the tension in the connecting thread? (O. & C.)

21. State the principle of Archimedes, and describe precisely its application to the determination of the density of (*a*) a solid, (*b*) a liquid.

A cylinder of density σ, length l, and cross-section A floats with its axis vertical in a liquid of density ρ. The cylinder is pushed vertically down until it is just immersed. Find the work done when the liquid is contained in a very large vessel, and also when it is contained

in a cylindrical vessel of cross-section B, comparable in magnitude to A. (O. & C.)

22. A piece of wax is found to float in water with one-twelfth of its volume above the surface. A block of the same wax, of mass 100 gm., has a lump of metal embedded in it of specific gravity 2·75. It is found that the system floats in water just totally immersed. Find (*a*) the specific gravity of the wax, (*b*) the mass of the lump of metal. (O. & C.)

23. Describe how you would determine the density of air at laboratory temperature and pressure.

A closed balloon contains 1800 kg. of hydrogen when partially inflated at sea-level, where the densities of air and hydrogen are respectively 1·29 gm. per litre and 0·09 gm. per litre. Find the greatest load it can lift ; and show that this lift does not alter until the balloon has reached such an altitude that it is fully inflated. (O.)

24. A balloon of a constant volume of 1 cubic metre is filled with hydrogen and is provided with an escape valve so that as the balloon rises the pressure inside and outside is equalised. The weight of the balloon itself is 30 grams. On the assumption that the atmosphere is isothermal at a temperature of 15° C., what will the pressure (in cm. of mercury) be when the balloon ceases to rise? (Densities of air and hydrogen at S.T.P. are 0·00129 and 0·00009 gm. per c.c. respectively.) (O. & C.)

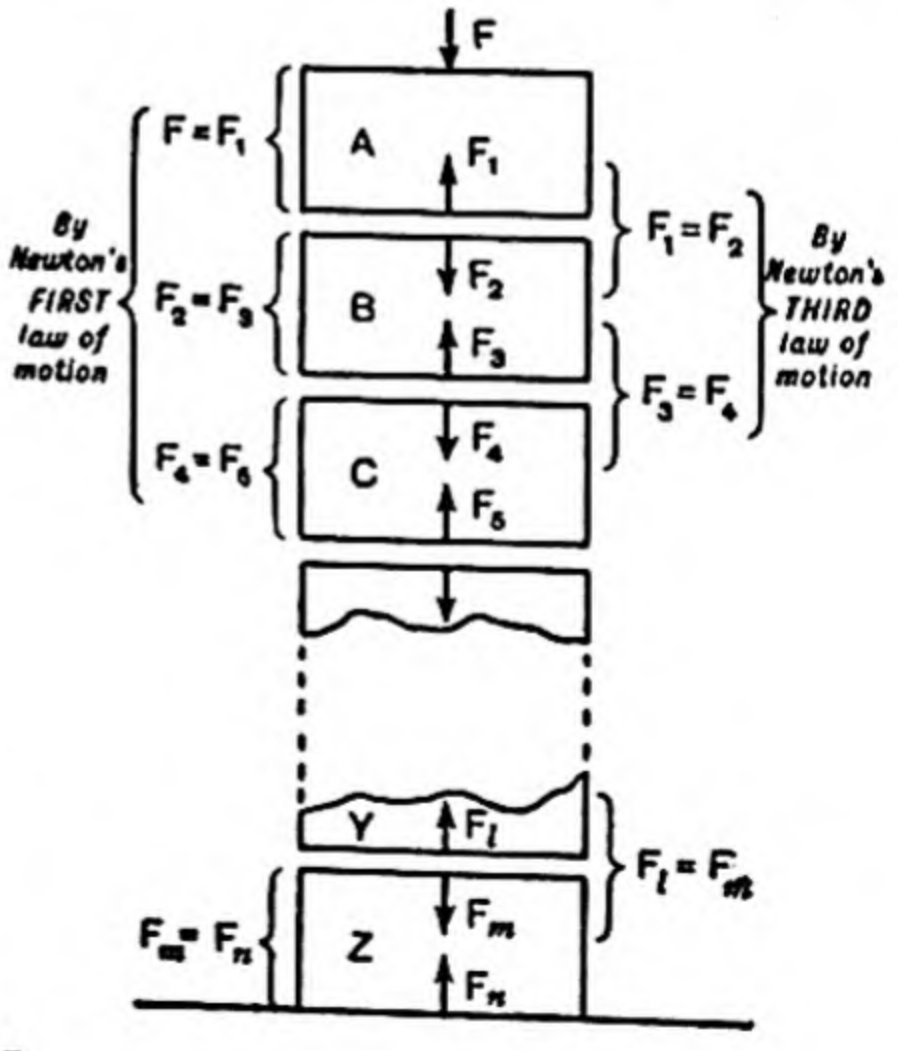
CHAPTER 8

STRESSES AND STRAINS

Properties of Solids

When an external force is applied to a solid body so constrained that it cannot move, the constraint exerts an equal and opposite external force applied at some quite different part of the body. The two forces are not the "action and reaction" of Newton's Third Law, since both act on the *same* body and they are only equal because the body is not accelerating; this point has been discussed in Chapter 3. The Laws of Motion will be applied to *portions* of the body now that we wish to consider what is happening inside the body itself.

Suppose the body is a solid pillar loaded from above with a force F, and supported from below by an equal force F_n. Imagine it cut into sections A, B, C.... The section A is in equilibrium under the applied external force F acting down, and the reaction F_1 of B acting upwards; B experiences a force F_2 (which is the "Third Law" reaction to F_1) downwards, and the reaction of C on it, F_3, acting upwards, and $F_2 = F_3$ since B is at rest. Proceeding down the pillar to the lowest section Z, this experiences a force F_m from above which is the reaction of Y on it, and an external force F_n which is equal and opposite to F_m, since Z is at rest; while the support experiences an external force which is the "Third Law" reaction to F_n. The important point is, that each section experiences a force from above and another from below, tending to compress the section. The



subdivision can be carried further, until we obtain "sections" of molecular dimensions, and it is evident that the external forces, though applied at extreme points on the body, are really having an effect throughout the whole of the body.

In a colloidal solid (as with a fluid) the ultimate particles for this purpose are the molecules, while in a crystalline solid they are electrically charged atoms called ions. The fact that a solid holds together at all is evidence of attractive forces (forces of cohesion) between the particles, and these internal forces must be *large* compared with the external forces applied in ordinary experiments, which rarely aim at destroying the specimen or the apparatus. The particles vibrate about their fixed stations, the energy of vibration constituting the heat content of the body; but thermal vibrations play little part in the stress-behaviour of solids as discussed here.

We must suppose that a solid body preserves its size and shape under ordinary conditions, because the internal forces between the particles, probably electrical in origin, and certainly depending on the spacing of the particles, are balancing in equilibrium when the distances between the particles are such as to give just that size and shape. Indeed, the characteristic crystal shape of pure crystalline substances is determined by the interplay of these internal forces, which range the ions in a regular three-dimensional pattern called a space-lattice. For each pattern there are certain directions in which lie planes relatively densely populated with ions, and the forces of cohesion within such a plane are greater than the forces between adjacent planes, so that crystals cleave, or in certain circumstances yield with slipping, parallel to these directions. A simple analogy is a "book" of postage stamps, which cleaves more readily as a rule along the perforation lines and between adjacent pages than in other directions.

Additional internal forces arising from externally applied forces disturb this equilibrium, which is readjusted by an alteration of the distances between the particles, and an alteration of size and shape, and so external forces always produce some deformation of the body to which they are applied. But if the deformation has not been so great as to alter the arrangement of the particles, removal of the external forces should leave everything as it was before, with the original shape regained under the action of the internal forces alone.

Matters are far too complex to enable us to write down any relation between the spacing Q of the particles and the internal force P between adjacent ones, but we can nevertheless forecast what may be observed if the external applied forces are themselves so small that they can cause only a small change δP in P. For, if we imagine that a graph of Q against P could be plotted, a sufficiently short portion of this could be regarded as very nearly straight. So whatever the form of the relation between Q and P, the small change δQ in Q resulting from a small change δP in P should be very nearly proportional to δP; and we may expect therefore that the observed deformations should be proportional to the applied external forces *if both are small.*

Simple observations show that this deformation depends on the force per unit area rather than on the force alone. For example, a given load will extend a thin wire more than a thick one, because in the former case the area of cross-section is less and the force per unit area greater. This must be because the deformation results from the adjustment of the internal forces between individual particles; the extent of this adjustment depends on the additional forces experienced by each of the particles affected—that is, on the additional force per particle. But each particle has a definite effective area (usually of the order 10^{-16} cm.^{2}), so *the additional force per particle is proportional to the external force per unit area.*

Stress is the name given to **force per unit area, regarded as being transmitted throughout the whole material of a body at rest.**

As a body is always under stress due to the internal cohesive forces, and is usually under an external stress due to atmospheric pressure, any deliberately applied stress might be described as a *stress increment*, since it is a change in an already existing condition.

In this chapter, the term stress increment is used to denote the effect of successive additions to existing loads made in the course of an experiment, as on pp. 211 and 215.

Strain is the name given to the **change in dimension per unit dimension,** or **the deformation of a specimen of unit dimensions,** experienced by a stressed body.

The term *strain increment* will be used for the small additional strain that accompanies a stress increment.

Elasticity. In any given case, the behaviour of a body when subjected to externally applied stress depends on a variety of factors: the nature of the material, the magnitude of the stress, the previous history of the specimen, and the time for which the stress acts all being important. If the strain depends appreciably on the time, the material is obviously showing a fluid-like propensity, as it is tending to flow. Most solids will do this under great enough stress; it is a very important property, enabling metals to be hammered into flat sheets (when they are called **malleable**) and drawn into wires (when they are called **ductile**); work has to be expended irrecoverably in performing such operations.

But for small stresses, it usually happens that the deformation is independent of the time for which the stress is applied, or nearly so; in this case it may happen that the body regains its original form completely if the stress is removed, or it may be permanently deformed to some extent. **The property of recovering completely from strain when the associated stress is removed is called elasticity,** and most true solids behave in this way provided the stress and strain are not too great. The **elastic limit** of a material under a given type of stress may be defined as the maximum stress from which a body is able to regain its original dimensions when all stress is finally removed.

It is found experimentally that, for small applied stresses and strains, provided the elastic limit is not exceeded, there is a considerable range of values for which the observed strain is proportional to the applied stress producing it. That is,

$$\frac{\text{stress}}{\text{strain}} = \text{a constant}$$

for a given type of stress and strain for a given material.

This ratio is called the **modulus of elasticity** of the material for that type of deformation.

It may happen (see p. 204) that a considerable stress S is needed to initiate strain. This may be due to abnormal internal stresses resulting from previous treatment, for repeated loading and unloading within the elastic limit remove or diminish S. If the stress-strain graph is linear, we usually take the *applied stress that is producing strain* to be (applied stress $- S$); this is proportional to the strain.

The really fundamental distinction between elastic behaviour and all other possible responses to stress is the complete recovery. This means also that the work done on the body by the external forces in producing the strain is completely recoverable as work done by the body on its external constraint, having been stored as potential energy in the strained material. If the elastic limit has been exceeded there will usually be partial recovery ; in this case, part of the work done on the body is recovered, while part has been expended irrecoverably in overcoming the internal forces between the particles and producing some permanent change in their configuration. This association of elasticity and complete restoration of strain energy has already been encountered in cases of elastic impact (p. 133).

The three types of stress and strain. For the purpose of classifying the behaviour of different materials, three types of stress and strain are recognised :

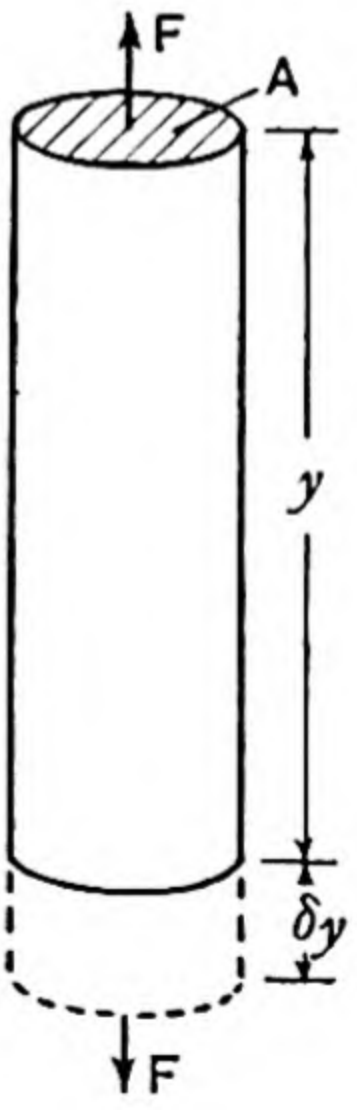
(1) LONGITUDINAL : External forces F dynes applied as in the diagram normally over an area A sq. cm. give a *longitudinal stress* of F/A dynes per sq. cm. throughout the material on which they act.

If an elongation, δy cm., is produced in a specimen of length y cm., the *longitudinal strain* is $\delta y/y$.

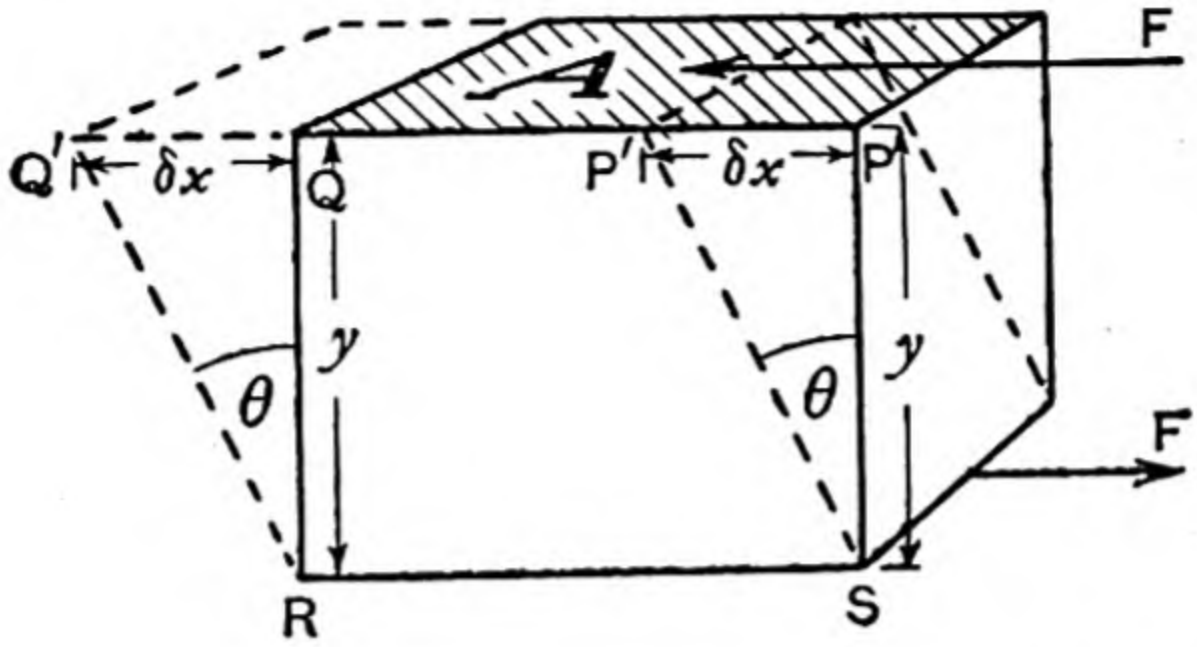
For small elastic displacements, within the region of perfect elasticity,

$$\frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \text{a constant}$$

known as **Young's modulus**, symbol Y, for the material.



(2) SHEAR : Shear or tangential forces applied in the planes of the areas concerned, as in the diagram, constitute a *shear stress*, acting through the material. Matters are really not quite as simple as this, for the two forces also constitute a couple, which must in some way be opposed by a couple of equal moment provided by the constraints if the body is not to rotate. If F dynes is the magnitude of one of



the forces, and A sq. cm. the area of the surface in the plane of which the force acts, the shear stress is F/A dynes per sq. cm. The resulting deformation alters the shape of the rectangular figure seen in elevation as $PQRS$ to $P'Q'R'S'$, displacing P to P' and Q to Q', and rotating SP and RQ through θ radians to SP' and RQ'. The *shear strain* is the lateral displacement in the direction of the stress per unit distance measured perpendicular to the stress; that is, $\delta x/y$ in the figure. As any elastic displacement is necessarily small, $\delta x/y$ is for all practical purposes the same as θ; θ is referred to either as the **shear strain**, or as the **angle of shear.**

For small shears, within the region of perfect elasticity,

$$\frac{\text{shear stress}}{\text{shear strain}} = \text{a constant}$$

called the **modulus of rigidity**, symbol n, for the material.

(3) BULK OR VOLUME. Uniform pressure, exerted equally from all sides, as in a fluid, and as experienced by a solid immersed in a fluid, constitutes a bulk or volume stress. If the pressure is p dynes per sq. cm., then the stress transmitted throughout the material of the body is p dynes per sq. cm. A change δp in the value of p causes a stress increment of δp dynes per sq. cm.

The resulting change in volume per unit volume is called a **bulk or volume strain.** Thus, if volume v c.c. is changed by δv c.c., the strain is

$$\delta v/v.$$

For small volume changes, within the region of perfect elasticity,

$$\frac{\text{pressure increment}}{\text{volume strain increment}} \text{ or } v\frac{\delta p}{\delta v} = \text{a constant}$$

called the **bulk modulus**, symbol k, for the material.

The reciprocal of k, or $\frac{1}{v}\frac{\delta v}{\delta p}$, is called the **compressibitity**. If the pressure unit used when this quantity is tabulated is the *bar*, then compressibility C per bar $= \frac{10^6}{k}$.

It is not suggested that all possible stresses and strains fall conveniently under one of these three headings; on the contrary, it is clearly impossible to extend the length of a specimen in one direction without at the same time altering its shape or its volume or both to some extent, and matters may be very complicated in most cases. But it is possible to devise experiments in which one form of strain predominates, and thus to measure the values of Y, n, and k.

All stresses have the dimensions of force/area, or $ML^{-1}T^{-2}$, and all are measured in absolute C.G.S. units in dynes per sq. cm.

All strains are dimensionless ratios between like quantities.

All three elastic moduli, with dimensions of stress/strain, or $ML^{-1}T^{-2}$, the same as those of stress, are also measured in dynes per sq. cm.

Poisson's ratio. A body extended by a longitudinal pull undergoes lateral contraction as a result. For small elastic displacements it is found that the ratio

$$\frac{\text{lateral contraction per unit breadth}}{\text{longitudinal elongation per unit length}} = \text{a constant}$$

called **Poisson's ratio**, symbol σ, for the material.

For a homogeneous solid which is isotropic (that is, the same in all directions) relations between σ and the elastic moduli Y, n, and k can be found: $n = \frac{Y}{2(1+\sigma)}$, and $k = \frac{Y}{3(1-2\sigma)}$. For ordinary solids σ cannot exceed $+\frac{1}{2}$, or k would be negative, and cannot be less than -1, or n would be negative.

The ideal elastic material; Hooke's Law. It usually happens that for small elastic displacements the displacement is directly proportional to the force producing it; for example, for a loaded wire, $\frac{\text{load}}{\text{extension}}$ is constant, and correspondingly for other types of deformation. This *direct proportionality* is known as **Hooke's Law**. It follows

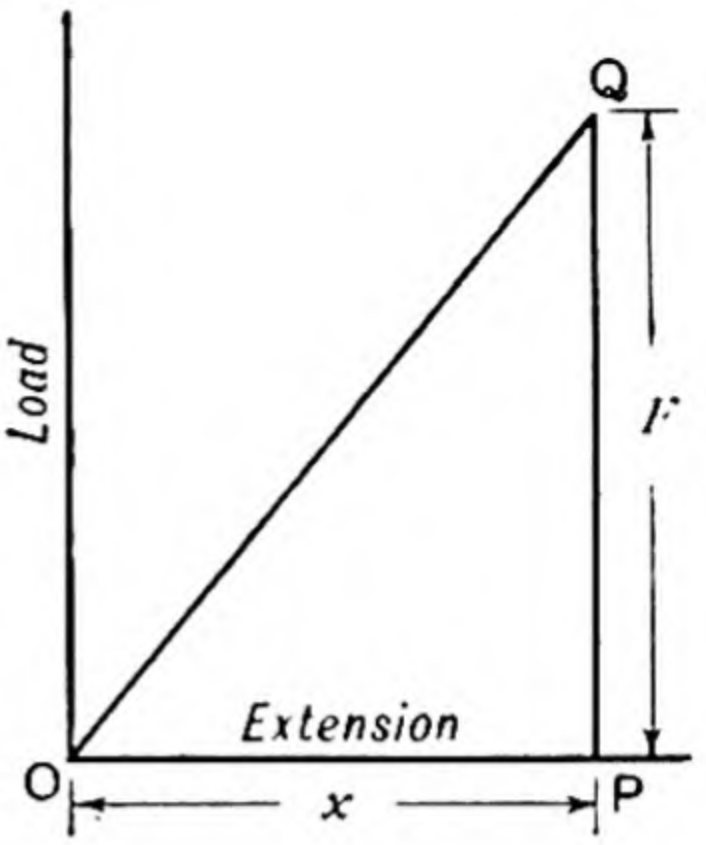
that in all such cases (if the area over which the force acts is unaltered, and the strain itself is small), stress/strain is constant.

Hooke's law is assumed in defining an elastic modulus, and also in the statical experiments to measure elastic moduli, where observations start with the specimen already under stress, so that what is actually measured is the ratio stress increment/strain increment, which must be the same as stress/strain if the stress-strain graph is a straight line through the origin. The word "modulus" is not used in cases (such as a gas) where the ratio stress increment/strain increment depends on the initial stress—"coefficient of elasticity", or "elasticity", is then the proper term. That is, elastic behaviour is something quite distinct from Hooke's law, but a modulus of elasticity only exists if Hooke's law is followed.

Hooke published the law as an anagram *ceiiinosssttuu*, and later decoded this as *ut tensio sic vis* in an essay *De Potentia Restitutiva* (1676), where he describes experiments with helical wire springs, flat watch springs, straight wires, and flexed beams. Here *tensio* means extension or elongation, and *vis* the tension or restoring force exerted *by* the elastic body rather than the load imposed upon it from without. The law states that restoring force is proportional to extension. The actual observations of extensions under given static loads he treated as the converse of the law—*ut pondus sic tensio*, or extension is proportional to weight. Indeed, Hooke states that he designed the spring balance for the express purpose of applying the law to detect changes of weight with altitude (p. 380); he was unsuccessful in this, as the altitude changes he tried (for example, climbing Westminster Abbey) were far too small.

The general nature of the law was realised by Hooke, but in his theoretical discussion he appears to have considered it as a generalisation of Boyle's law, for he supposed that a body when extended pushed back the "ambient ether", which behaved like a compressed gas and gave rise to the restoring force apparently exerted by the body. More interesting perhaps is the assumption of the Third Law of Motion, in equating *vis* exerted by the spring on the load to *pondus* exerted by the load on the spring; no doubt this principle was widely applied intuitively before Newton's formal enunciation.

Strain energy. Consider a wire of length l cm., area of cross-section A sq. cm., loaded with a tension F dynes producing an extension x cm. If Hooke's law is obeyed, the load-extension graph is a straight line, as in the diagram, whence the work done (represented by the area between the graph and the extension axis) is given by the area OPQ, or $\frac{1}{2}OP \,.\, PQ$, representing $\frac{1}{2}Fx$ ergs.



Now the stress is F/A dynes per sq. cm., and the strain x/l, so

$$F = \text{stress} \times A,$$

$$x = \text{strain} \times l,$$

and

$$\tfrac{1}{2}F \,.\, x = \tfrac{1}{2} \text{ stress} \times \text{strain} \times Al,$$

$$= \tfrac{1}{2} \text{ (stress} \times \text{strain} \times \text{volume of wire) ergs.}$$

Thus, $\frac{1}{2}$ stress × strain represents the work done per unit volume in extending the wire, or as this is elastically recoverable, $\frac{1}{2}$ stress × strain represents the potential energy in ergs stored in unit volume of the strained material.

Since, if *stress/strain* is constant, this ratio is itself Young's modulus, Y, alternative expressions for the potential energy per unit volume are :

$\frac{1}{2}$ *stress* × *strain* erg per c.c.

$\frac{1}{2}$ $Y \times (strain)^2$ erg per c.c.

$\frac{1}{2}$ $(stress)^2/Y$ erg per c.c.

The above applies in its entirety only to a wire under longitudinal tension. But if such a wire is twisted by a couple applied as in the diagram on p. 215, and the angle in radians through which the extremity of the wire is turned is found to be proportional to the couple, we again say that Hooke's law is obeyed. If C be the couple in dyne cm. required to produce a twist θ radians, then the work done is

$$\tfrac{1}{2}C\theta \text{ ergs.}$$

If α dyne cm. be the couple required to produce one radian twist, then

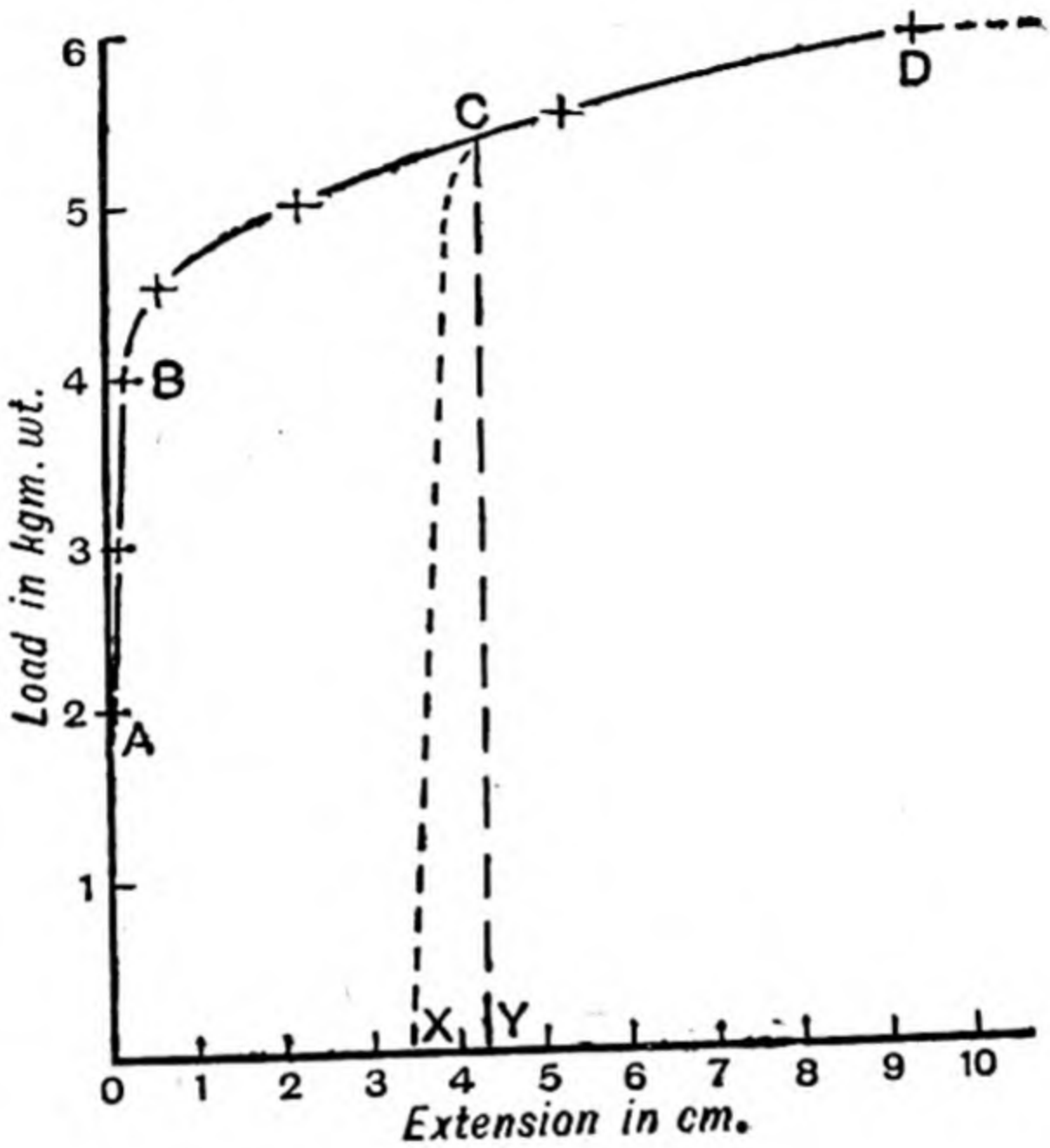
$$C = \alpha\theta,$$

and the work done in producing the twist is

$$\tfrac{1}{2}\alpha\theta^2 \text{ ergs.}$$

Actual behaviour of metal wires.

(*a*) *Copper.* A length of about 2 metres of No. 22 S.W.G. copper wire, firmly attached to an upper support, and provided with an index moving over a scale at its lower end, was loaded, with the results indicated in the graph.



With the 6 kilogram-weight load, the extension was seen to be gradually increasing as time went on ; the 10·3 cm. extension was eventually far exceeded, and the wire broke.

The graph of load against extension, as can be seen, can be divided into three parts. Over the first part, *OA*, no observable extension occurred for loads less than 2 kgm. wt. The explanation of this is by no means obvious ; it is often alleged that this kind of lag is caused by small kinks, which must be removed before the wire behaves normally, but no work is being done, as would be expected if this were true. Also, the same behaviour is observed when one has taken steps to remove all visible kinks. It seems simpler to state just that Hooke's law is *not* obeyed ; a proper explanation of the reason is probably a very difficult task.

Next, from A to B the graph is linear ; thus

$$\frac{\text{load increment}}{\text{extension increment}} \text{ is constant over this range,}$$

and so therefore is $\dfrac{\text{stress increment}}{\text{strain increment}}$.

Lastly, beyond B, each successive increase in load gives a correspondingly greater increase in extension, until somewhere about D the wire continues to extend without further addition to the load and flows plastically. The point on the graph (or more precisely, the load value) for the stage at which this plastic flow begins, is called the **yield point** for the particular wire under test.

This specimen being expended, a second piece of the same wire was subjected to smaller loads up to 4 kgm. wt. After each load had been added, and the extension recorded, the whole load was removed. It was found that the wire returned to its original length in each case. Thus, *for loads up to 4 kgm. wt., the stage represented by the point B on the graph, the wire behaves elastically.* For loads in excess of 4 kgm. wt., it was found that there was a permanent extension when the load was removed, although part of the strain disappeared. The return, when unloaded at some stage such as C, may be represented by the line CX ; OX gives the permanent extension.

Thus, for this material, it appears that the elastic property of recovering completely when the load is removed is possessed only over the stage from A to B. The stage represented by B is called the **elastic limit** for the specimen. For loads less than that for the elastic limit, the wire returns to the original length when unloaded. For loads in excess of that for the elastic limit, complete recovery does not occur, and the wire is permanently extended.

The elastic limit should not be confused with the yield point ; it can be seen here that there is a wide range of loads, in excess of that for the elastic limit, but less than that for the yield point, which can be *used* to produce a permanent extension ; this wide separation of elastic limit and yield point is a characteristic of a *ductile* material such as copper. Referring to the curve again, suppose the wire to be extended elastically to B, or any point between A and B ; the work done on the wire, represented by the area between the graph and the

extension axis, is all recovered as work done by the wire when it returns.

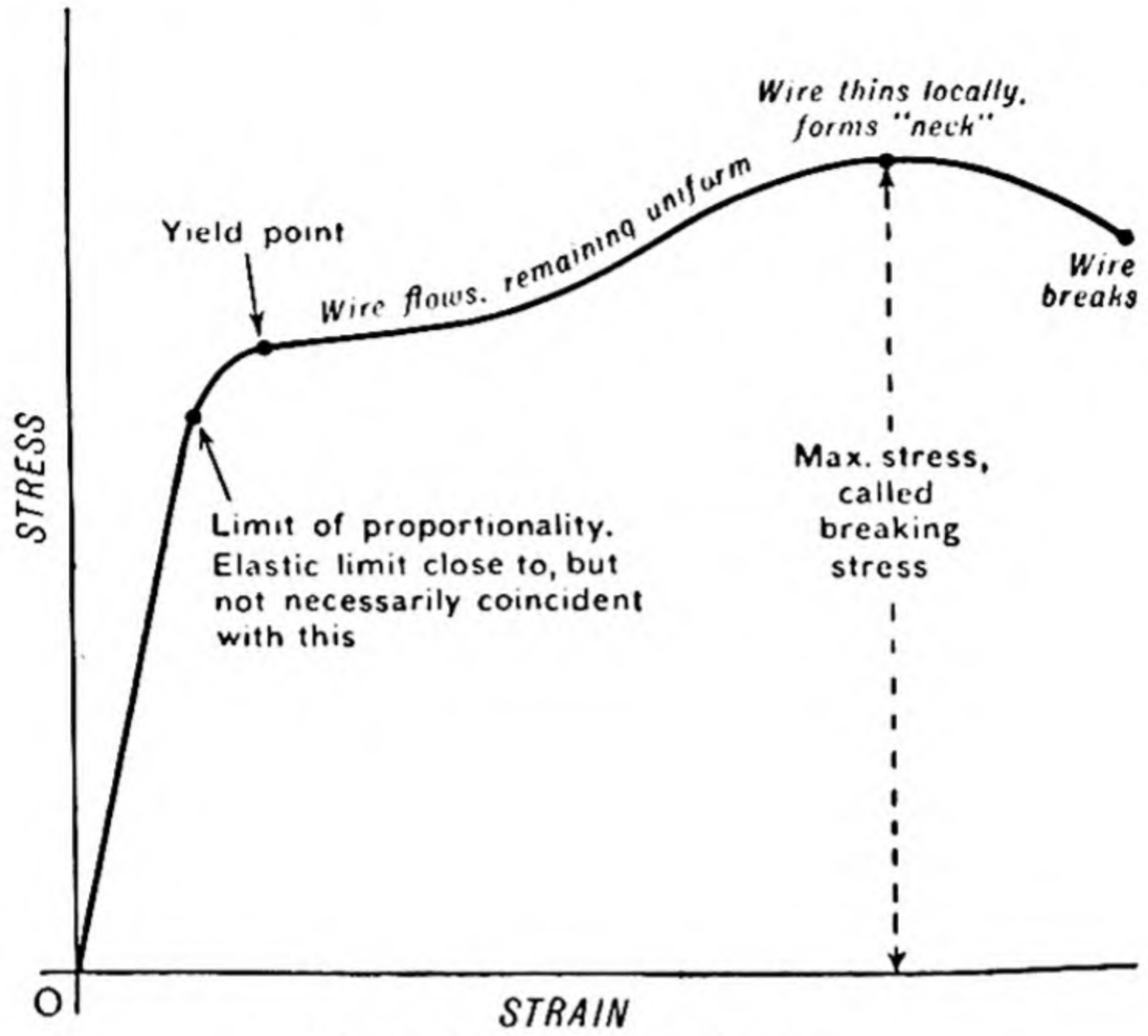
If the wire is extended to C and then unloaded, the work done on the wire is represented by the area $OABCYO$; part of this only is recoverable, namely, that represented by the area CYX beneath the line CX indicating the unloading. The remainder represents the work done in overcoming the internal forces binding the particles together, and is dissipated as heat.

(*b*) *Steel.* A longer specimen of steel piano wire, of slightly narrower gauge than the copper, suspended in the same way, gave elastic displacement for loads up to 12·5 kgm. wt.

Load in kgm. wt.	Extension in cm.
0	0·0
3·1	0·19
6·3	0·42
9·0	0·66
12·5	0·95

The reader should plot the graph of load against extension, which exhibits (though to a less extent than does the corresponding graph for copper) a lag under small loads; and although Hooke's law is very nearly obeyed, the points are best represented by a curve the greater part of which is a straight line not passing through the origin. Here no attempt was made to extend the wire to the elastic limit; the readings in fact were a preliminary set made before starting readings in a Young's modulus determination (see p. 211) in order to be sure that the elastic limit would *not* be reached in this actual experiment. The expected course of the load-extension curve with greater loads would, however, be of the same general form as that for copper, except that the elastic limit and the yield point lie rather closer together for this tougher and less ductile material.

(*c*) *General.* The various stages to be expected in the loading of a wire are illustrated in the general stress-strain diagram. For small stresses, Hooke's law should be obeyed, and the end of the linear part is called the **limit of proportionallity** beyond which the gradient of the curve decreases. Complete recovery occurs on unloading

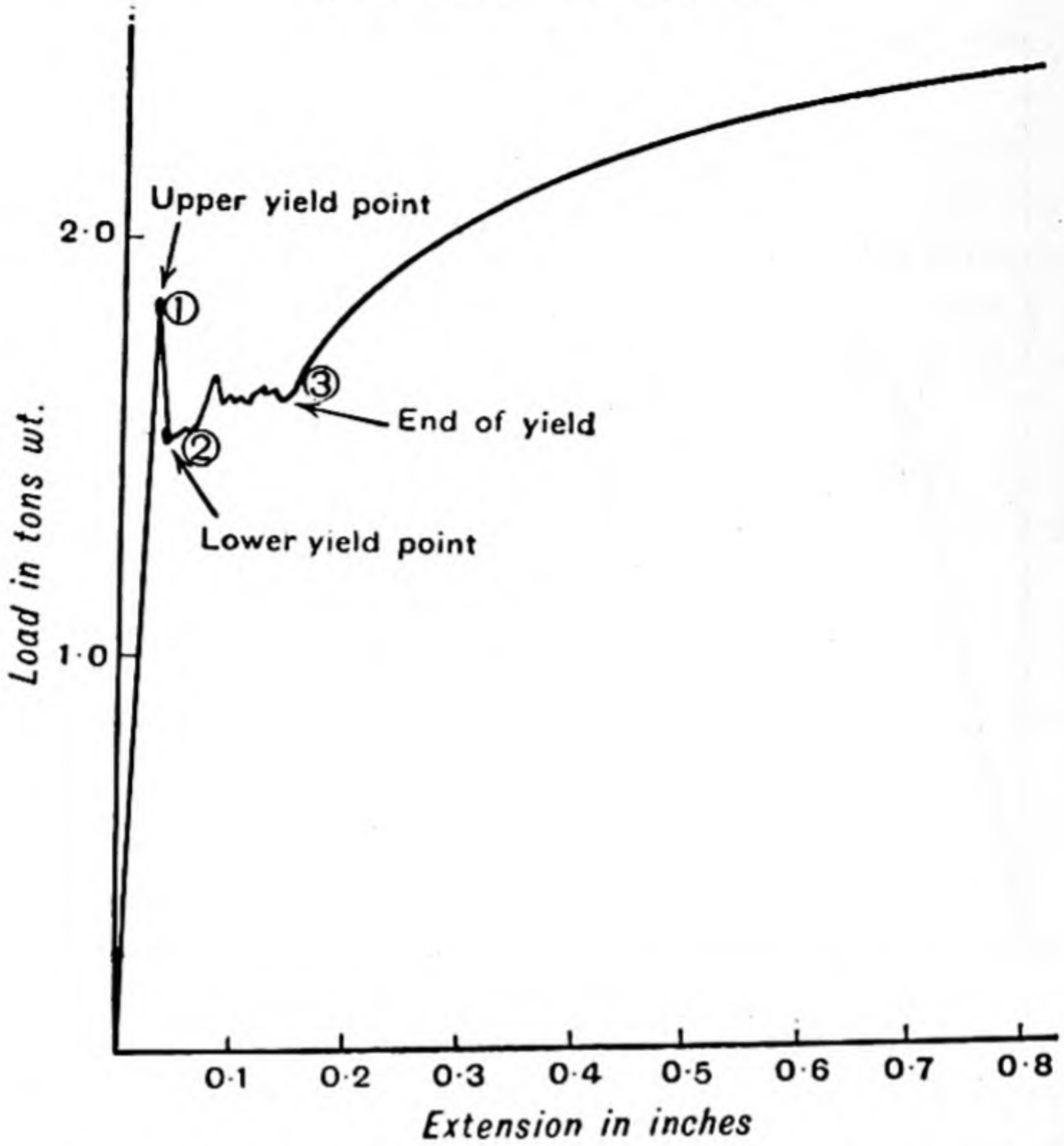


General stress-strain diagram

from a point on the linear part, while permanent extension is observed on unloading from beyond this; so the elastic limit lies close to the limit of proportionality, although the two need not necessarily coincide, since Hooke's law and the elastic behaviour usually associated with it are two quite distinct things. At the yield point, plastic flow begins and the wire flows, thinning uniformly. The stress rises to a maximum, which we will call the breaking stress (though this is usually calculated with respect to the original area of cross-section of the wire, instead of the thinned area). Then the wire thins locally, and breaks with a release of stress.

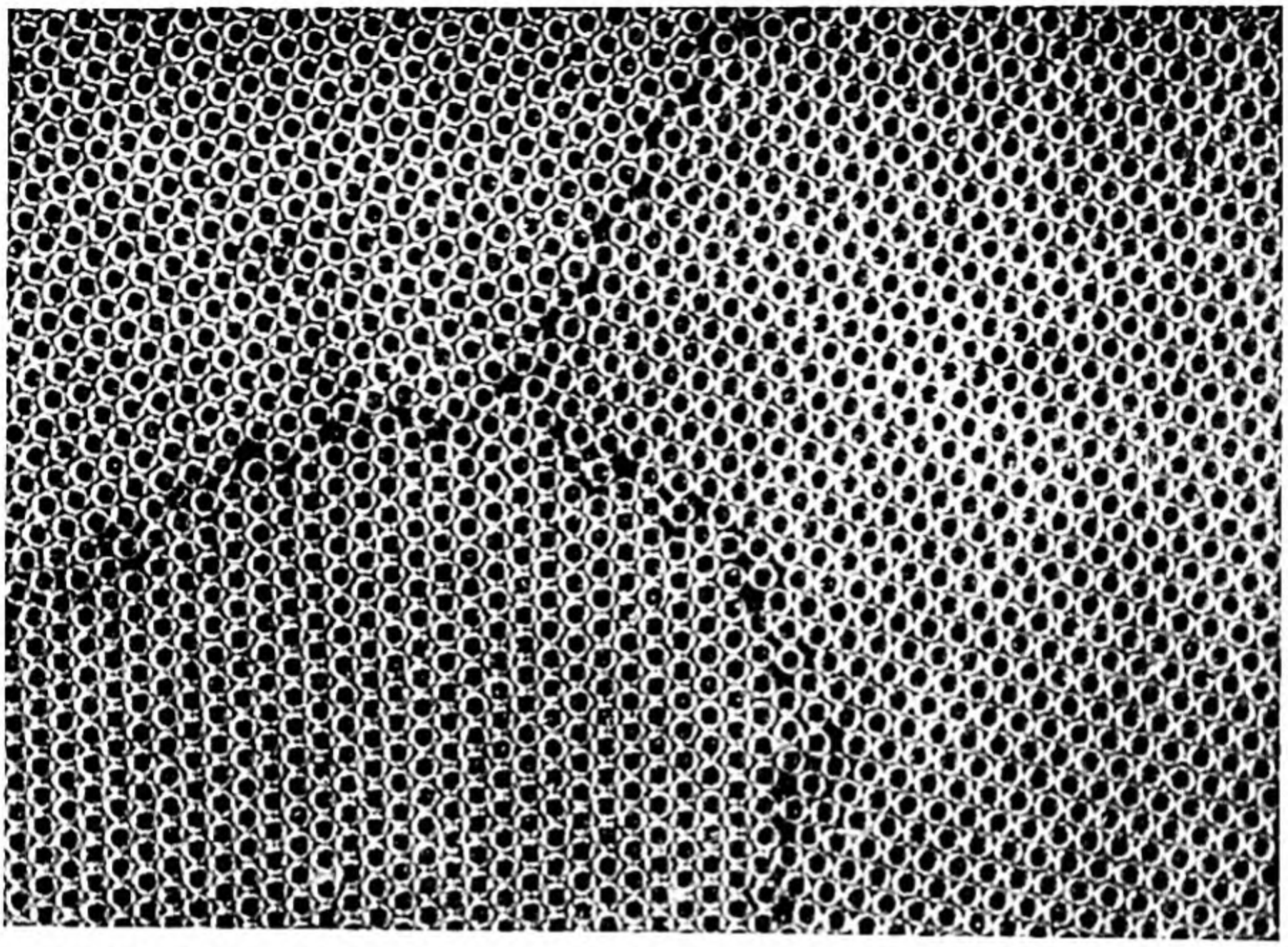
The yield. A sudden change in the behaviour of a stressed metal occurs at the yield point; and plastic flow begins. But investigations with self-recording testing apparatus show that there is a sharp drop in the load, and a plastic yield under this lowered load, before the further plastic flow which ends in fracture.

The diagram on p. 208, from a paper by C. F. Elam (*Proceedings of the Royal Society*, vol. 165, 1938), shows the behaviour of a steel



specimen 6 in. long, of diameter 0·40 in., under a continuously-applied load in a testing machine. At 1 there is a sharp drop in the load, to 2; at 3 this initial yield has finished, and further load is needed to complete the fracture.

The generally accepted picture of elastic extension is that under relatively small stress the crystal lattice of the metal is slightly distorted, but the structure is unimpaired, so that it recovers when the stress is removed. But if the stress exceeds a certain value, sliding occurs along the natural cleavage planes of the crystal, which glide over one another in much the same way as the individual volumes do whenever one tries to carry a pile of books. Before gliding occurs, the whole crystal lattice is strained; after gliding, the tangential stress at the slip planes has been partially released, and the stress and strain are reduced. This explains the fall on the load-extension graph at the start of the yield. As happens with a pile of books, it may be possible for there to be several intermediate stages



From a paper by Sir Lawrence Bragg and J. F. Nye, *Proc. Roy.* Soc., A, 190, 474(1947)
Bubble-raft with bubbles 0·76 mm. in diameter.

before the structure breaks down, and subsidiary peaks indicating this are found in the region between 2 and 3.

The pile of books may appear a very homely analogy ; a better one, in the form of a working two-dimensional model, has been devised by Sir Lawrence Bragg. This is simply a raft of small equal soap-bubbles floating on the surface of a vessel of soap solution, and ending on two parallel rods, one of which can be moved to shear the pattern, while the other records tangential pull. The mechanism of increase of strain, relaxation of strain with glide, and further increase of strain, can be clearly demonstrated. The accompanying illustration shows a portion of such a bubble-raft, in which the bubbles are uniformly distributed like the atoms in a metallic crystal, except at the boundaries of adjacent "crystals."

The splintered appearance of the edges of a fractured specimen, and the observable "stepping" on the outer surface of a single crystal that has been subjected to strain beyond the yield point, also bear out this view of the mechanism of elastic extension.

O

N.P.

Elasticity of rubber. Rubber owes its usefulness to the very low values of its elastic constants ; Y is about one fiftieth, and n about one ten thousandth, of the corresponding values for the average metal. But the effect of temperature change is even more remarkable. The elasticity of metals decreases as the temperature rises, the temperature coefficient (which is the fractional change per C. degree rise) being small, of the order 10^{-4}, somewhat greater than the coefficient of thermal expansion. The elasticity of rubber increases as the temperature rises, and is roughly proportional to the absolute temperature, so that the temperature coefficient of elasticity is approximately the same as the coefficient of thermal expansion of a gas. While, therefore, a simple statical picture (p. 209) helps us to explain the elastic behaviour of metals, something analogous to the kinetic theory of gases (p. 217), invoking moving particles with energies of thermal agitation that are proportional to the absolute temperature, seems necessary for rubber.

In natural rubber, and all synthetic materials showing rubber-like properties, the material is not crystalline in the usual sense of the word,* but is made up of molecules constructed on long chains of carbon atoms. The molecule of natural rubber, with a molecular weight of about 300,000, has for its " backbone " a chain of some 20,000 carbon atoms end to end. The general picture of the structure of a mass of rubber is a three-dimensional network of such chains, interlocking with one another at a few points along the length of each, but loose and free to swing round like a skipping rope in between these points. Thermal agitation causes the free parts to rotate in some such way. An externally applied stress has only to compete with this thermal agitation, which offers much less resistance than the forces between the atoms of a metal ; the result is that large perfectly elastic strains can be produced with relatively small stresses at low temperatures. Increasing the external stress brings an elastic limit, at which the chains are " taut ", and any further extension means that work has to be done against the attractive forces between the atoms in the molecules, so that increasing the stress beyond this causes only small increases in strain. At this stage, the material is very near its breaking point.

*Rubber will crystallise if maintained at a low temperature or under great strain, but then loses its characteristic elastic properties.

At high temperatures, the effect of thermal motion of the unattached portions of the chain will be greater, and so a smaller strain is produced for a given stress (or a greater stress is needed to give the same strain). It is found experimentally that the stress required to maintain a fixed extension is directly proportional to the absolute temperature.

This section is a brief summary of some of the main points of an article by Dr. L. R. G. Treloar in the Physical Society's *Reports on Progress in Physics* (1942–43).

Measurement of Young's modulus for the material of a wire. In making this measurement, it must be remembered that :

(*a*) the elastic limit must not be exceeded ; and

(*b*) the ratio of stress increment to strain increment for the range in which the stress-strain relation is linear is the quantity required.

The observations needed are the length l cm. of the specimen, and its diameter d cm., together with the extension increment e cm. produced by a load increment of m gm. wt., or mg dynes.

Then the stress increment is $\frac{mg}{\pi d^2/4}$ dynes per sq. cm. ; the strain increment is e/l ; and $Y = \frac{\text{stress increment}}{\text{strain increment}}$

$$= \frac{4\pi mgl}{d^2 e} \text{ dynes per sq. cm.}$$

It is easy enough to obtain a *rough* value of Y, and if we are aiming at an accurate result we shall be guided by this rough value, and also the value of the breaking stress, in planning the experiment.

Y must be of the order 10^{12} dynes per sq. cm., and the breaking stress between 10^9 and 10^{10} dynes per sq. cm.; whence the maximum strain the wire can be expected to stand without breaking will be of the order 10^{-3} to 10^{-2}, while to be sure that the elastic limit is not exceeded the strain must be considerably below the maximum. Suppose, as a guide to the order of extension to be measured, a maximum strain of 0·005 is prepared for ; then a wire of length 2 metres will extend 1·0 cm., and a vernier measuring to within 0·01 cm. will measure this with 1 per cent. accuracy.

The corresponding stress is $10^{12} \times 0{\cdot}005 = 5 \times 10^9$ dynes per sq. cm.

(about 5×10^3 kgm. wt. per sq. cm.). Suppose the maximum convenient load is 50 kgm. wt., the area of cross-section of the wire must be 10^{-2} sq. cm., or 1 sq. mm., whence the diameter will be about 1·2 mm. (about 18 S.W.G.). This diameter can be measured to about 1 per cent., using a micrometer screw gauge reading to the nearest 1/100 mm., so that the cross-section area, involving d^2, is found with an accuracy of one part in 50. One would be very content with smaller loads, in which case a thinner wire would be better, but there will be correspondingly less accuracy in measuring the diameter of the wire.

In the usual simple form of school apparatus, the wire under test is fixed to a beam in the roof, and carries a vernier securely fixed near its lower end. Suspended from the same support is another wire of the same material carrying the reference scale, and also a sufficient load to keep it taut. Yielding of the support, and thermal expansion due to temperature change, affect both wires equally and so lead to no error.

A preliminary test is first made, reading the vernier, putting on the greatest load it is proposed to have, removing the load, and reading the vernier again. If the two readings agree within one or two tenths of a millimetre, the elastic limit has not been exceeded, and the experiment can proceed. A set of readings as indicated in the following table is taken.

	Load :	Vernier readings			Mean extension increment for load increment of 8 kgm. wt.
		Loading	Unloading	Mean	
(*a*)	0				
(*b*)	2				
(*c*)	4				(*e*)–(*a*)
(*d*)	6				(*f*)–(*b*)
	.....	.....	.....	.....	.....
					(*g*)–(*c*)
(*e*)	8				(*h*)–(*d*)
(*f*)	10				
(*g*)	12				
(*h*)	14				

To work out the result in the most economical manner without drawing a graph, Allen and Moore's *Textbook of Practical Physics*

recommends dividing them into two sets—here the first four and the last four. Take readings (a) and (e), (f) and (g), etc., and from them obtain four separate readings for the extension increment for a load increment of 8 kgm. wt.

Measure the length of the wire from the upper support to the point of attachment of the vernier. Measure the diameter of the wire in several places by means of the screw gauge; measuring at different places along the wire corrects for tapering of the wire, and at each place two diameters at right angles correct for possible elliptic section.

Young's modulus is then calculated from the formula of p. 211.

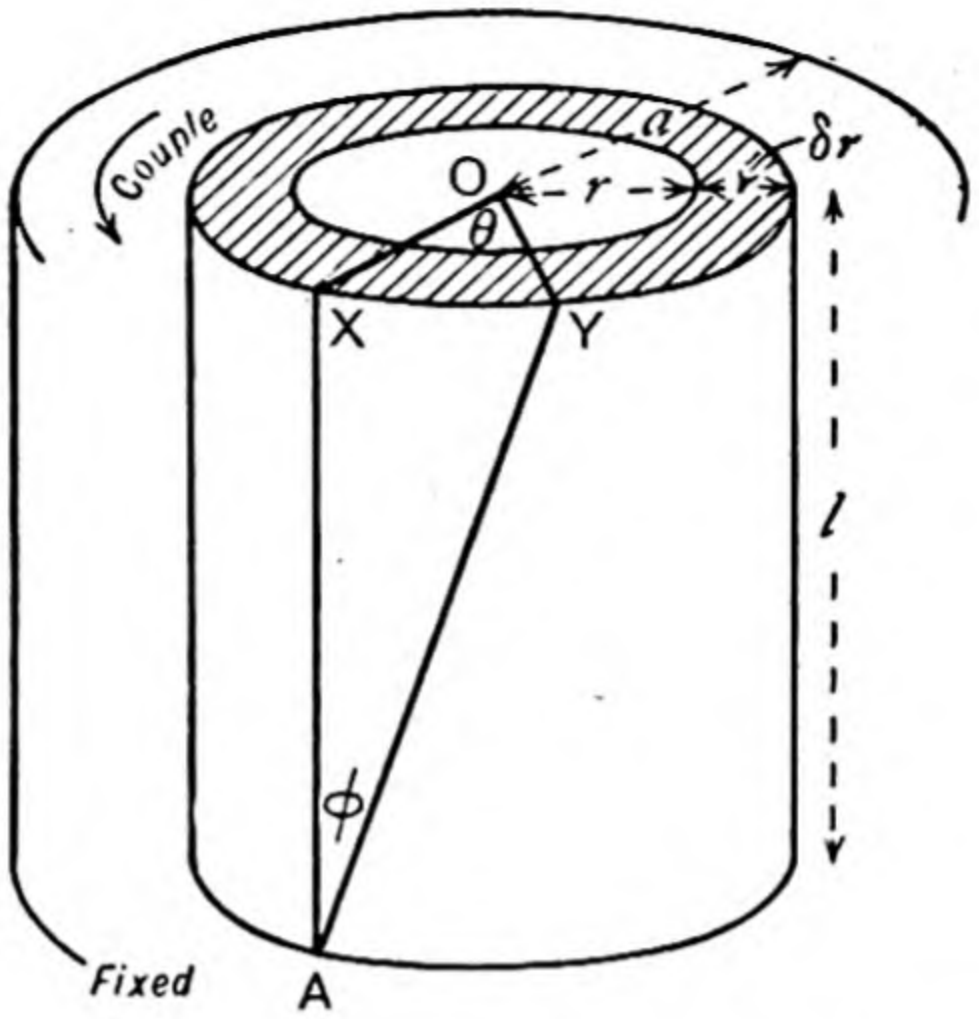
In Searle's apparatus, the extension is measured by more refined means. The wire under test and the comparison wire both carry rectangular frames which are connected by a horizontal member M bearing a spirit-level. One end of M is pivoted freely, while the other end rests on the point of a micrometer screw. When the specimen is strained, M is brought back to the horizontal by turning the screw through a distance equal to the extension, which can thus be measured to 0·001 cm., or perhaps 0·1 per cent. It might appear that this, and other more sensitive methods of measuring the extension, are really wasted if the diameter of the specimen can only be measured to 1 per cent. But they allow thicker specimens to be used, so that the percentage error in measuring the thickness is also reduced.

Torsion of a cylinder. Consider a thin hollow cylinder, of length l cm., internal radius r cm., thickness δr cm., made of material of modulus of rigidity n dynes per sq. cm. The end area, shaded in the figure, is $\pi(r+\delta r)^2 - \pi r^2$, which is $2\pi r\,\delta r$ sq. cm. if δr is sufficiently small.

The lower end is fixed, and the upper end experiences a couple C in the plane of the shaded area which, together with that of the constraint, rotates the upper end through a small angle θ radians, and produces a shear strain ϕ radians, since each generator of the cylinder is turned through an angle ϕ, and if it were cut along a generator and straightened out it would be exactly as the figure of p. 200.

The shear stress exerted on the upper face is then

$$n\phi \text{ dynes per sq. cm.}$$



The force exerted on each element of area of the upper face is then $n\phi \times$ *area of element* dynes, and the moment of this about the axis of the cylinder is $n\phi \times$ *area of element* $\times r$ dyne-cm. The sum of all these moments must equal the applied couple C, so

$$C = n\phi \times \textit{total area of end} \times r,$$
$$= n\phi \times 2\pi r\,\delta r \times r \text{ dyne-cm.}$$

Now the arc XY $= r\theta = l\phi$, so $\phi = r\theta/l$;

whence
$$C = \frac{2\pi n \,.\, r^3\,\delta r}{l} \,.\, \theta \text{ dyne-cm.}$$

A solid cylinder of radius a cm. can be regarded as an assemblage of coaxial thin cylinders, of radii varying from 0 to a, all undergoing the same rotation θ. Since the moments of couples add algebraically, the couple required to twist such a cylinder in the same way is

$$C = \frac{2\pi n}{l}\,\theta \,.\, \sum_0^a r^3\,\delta r.$$

Integration shows that the value of $\sum_0^a r^3\,\delta r$ is $\frac{1}{4}a^4$, so for a solid cylinder of length l cm. and radius a cm., the couple required to twist it through an angle θ radians is

$$C = \frac{\pi n a^4}{2l}\,\theta \text{ dyne-cm.}$$

The " torsion constant " for a cylindrical wire, that is, the couple k required to cause a twist of one radian is then

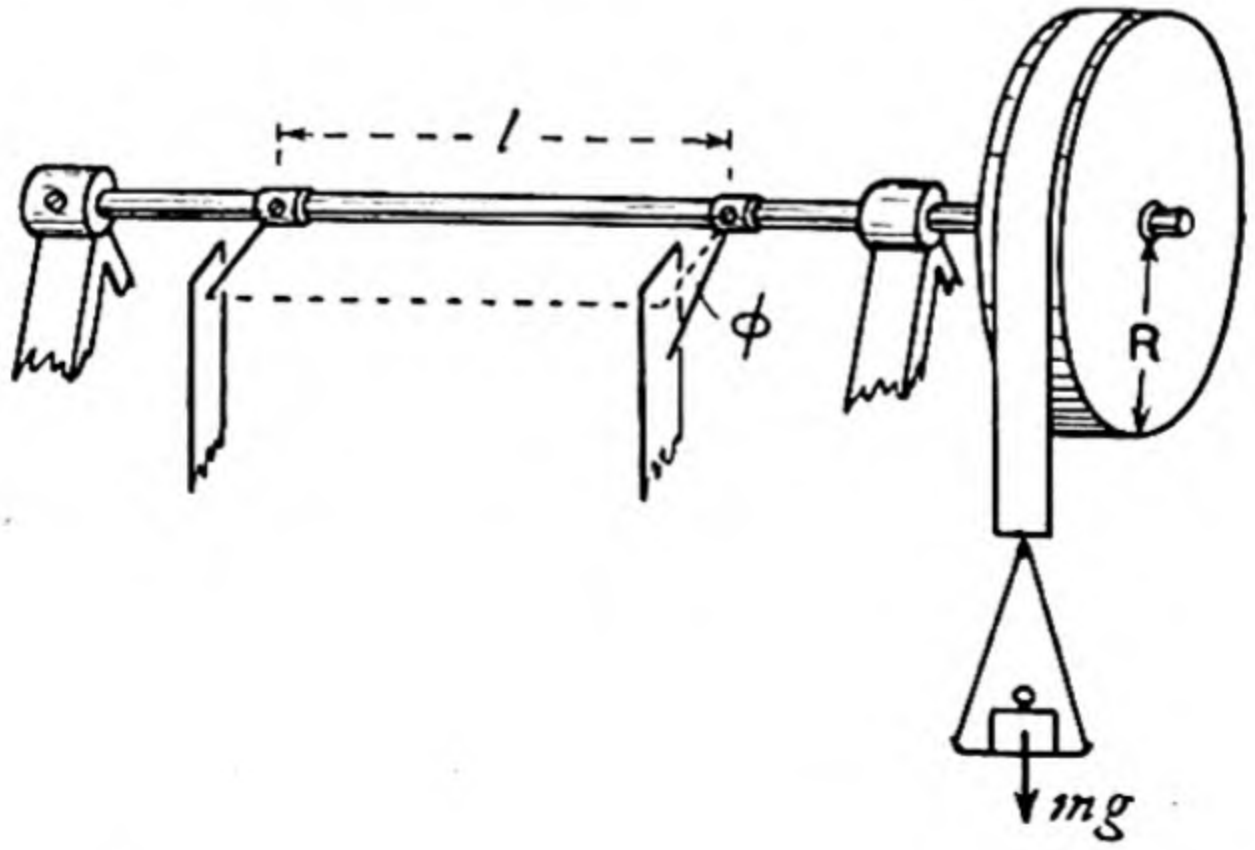
$$k = \frac{\pi n a^4}{2l} \text{ dyne-cm.}$$

For a thick hollow cylinder, of external radius a cm. and internal radius b cm., a similar calculation applies, but here the summation $\Sigma r^3 \delta r$ is from $r = b$ to $r = a$, and this works out to $\frac{1}{4}(a^4 - b^4)$, giving

$$C = \frac{\pi n (a^4 - b^4)}{2l} . \theta \text{ dyne-cm.}$$

It is interesting to note that the couple required to twist a hollow cylinder is a surprisingly large fraction of that for a solid cylinder. Thus, if $b = 0{\cdot}5a$, $(a^4 - b^4)$ is nearly $0{\cdot}9a^4$, while if b is $0{\cdot}8a$, $(a^4 - b^4)$ is nearly $0{\cdot}6a^4$.

Measurement of modulus of rigidity. Two methods are available. The statical method, illustrated in the figure, applies a measured couple C dyne-cm. by means of weights hanging tangentially from the rim of a wheel attached rigidly to the specimen. If m gm. wt. is applied to a wheel of radius R cm., then $C = mgR$ dyne-cm. Two indexes fixed to the wire at a measured distance l cm. apart enable the twist of this length of wire to be found, without error due to any yielding of the fixed end clamp. The twist is represented by ϕ in the figure, this symbol replacing the θ of the last paragraph. The



diameter of the specimen is measured with a screw gauge, and a found. Then, as

$$mgR = \frac{\pi n a^4}{2l}\phi,$$

$$n = \frac{2lmgR}{\pi a^4 \phi} \text{ dynes per sq. cm.}$$

In the dynamical method, a heavy disc, or other suitable body of known moment of inertia I gm. cm.^{2} about the axis of rotation, is attached to the lower end of the wire and set oscillating about a vertical axis as a torsion pendulum (p. 298). If T sec. be the period,

then $T = 2\pi\sqrt{\frac{I}{k}}$, where $k = \frac{\pi n a^4}{2l}$,

so $$n = \frac{4\pi^2 I \,.\, 2l}{T^2 \,.\, \pi a^4} = \frac{8\pi l I}{T^2 a^4} \text{ dynes per sq. cm.}$$

Properties of Liquids

A liquid possesses no definite shape, and transmits a locally applied stress equally in all directions. Only volume stresses and volume strains are possible, and the only modulus of elasticity is the bulk modulus, k, often called simply the " elasticity " of the liquid. For many purposes we choose to imagine liquids as incompressible (that is, k infinite and C zero); but water, for example, has a compressibility C of about 4×10^{-5} per bar, which is about a hundred times as great as that of an ordinary metal.

While the internal forces between the molecules of a liquid are not strong enough, in competition with thermal agitation, to marshal the molecules into any kind of pattern or indeed to keep them from migrating freely, they are responsible for the cohesion of drops and for the surface effects discussed in the next chapter. These internal forces also enable a column of liquid in, say a glass capillary tube to withstand a considerable outward stress (applied as tensile stress) without breaking. Results for the breaking-stress of air-free water in glass vary considerably, and the matter has been reinvestigated by H. N. V. Temperley, who considers that the value is of the order of 30–50 atmospheres, and calculates that the intrinsic pressure $\frac{a}{V^2}$

of van der Waals's equation (p. 221), supposing this to be applicable, should give values considerably greater than this.

Properties of Gases

Elasticity of gases. A gas occupies completely the volume of its containing vessel, and is continually under a stress, which is the uniform pressure between the gas and the walls of its container. In the case of a solid or a liquid, we tend to regard a stress throughout its bulk as a kind of defensive reaction against a deliberately applied external load, and distinguish between internal stresses (which are usually disregarded) and stress increments due to such loading. A stretched wire, for example, exerts forces on the constraints at its ends *in response to* the forces there applied to it—or at least that is the way in which we describe the matter, the reason being that we are used to seeing solids free from large external stresses and regard the introduction of a stress as something that must be *done to* the solid body. But a gas takes the offensive itself, and exerts a pressure on the walls of the container by continual bombardment of the walls by its molecules. *A gas is always exerting a pressure, is always stressed, and is always in a state of strain.*

The one thing common to all strained bodies is the possession of strain energy, which is liberated in the performance of external work when the strain is removed. In strained solids, this is potential energy stored in the solid by work done on the body by an external force. In gases, it is kinetic energy of translation of the individual molecules moving rapidly about, colliding with one another and with the walls of the container. The kinetic theory of gases is usually expounded fully in works on Heat, so that only an outline will be given here, starting with the assumption made by this theory.

The molecules of a gas are continually in rapid random motion. The volume of the molecules themselves is very small compared with the space occupied by the gas. The molecules of a given gas are all alike, and all of the same mass. They behave as if they were perfectly elastic spheres, so that no energy is lost when they collide with each other, or with the walls. The attraction of the molecules for one another is very small, so that the potential energy (using the term here to mean energy stored in a body of gas by work done

against molecular attractions) is negligible. The average kinetic energy of the molecules is proportional to the absolute temperature.

Accepting these assumptions, the physical behaviour of the ideal gas to which they are applicable can be forecast in considerable detail.

A perfectly elastic sphere of mass m and velocity u impinging perpendicularly on a fixed rigid wall has its momentum reversed from mu to $-mu$, and the wall receives an impulse which is equal to the change in momentum of the sphere, which is $2mu$. If, during one second, a large number of such impacts occur, the average force on the wall, by Newton's Second Law of Motion, is equal in absolute units to the total change of momentum in that second. The pressure is then obtained by taking the force on unit area of the wall.

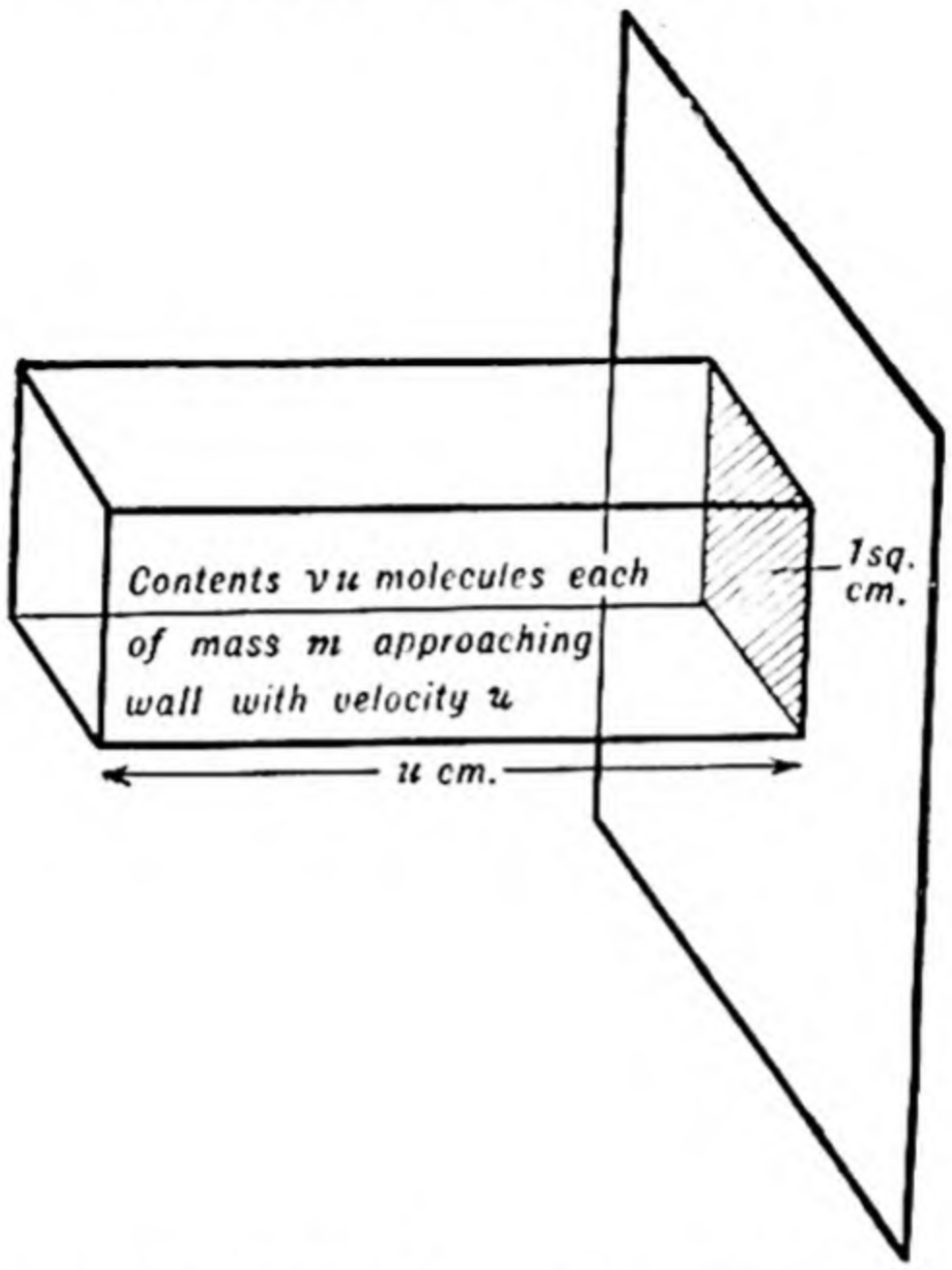
Suppose that a closed vessel of volume V c.c. contains a fixed mass of gas at pressure p dynes per sq. cm., the total number of molecules being n. The number of molecules per c.c., ν, is then given by $\nu = n/V$; and let m gm. be the mass of each molecule.

Imagine for a moment that all the molecules are moving in a direction perpendicular to one wall of the container with velocity u cm. per sec., and consider 1 sq. cm. of this wall. All the molecules in a prism of length u cm. and base area 1 sq. cm. strike the selected area in one second. The volume of this prism is u c.c., and the number of molecules it contains is νu. Each molecule undergoes a momentum change $2mu$ at right angles to the wall when it strikes it, giving the wall a corresponding impulse of $2mu$ units. Hence the force on unit area is $\nu u \times 2mu$ dynes, so the pressure is

$$p = 2\nu m u^2 \text{ dynes per sq. cm.}$$

In reaching this conclusion, two extremely unlikely assumptions have been made.

(1) If the molecules are moving at random in all directions, it is highly improbable that at any given instant they will all be simultaneously moving in the same direction. It is more reasonable to suppose that they will, on the average, be distributed equally in six directions along any set of rectangular co-ordinate axes we like to choose, as in the diagram. On the average, then, only one sixth of



the νu molecules considered in the prism will be moving towards the wall in a direction at right angles to it. So it would be nearer the truth to write

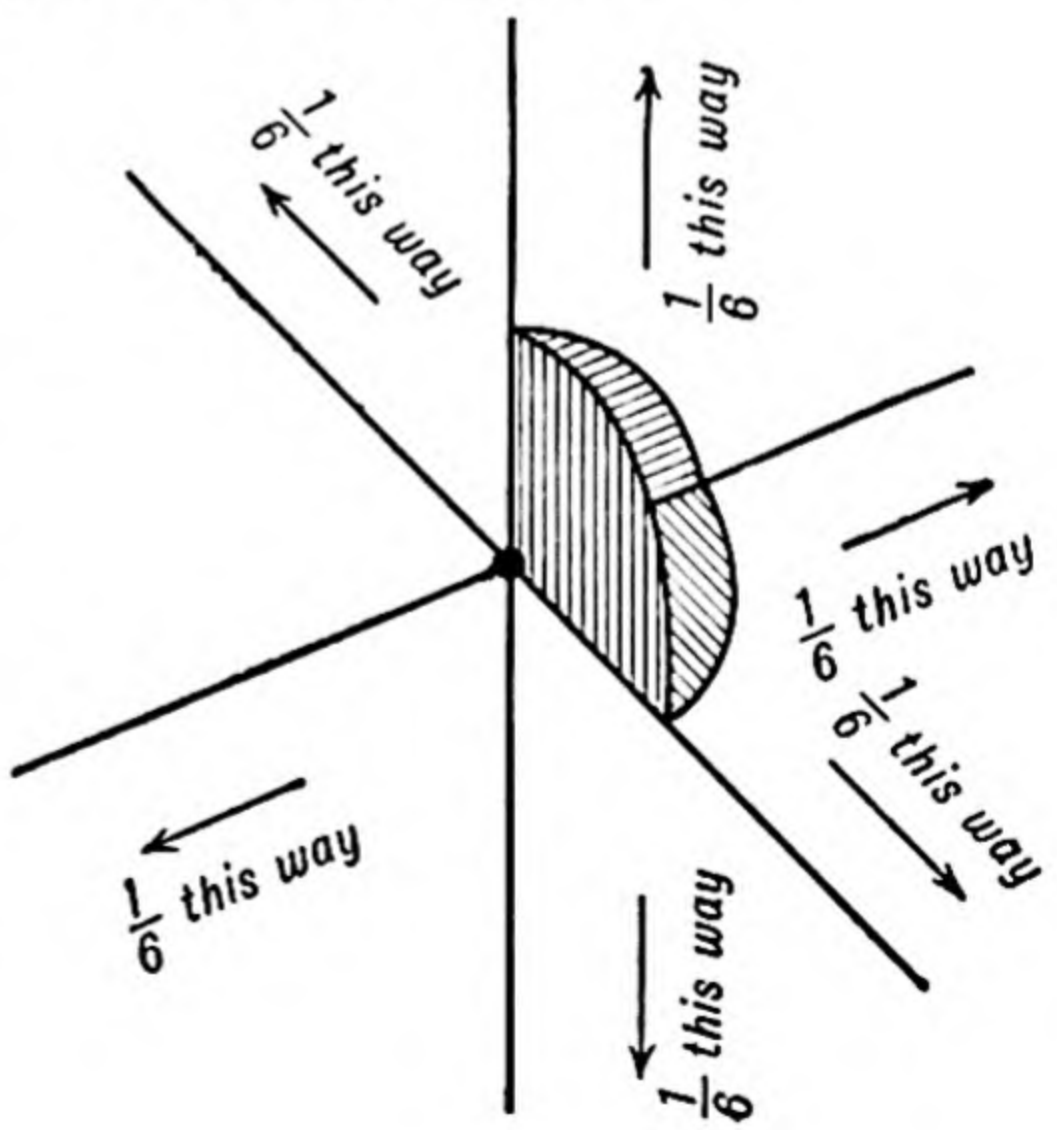
$$p = \tfrac{1}{6} \times 2\nu m u^2 = \tfrac{1}{3}\nu m u^2 \text{ dynes per sq. cm.}$$

(2) All the molecules do not have the same value of u, and we should really consider the average value of u for all the molecules. This is called the **mean square velocity**, and is given the symbol $\bar{c}^2$. Thus, the expression for the pressure should really be

$$p = \tfrac{1}{3}\nu m \bar{c}^2 \text{ dynes per sq. cm.}$$

As $\nu = \dfrac{n}{V}$, $p = \tfrac{1}{3}\dfrac{nm}{V}\bar{c}^2$,

or, $pV = \tfrac{1}{3}nm\bar{c}^2$.



If the average kinetic energy of a molecule, $\frac{1}{2}m\bar{c}^2$, is proportional to the absolute temperature T, then $\frac{1}{2}m\bar{c}^2 = \alpha T$, where α is a constant,

and $$\tfrac{1}{3}nm\bar{c}^2 = \tfrac{2}{3}n\alpha T, \text{ or, } pV = CT,$$

where C, written for $\frac{2}{3}n\alpha$, is a constant for the particular mass of gas taken; if this mass is one gram-molecule, the symbol R is used instead of C, and the constant R is called the **Gas Constant.**

How will such an ideal gas behave? To begin with, if we have a fixed mass (n fixed, and therefore C fixed), (1) at a constant temperature, the product pV is constant.

Or (2), if we keep p constant and vary T, V is directly proportional to T.

Again (3), if we keep V constant and vary T, p is directly proportional to T.

These three statements represent the behaviour of all the ordinary "permanent gases" (air, oxygen, nitrogen, helium, hydrogen, etc.) over a wide range of varied conditions, with a fairly high degree of accuracy, and it is usual to treat them for all elementary purposes as following the ideal gas equation under most conditions.

Conversely, it is believed that the assumptions made in building up the equation must be fairly near the truth (though some cannot possibly be really accurate) for the permanent gases.

The value of $\sqrt{\bar{c}^2}$, which we will write as c, can be found from the equation $pV = \frac{1}{3}nm\bar{c}^2$; for nm is the total mass of gas occupying volume V, whence nm/V is simply the density ρ gm. per c.c.; so $p = \frac{1}{3}\rho\bar{c}^2$, and $c = \sqrt{3p/\rho}$.

For hydrogen at N.T.P., ρ is 0·00009 gm. per c.c., and p is

$$76 \times 13{\cdot}6 \times 981 \text{ dynes per sq. cm.,}$$

so $$c = \sqrt{\frac{3 \times 76 \times 13{\cdot}6 \times 981}{0{\cdot}00009}},$$

$$= 1{\cdot}86 \times 10^5 \text{ cm. per sec.}$$

In the equation $pV = RT$, the quantity R, which is proportional to n, the total number of molecules present, is called the **Gas Constant** for the particular mass of gas concerned. Usually, when its value is quoted, it refers to one mole (one gram-molecule); for one mole of every gas has the same V at the same pressure and temperature,

and so R has the same value for one mole of every gas supposed ideal. The following table, taken from Kaye and Laby's "Tables", gives the values of c at 0° C. for the common gases.

Gas	$c=\sqrt{\overline{c^2}}$
Hydrogen	$18{\cdot}39 \times 10^4$ cm. per sec.
Helium	$13{\cdot}1 \times 10^4$,,
Nitrogen	$4{\cdot}93 \times 10^4$,,
Oxygen	$4{\cdot}61 \times 10^4$,,
Carbon dioxide	$3{\cdot}92 \times 10^4$,,

The equation which has been developed on page 220, namely,

$$pV = RT$$

is called the **Ideal Gas (or Perfect Gas) Equation.** In its derivation it is assumed that the volume of the molecules themselves, and the effect of attractions between them, are both negligibly small effects, which is a fair approximation if the density of the gas is very low. But if these effects are not negligible, which must be reckoned with in most experiments with actual gases (for how can any gas liquefy or solidify at all if the molecules do not attract one another, and how can the liquid or solid have such considerable volume if the molecules themselves are inappreciably small?), then the measured pressure on the walls of the container is an under-estimate of the true pressure within the gas, and the volume of the container an over-estimate of the space available for the molecules to traverse. Of several approximate equations which have been proposed to modify the equation $pV = RT$ to apply to actual gases, the most convenient is **van der Waals's equation** :

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT,$$

where a and b are "constants" characteristic of the particular gas concerned, and are different for different gases.

Here $\left(p + \frac{a}{V^2}\right)$ represents the true bombardment pressure, which is greater than p as measured at the container wall, and a/V^2 is the effect of attraction between the molecules, or the internal stress within the gas. If a/V^2 is sufficiently great, cohesion is manifested and the substance is not a gas at all, but a liquid ; if a/V^2 is greater still, p is zero

and the container can be dispensed with, so the substance is a solid. The differences between the three states of matter, as far as response to external stress increment is concerned, are thus in a general way attributable to differences in the magnitude of the internal stress, and one of the great merits of van der Waals's equation is that it enables us to picture the differences between the three states of matter as differences in *degree* rather than differences in *kind*. The quantity $\frac{a}{V^2}$ is usually called the **intrinsic pressure** of the material when the equation is applied to explain the behaviour of *liquids* (cf. pp. 216 and 260).

Adiabatic and isothermal change. The total translational kinetic energy in ergs of the n molecules in a chosen mass of gas is $n \times \frac{1}{2}mc^2$, which is $\frac{3}{2} \times \frac{1}{3}nmc^2$, or $\frac{3}{2}pV$, or for one mole of gas, $\frac{3}{2}RT$.

Suppose the volume of the gas increases. The pressure of the gas does external work in moving back the walls of its constraint. This must be drawn from the kinetic energy of the gas molecules, which, by the law of conservation of energy is therefore reduced. Thus the average kinetic energy falls, and there is a fall in temperature. Similarly, if the volume of the gas is reduced, work is done on the gas by an external agent; this reappears as an equivalent increase in internal kinetic energy of the gas, and there is a rise in temperature.

Ideally, there are two separate ways in which the stress-strain relation for a gas can be studied:

(1) We can prevent any escape of heat from, or access of heat to the gas, when the change is said to be **adiabatic.**

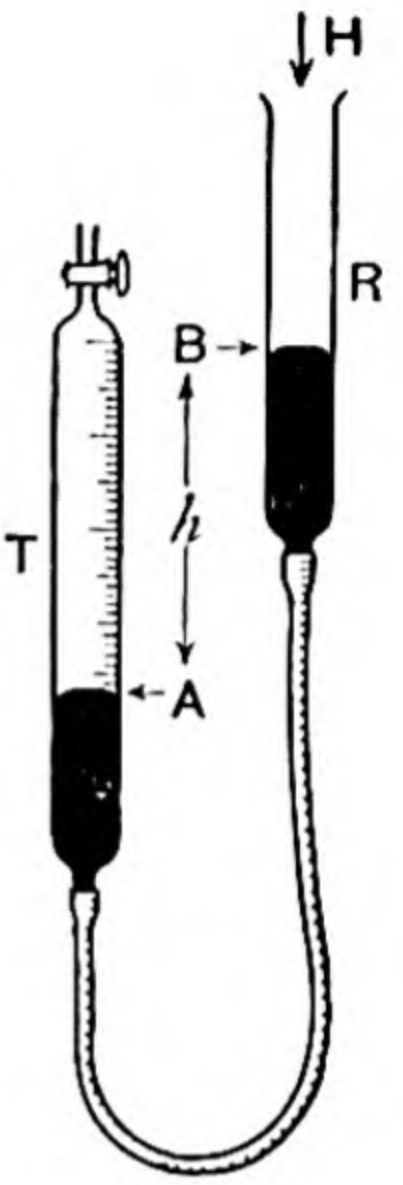
Or (2), we can arrange to remove the heat generated by reduction of volume, or replace the heat abstracted by expansion, by maintaining the gas in a constant-temperature enclosure and waiting a little after the change for temperature equalisation to occur, when the change is said to be **isothermal.**

Boyle's Law

Boyle's Law states: **For a fixed mass of gas maintained at constant temperature, the pressure and volume are inversely proportional** (or, pressure × volume is constant for the particular mass of gas concerned).

This law may be verified with the apparatus illustrated, in which a fixed mass of dry air, or other gas, is trapped by a mercury column

in a graduated tube, and the pressure may be varied by raising or lowering a mercury reservoir communicating with the gas tube. The pressure of the atmosphere, H cm. of mercury, is first read. A series of readings of the mercury levels A and B, and the volume V of the trapped gas, is taken, some with B above A, some with B below A. Writing h cm. for the difference in levels $(B - A)$, which is positive if B is the higher, negative if B is the lower, the total pressure of the gas is $p = (H + h)$ cm. of mercury. The product pV is found to be approximately constant, and the graph of p against $1/V$ is a straight line through the origin, showing that under these conditions Boyle's law is fulfilled. It is important to allow a little time after adjusting the apparatus before taking a reading, so that temperature equalisation can take place.

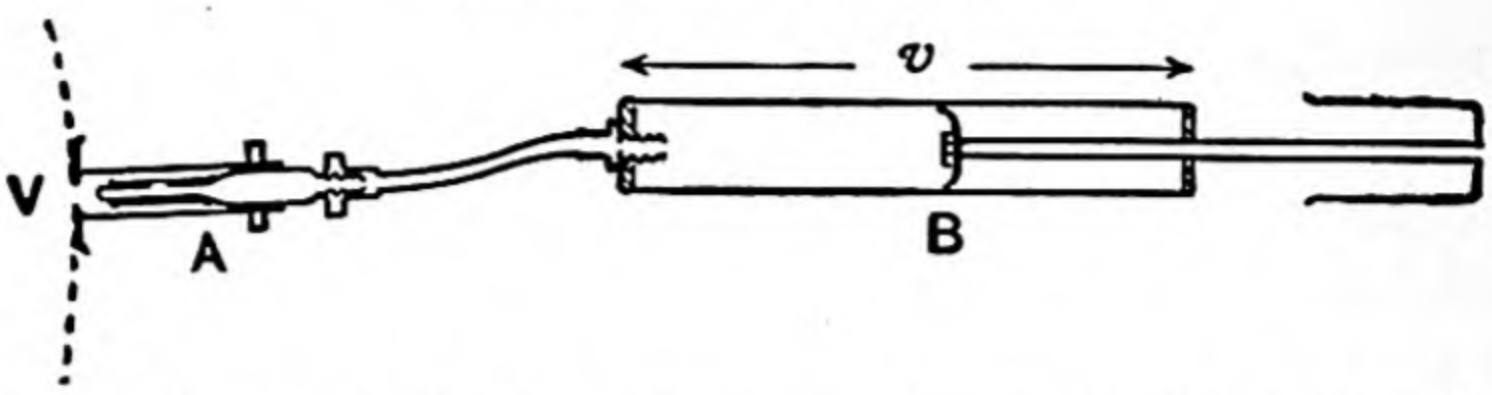


For air, oxygen, hydrogen, and others of the " permanent gases ", Boyle's law is closely obeyed at ordinary air temperatures, for pressure ranges between about half an atmosphere and two atmospheres. At higher pressures experiments have shown that the product pV is not constant for a given mass of gas, but varies with p in a manner which depends on (*a*) the nature of the gas, and (*b*) the temperature at which the observations are made.

Investigations over wide ranges of temperature have shown that for every gas there is a certain temperature, called its **Boyle temperature**, at which Boyle's law is closely followed over a wide range of pressures.

Applications of Boyle's Law

Air compressor. In the cycle tyre pump, illustrated in the diagram, the tyre valve A and the cup washer B on the piston are both made to allow air to pass only into the tyre (from right to left in the figure, p. 224). On the forward stroke of the piston, B presses against the wall of the barrel and the volume of air in the pump is reduced so that its pressure increases ; and when this pressure exceeds that in the tyre by sufficient to force air past A, most of the original pumpful of air enters the tyre.



Let the volume of the tyre (supposed constant) be V and that of the pump v, so that each stroke of the pump sends into the tyre a mass of air that occupies volume v at atmospheric pressure. Suppose the initial pressure in the tyre is one atmosphere, and let the final pressure after n strokes be p_n atmospheres. Then, at the end of the n strokes, a fixed mass of air which occupies $V+nv$ units of volume at one atmosphere pressure occupies V units at p_n atmospheres. So, by Boyle's Law,

$$p_n V = 1 \times (V + nv),$$

$$\therefore \quad p_n = \left(1 + \frac{nv}{V}\right) \text{ atmospheres.}$$

The increase in pressure is thus directly proportional to the number of strokes.

As the pressure in the tyre increases, so does the pressure in the pump barrel needed to open the valve A, which must therefore open at a later stage in each successive stroke. There is a limit to the pressure attainable with this type of pump, quite apart from that set by the strength of the barrel ; if, for example, the space beyond the piston when this is thrust fully home is 1/20 of the volume of the barrel, the minimum volume to which a pumpful of air can be reduced is 1/20 of its original volume, whence the maximum pressure attainable will be 20 atmospheres.

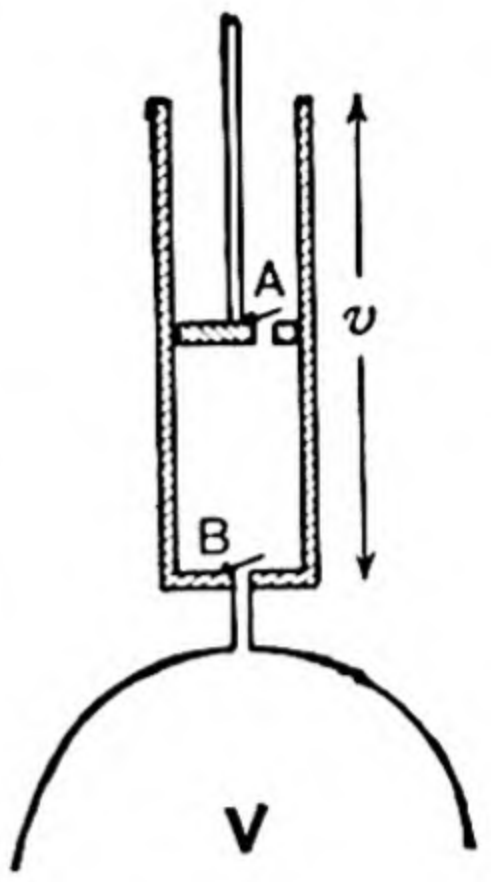
Exhaust pump. The valves of the simple piston exhaust pump are arranged to allow air to pass only outwards from the vessel to be exhausted (that is, upwards in the figure).

Let the volumes of vessel and pump be V and v, and let the initial pressure in the vessel be p_0 units. Starting with the piston at the bottom of its stroke, on the first upstroke the volume of the space in the pump increases, the pressure decreases, and the valve B is opened by the excess pressure of the air in the vessel over the now diminished pressure in the pump barrel. At the top of the stroke, V units of volume at pressure p_0 have become $(V+v)$ units at a

lower pressure p_1; whence, applying Boyle's Law to this fixed mass of air,

$$p_1(V+v)=p_0V,$$

$$p_1=p_0\frac{V}{V+v} \text{ units of pressure.}$$



On the downstroke of the piston, B closes at once, since the pressure in the barrel immediately increases, while A opens as soon as the pressure in the pump barrel exceeds that of the atmosphere, after which the barrelful of air escapes to the outside. On the second cycle of operations, the process is repeated, and the pressure after two strokes is given by

$$p_2=p_1\left(\frac{V}{V+v}\right)=p_0\left(\frac{V}{V+v}\right)\left(\frac{V}{V+v}\right),$$

$$\therefore\ p_2=p_0\left(\frac{V}{V+v}\right)^2 \text{ units of pressure.}$$

The calculation can be carried on for each successive stroke, each time applying Boyle's Law to a different fixed mass of air—the mass in the vessel at the beginning of the stroke. After three strokes, the pressure p_3 is

$$p_0\left(\frac{V}{V+v}\right)^3,$$

and after n strokes p_n is

$$p_0\left(\frac{V}{V+v}\right)^n.$$

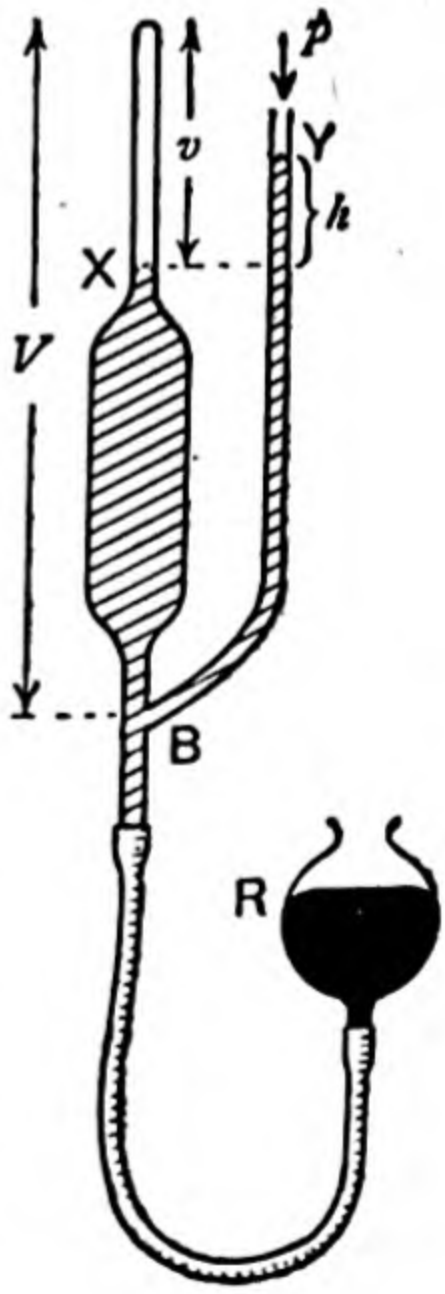
Successive pressures thus form a geometrical progression, and even with a perfect pump operating for an indefinitely long time it would be impossible to reduce the pressure in the vessel to zero. Apart from obvious mechanical difficulties, such as leakage past the valves, the chief factor limiting the vacuum attainable is that the valve A can only be opened by superior pressure from below. If the "dead space" above B is, say, one thousandth the total volume of the pump, and the pressure has been brought down to one thousandth of an atmosphere, when the piston next moves down, the barrelful of air at 1/1000 atmosphere pressure will be compresssed to 1/1000 its volume, and will just be at one atmosphere pressure, but this will not

give the excess required to open the valve. This difficulty is overcome in modern piston-pumps by having a layer of oil over the top of the piston, so that what is effectively an oil piston sweeps out all the air above it ; and in a similar manner by the Toepler pump which employs a mercury column as a piston.

The McLeod gauge. This pressure gauge is used for the measurement of low pressures down to about 1/100 mm. of mercury. A narrow capillary tube is attached to a bulb which communicates at B, through a long vertical tube, with the vessel containing the gas of which the pressure p is to be measured. The volume V of the bulb and capillary together, and the volume v of the capillary down to a fixed mark X, are known. By raising the mercury reservoir R, a volume V of gas at the unknown pressure p is cut off at B and compressed to volume v, when the mercury levels stand at X in the capillary and Y in the connecting tube. The difference in levels h between X and Y is observed. Then the fixed mass of gas cut off as a volume V at pressure p cm. of mercury has been compressed to volume v at pressure $(p+h)$ cm. So, by Boyle's law,

$$pV = (p+h)v,$$

$$p = \frac{hv}{V-v} \text{ cm.}$$



As p is proportional to h, a direct reading scale is attached to the side of the connecting tube.

It is assumed in this calculation that the volume of the apparatus containing the gas under measurement is so large that the intrusion of the mercury column up the connecting tube causes no appreciable increase of pressure in it. The above theory of the McLeod gauge assumes that Boyle's law is accurately obeyed over a considerable range of pressure ratios, a condition which is more nearly fulfilled at low pressures than at higher ones.

Molecular gauges and pumps. The mean free path λ of a gas molecule is the average distance travelled between collisions with other

molecules. An approximate value for λ can be calculated simply, in terms of the molecular diameter, σ, and the number of molecules per unit volume, ν. Any two molecules whose centres approach within a distance σ of one another collide; so we can imagine any one molecule as surrounded by a sheath of radius σ, collision occurring when the centre of another molecule touches this sheath. If our chosen molecule travels a distance x in a straight line, the sheath sweeps out a cylinder of volume $\pi\sigma^2 x$, and the number of other molecules whose centres lie within this volume is $\nu\pi\sigma^2 x$. But this is all fictitious; what we are trying to say is that in travelling a distance x the chosen molecule collides with $\nu\pi\sigma^2 x$ others, so that the average distance between collisions is $\frac{x}{\nu\pi\sigma^2 x}$, or $\lambda = \frac{1}{\nu\pi\sigma^2}$ cm.

For an ideal gas, the assumption of p. 217 that $\sigma \simeq 0$ means that the mean free path would be extremely large; in fact an ideal gas would presumably have no interesting mechanical properties apart from those of possessing internal kinetic energy and exerting a pressure on the walls of its containing vessel. But for real gases, the collisions of molecules with one another are responsible for many of the physical properties which distinguish one gas from another, and the mean free path is a very important quantity.

As ν is directly proportional to the pressure p, λ is inversely proportional to p. For nitrogen at atmospheric pressure λ is of the order 10^{-5} cm.; at a pressure of 1 mm. of mercury, of the order 0·7 mm.; and at a pressure of 10^{-3} mm. of the order 70 cm.; which means in effect that at very low pressures a molecule rebounds from the enclosure walls many times between collisions with other molecules, so that the mean *free* path is then really of the order of the diameter of the vessel, and not that calculated from the formula.

Theory shows that the coefficient of viscosity η of a gas is proportional to $\rho c\lambda$, where ρ is the density and c the root-mean-square velocity (p. 220); and the coefficient of thermal conductivity κ is proportional to ηC_v or to $C_v\rho c\lambda$, where C_v is the specific heat at constant volume. The diffusion coefficient D (which is the mass of gas crossing unit area in unit time under unit concentration gradient) is proportional to λc. These relations cannot be derived here, but the method of dimensions (p. 243) will show that those for η and for D are physically possible expressions.

Since ρ is proportional to ν, and λ is proportional to $1/\nu$, the product $\rho\lambda$ is independent of ν, and thus independent of the pressure. But this conclusion is only true so long as the pressure is above the value at which the mean free path is limited by the dimensions of the vessel; at pressures below this, λ is effectively fixed and cannot increase as p decreases further; so that at very low pressures $\rho\lambda$ is proportional to the pressure, whence η and κ are at these low pressures proportional to the pressure. Pressure gauges for the measurement of very low pressures (less than 10^{-2} cm. of mercury), apply these results, employing the viscous damping of an oscillating system, or the rate of emission of heat by conduction from a wire carrying an electric current, as the effects by which p is estimated.

The method of action of the modern **high-vacuum pump** is to some extent analogous to the simple experiment in which a flask is "evacuated" by placing a little water in it, boiling until there is a steady efflux of steam, and then removing the flask, sealing it quickly, and cooling it. The air has been replaced during boiling by steam at atmospheric pressure; and when the flask is cooled, the pressure within it becomes the saturated vapour pressure of water at the final temperature. Matters are not quite so simple as this, for the problem is to evacuate a vessel attached to the boiler, and not merely the boiler itself; the air must be persuaded to diffuse out of the vessel into the boiler, where it can be dealt with, and must be prevented from diffusing back again; and a liquid with a vapour pressure much lower than that of water at ordinary temperatures must be used. Mercury, which has a vapour pressure at ordinary temperatures of 10^{-3} mm. and can be heated without fear of chemical decomposition, and the low vapour-pressure "Apiezon" oils developed at Messrs. Metropolitan-Vickers Electrical Co. Ltd., which have a much lower vapour pressure than mercury but cannot be heated to such high temperatures, are the liquids commonly used. (It must be remembered, incidentally, that the boiling points of all liquids under the pressures obtaining within such a pump are far below the normal boiling points at atmospheric pressure.)

With this analogy in mind, the action of the original **mercury-vapour pump** of Gaede can be explained as follows. A gas diffuses so that it always fills the available volume completely, whether other gases are present or not. Thus air will expand by diffusion into an

enclosure containing mercury vapour just as if the enclosure were evacuated. Suppose the vessel A to be evacuated is maintained at a high temperature, and placed in communication with a similar vessel B containing mercury vapour at the same high temperature. Mercury vapour diffuses into A and air into B, and after a short time the pressure of the air in A is half what it was at the start. Now remove B and replace it by a similar vessel C of mercury vapour; again, diffusion halves the partial pressure of the air in A, and it becomes one quarter of what it was to begin with. Continuing this process, the partial pressure of air in A is reduced to a very low value, for most of the air has been replaced by mercury vapour. If A is now cooled the mercury vapour is condensed, and the pressure in A will be very little above the saturated vapour pressure of mercury at the low temperature.

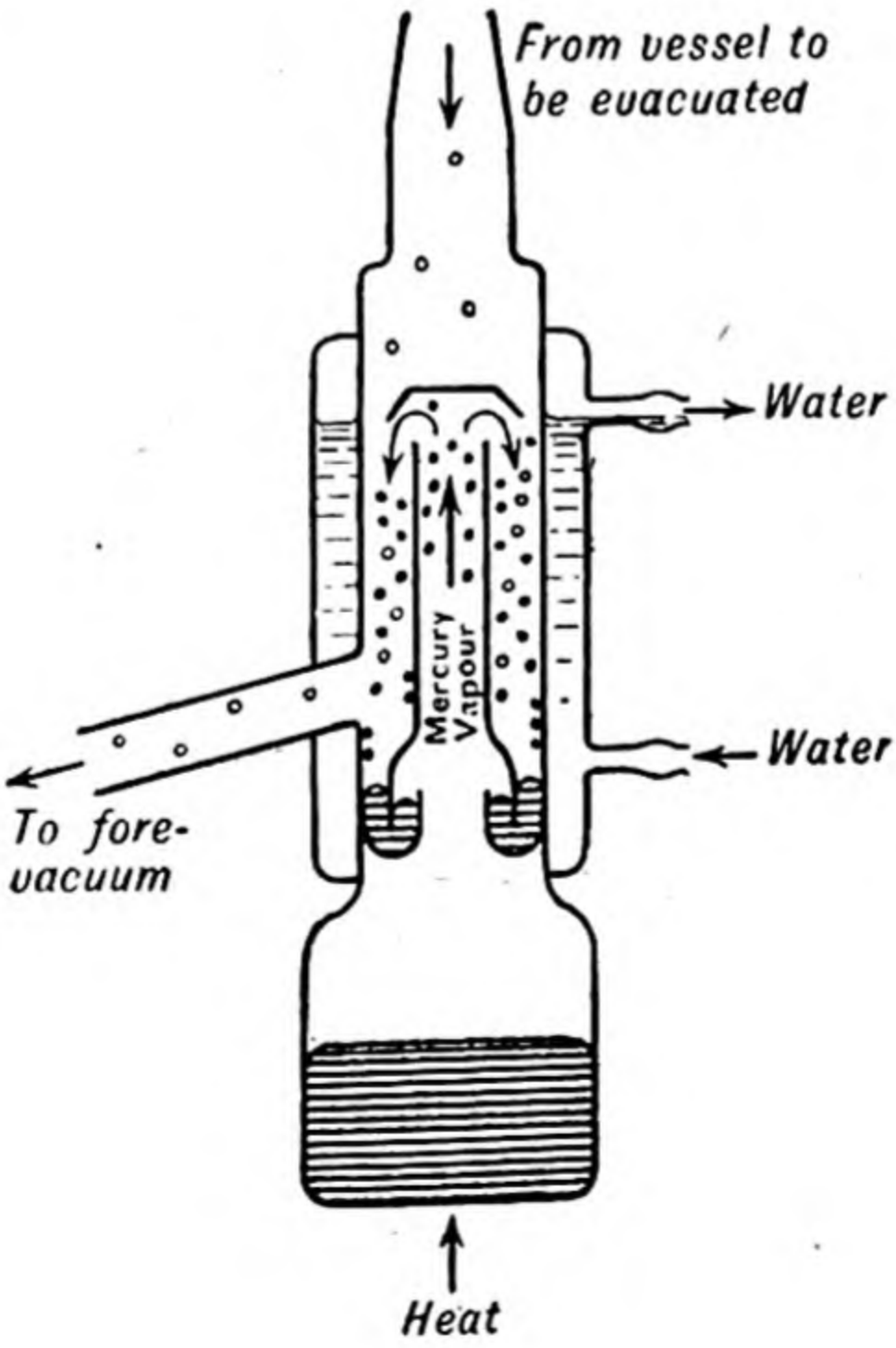
In practice, the effect of repeated exchange of the second vessel is achieved more conveniently by having a continuous stream of mercury vapour flowing past an orifice communicating with A, which need not really be heated itself, and by having the initial pressure so low that diffusion of air back from the mercury stream into A is very small indeed. That is, the pump will not work on its own, but must have the way prepared for it by a fore-vacuum or "backing" pump of the piston or other equivalent type, which gives a pressure of the order of 0·1–0·5 mm. of mercury.

The mean free path of air molecules at such a pressure is of the order of about 0·1 mm., and if the orifice is of about this width, air molecules diffusing into the mercury stream pass out of A unimpeded by collision, and once in the stream of mercury vapour are carried away by it. In Gaede's pumps, known as **diffusion pumps**, great importance was attached to the dimensions of the orifice, for it was supposed that the whole effect of the pump depended on unimpeded diffusion of the air through this into a slowly moving stream of mercury vapour. Any mercury vapour diffusing in the reverse direction condensed, and liquid air traps interposed between the pump and the vessel A prevented vapour from passing into A and kept the mercury vapour pressure there very low.

Langmuir showed that the dimensions of the orifice are not of critical importance if a rapid stream of mercury vapour is sent past it in the direction of the fore-vacuum; for if the speed of the stream

exceeds the mean velocity of the air molecules, those leaving the orifice are caught up and swept away with little chance of diffusing back. Pumps made on the same general lines as Gaede's, but with much wider communication between the vessel and the mercury stream, and with a much more rapid directed stream of mercury vapour, are called **condensation pumps.** The processes of diffusion and condensation play similar parts in both the Gaede and Langmuir pumps, but the condensation type is much more efficient, and all modern pumps are designed on these lines.

The diagram shows a typical single-stage condensation mercury vapour pump, made of metal. A backing pump, attached to the side tube at the left, maintains an outlet pressure of a fraction of a millimetre of mercury, and the vessel to be evacuated is attached to the wide opening at the top. Mercury vapour from the boiler comes up the central tube, and is deflected down by the umbrella-like cap, the whole of the annular space within the pump at this level forming the "orifice", into which air molecules diffuse downwards as



indicated, to be carried away into the fore-vacuum. In the lower part of the pump, mercury vapour is condensed by the water-jacket, and mercury returns to the boiler by the annular siphon-trap shown.

As a general rule, it can be stated that the lower the fore-vacuum pressure, the greater the efficiency of this type of pump becomes; thus, two such pumps connected in series, the first acting as fore-vacuum to the second, will be much more effective than one alone; and two or more pumps in series are often built up together in the same container over the same boiler, being known as "two-stage", etc., pumps.

At very low pressures, when the mean free paths of the molecules are considerable, most molecular collisions will be made with the walls of the containing vessel and the connecting tubing. If the number of these collisions is reduced by using short connecting tubing of wide gauge, the molecules will meet with much less obstruction in their passage out towards the pump, so that wide connecting tubing is invariably used; the upper opening of the pump in the figure is intended to join on to a tube of about two or three centimetres diameter.

Isothermal elasticity of a gas. The stress to which a gas is subjected is a uniform pressure p dynes per square centimetre. If a small increment δp produces a diminution δV in its volume V c.c., or an *increment* $-\delta V$, then the elasticity, which is stress increment/strain increment, is $\dfrac{\delta p}{-\delta V/V} = -V\dfrac{\delta p}{\delta V}$.

Considering very small increments indeed, this is written,

$$-V\frac{dp}{dV} \text{ dynes per sq. cm.}$$

If Boyle's law is obeyed,

$$pV = a\ constant,$$

at constant temperature. So, if the pressure change to $p+\delta p$, and the volume to $V+\delta V$ (where δV is actually negative),

$$(p+\delta p)(V+\delta V) = pV\ ;$$

or, neglecting the very small product $\delta p \,.\, \delta V$,

$$p\delta V + V\delta p = 0,$$

or,
$$-V\frac{\delta p}{\delta V}=p.$$

For very small increments this is usually written

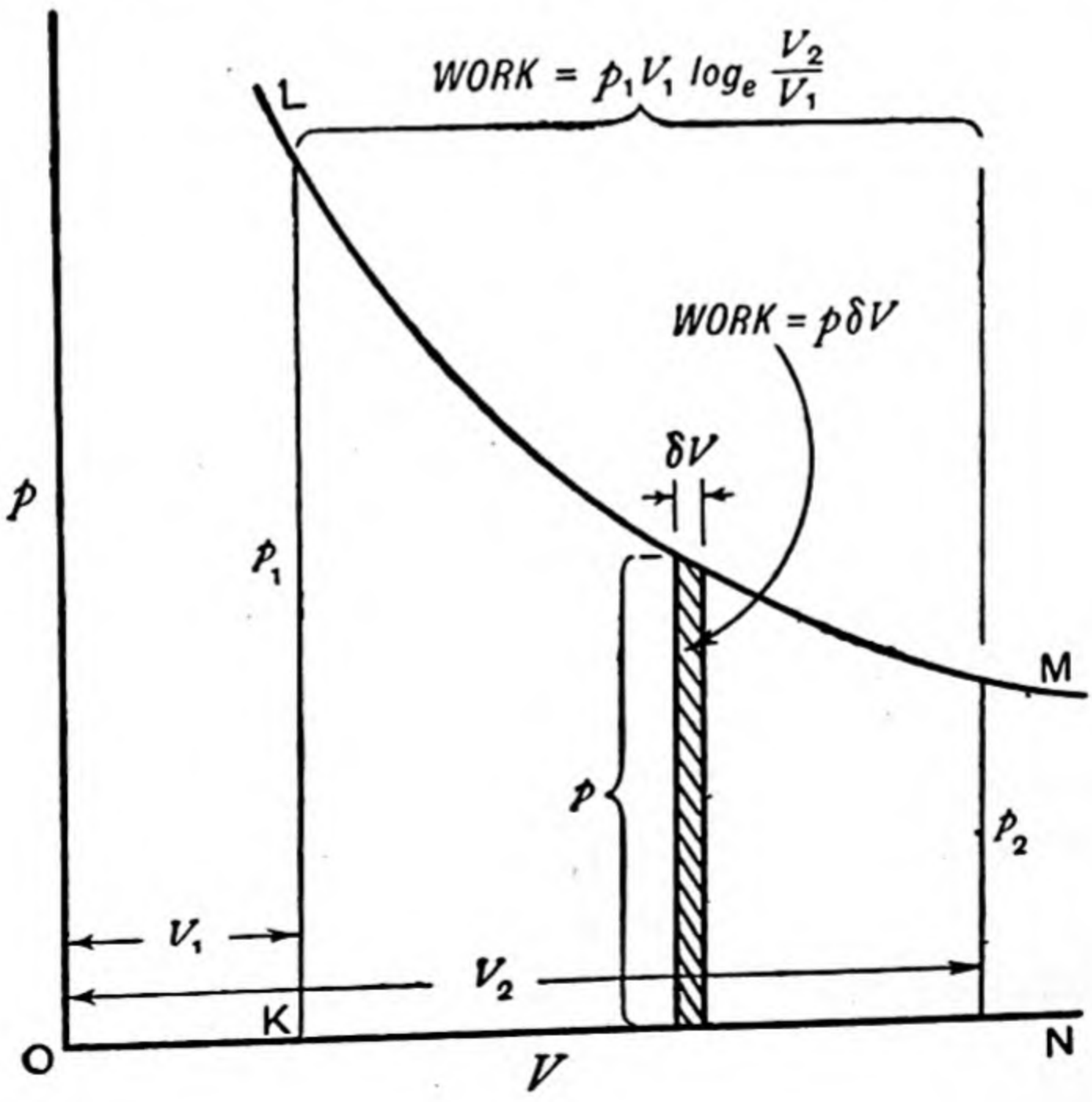
$$-V\frac{dp}{dV}=p \text{ dynes per sq. cm.}$$

Thus, the isothermal elasticity of a gas obeying Boyle's law is equal to the pressure.

It is usually called the " elasticity ", and not the " modulus of elasticity ", because the term " modulus " implies a *constant* ratio between stress increment and strain increment, as when Hooke's law is followed.

External work done by a gas during small expansion. Consider a fixed mass of gas, exerting a pressure p dynes per sq. cm. on the walls of its container, occupying volume V c.c.

The work done by the gas during an expansion cannot depend on the shape of the container, so suppose this is a cylinder of cross-sectional area A sq. cm., closed by a light frictionless piston. Let



the gas expand by a small amount, δV c.c., pushing back the piston a distance x cm., so that $\delta V = A\delta x$. The force exerted by the gas on this piston is pA dynes ; the work done by the gas is then

$$pA\delta x \text{ ergs} = p\delta V \text{ ergs},$$

and is represented on a p-V diagram by the area of a narrow strip, as shown shaded in the figure.

External work done by a gas during finite isothermal expansion. Let the temperature remain constant at T absolute, and the fixed mass of gas expand from volume V_1 c.c. at pressure p_1 dynes per sq. cm. to volume V_2 c.c. at pressure p_2 dynes per sq. cm. Then, as for each small volume change during the expansion the external work done is represented by the area of the corresponding strip beneath the graph and the V-axis, the total work done is the sum of all such areas, represented by the area $KLMN$ in the figure. The graph can always be drawn, and the area calculated from it ; but important formulae for the external work are obtained by using the calculus.

For all values of p and V, if Boyle's law holds, $pV = p_1V_1 = p_2V_2 =$ a constant, k say. The external work done during a small expansion δV c.c. when the pressure is p dynes per sq. cm. is $p\delta V$ ergs. As $pV = k$, $p = k/V$, so the external work for a small expansion is

$$\frac{k}{V}\delta V \text{ ergs.}$$

The external work done for an expansion from V_1 to V_2 is

$$\int_{V_1}^{V_2} \frac{k}{V}\,dV = k \log_e \frac{V_2}{V_1} \text{ ergs.}$$

As $p_1V_1 = p_2V_2$, $\dfrac{p_1}{p_2} = \dfrac{V_2}{V_1}$; hence the external work may be written

$$k \log_e \frac{p_1}{p_2} ;$$

or, remembering that $k = p_1V_1 = p_2V_2$, the formula can be written as

$$p_1V_1 \log_e \frac{V_2}{V_1} \text{ ergs} ;$$

or

$$p_2V_2 \log_e \frac{V_2}{V_1} \text{ ergs} ;$$

or $$p_1 V_1 \log_e \frac{p_2}{p_1} \text{ ergs};$$

or $$p_2 V_2 \log_e \frac{p_2}{p_1} \text{ ergs.}$$

If we are dealing with unit mass, or one mole, when the gas constant R can be appropriately introduced, since $p_1 V_1 = p_2 V_2 = RT$, any of the four expressions above may be written

$$RT \log_e \frac{V_2}{V_1}, \text{ etc.,}$$

while for any quantity x moles we get

$$x\, RT \log_e \frac{V_2}{V_1}, \text{ etc.}$$

If V_1 is greater than V_2, the external work done by the gas is negative; for, of course, external work must be done *on* the gas by an external agent to reduce its volume.

Adiabatic elasticity of a gas. When the pressure and volume of a fixed mass of gas change under adiabatic conditions, the relation between the pressure p dynes per sq. cm., and the volume V c.c., is

$$pV^\gamma = a\ constant,$$

where γ is the ratio $\dfrac{\text{specific heat at constant pressure}}{\text{specific heat at constant volume}}$,

and has the value 1·66 for monatomic gases, approximately 1·40 for diatomic gases (under ordinary conditions the value for air is 1·41), and is less than 1·40 but greater than 1 for gases of higher atomicity.

If a small pressure increment δp causes a volume increment δV (which is, of course, really negative), then

$$(p + \delta p)(V + \delta V)^\gamma = pV^\gamma.$$

Dividing both sides by pV^γ,

$$(1 + \delta p/p)(1 + \delta V/V)^\gamma = 1,$$

expanding the second bracket and neglecting powers of $\delta V/V$ higher than the first,

$$(1 + \delta p/p)(1 + \gamma \delta V/V) = 1;$$

whence, by rearranging,

$$-V \frac{\delta p}{\delta V} = \gamma p.$$

Considering infinitesimally small changes,

$$-V\frac{dp}{dV}=\gamma p.$$

The adiabatic elasticity of a gas at pressure p dynes per sq. cm. is thus γp dynes per sq. cm.

Hence $$\frac{\text{adiabatic elasticity}}{\text{isothermal elasticity}}=\frac{\gamma p}{p}=\gamma.$$

This is a general result, true for real gases as well as ideal gases, and true also for solids and liquids—that the ratio of the adiabatic and isothermal elasticities is equal to the ratio of the specific heats at constant pressure and constant volume.

The compressions and rarefactions of sound waves succeed one another so rapidly that each is an almost perfectly adiabatic change. The adiabatic elasticity is thus that concerned with the propagation of sound in gases.

External work done during adiabatic change. For a finite adiabatic expansion of a given mass of gas from pressure p_1 dynes per sq. cm. and volume V_1 c.c., to pressure p_2 dynes per sq. cm. and volume V_2 c.c., as at every stage pV^γ a constant, say c, and $p_1V_1^\gamma=p_2V_2^\gamma=c$, the external work done by the gas during a small expansion δV at pressure p is as before $p\delta V$ ergs $=cV^{-\gamma}V$ ergs. The area beneath the appropriate *adiabatic* pV graph represents the work done, and the integral calculus shows that this external work is

$$\frac{c}{\gamma-1}\left(\frac{1}{V_1^{\gamma-1}}-\frac{1}{V_2^{\gamma-1}}\right)\text{ ergs,}$$

or,
$$=\frac{p_1V_1^\gamma}{\gamma-1}\left(\frac{1}{V_2^{\gamma-1}}-\frac{1}{V_1^{\gamma-1}}\right)\text{ ergs,}$$

as $pV^\gamma=c.$

This is the work done by the gas in extending its boundaries against an external constraint. If the volume of the gas is reduced, the result is negative. We can then either write that the work done by the gas is this negative quantity, or, dropping the negative sign, say that the external work done on the gas by the external agent is positive.

EXAMPLES ON CHAPTER 8

1. State Hooke's law and describe, with the help of a rough graph, the behaviour of a copper wire which hangs vertically and is loaded with a gradually increasing load until it finally breaks. Describe the effect of gradually reducing the load to zero, (*a*) before, (*b*) after, the elastic limit has been reached.

A uniform steel wire of density 7·8 gm. per c.c. weighs 16 gm. and is 250 cm. long. It lengthens by 1·2 mm. when stretched by a force of 8 kg. weight. Calculate (*a*) the value of Young's modulus for the steel, (*b*) the energy stored in the wire. (N.)

2. Explain the terms *stress*, *strain*, *modulus of elasticity*.

Describe with full experimental details a method of determining Young's modulus for a steel wire.

The upper end of a rubber cord is fixed and the extensions caused by suspending various masses (*M* gm.) from its lower end are measured. The results are given in the following table :

Mass *M* gm.	100	150	200	250	300
Extension in cm.	6·5	10·5	14·5	19·5	25·0

Draw a load-extension graph and from it determine the work done in increasing the extension of the cord from 7·5 cm. to 22·5 cm. (N.)

3. Define *Young's modulus of elasticity* and explain how you would determine this modulus for a steel wire.

Calculate the extension of a steel wire of length 4 metres and diameter 2 mm. when loaded with a weight of 8 kilograms, if Young's modulus for the steel is 2×10^{12} dynes per cm.^{2}. How much strain energy would be stored in the loaded wire? (N.)

4. Give a general account of the phenomena of elasticity. State the general law governing the behaviour of elastic substances under stress, and show how it is applied to the quantitative definition of the three principal kinds of elasticity.

Describe how Young's modulus can be determined experimentally, explaining the precautions taken to ensure accuracy. (O. & C.)

5. Describe in detail an experimental method for the determination of Young's modulus.

A mass of 200 gm. is fastened to one end of a flexible wire 100 cm. long, the other end of which is attached to a horizontal axis. The mass is caused to whirl in a vertical plane. Calculate its least velocity at the top and bottom of its path if the wire remains taut. If the wire is 1 mm. in diameter, and its Young's modulus is 10^{12} dynes/sq. cm., find its increase in length when the mass is in its lowest position. (O. & C.)

6. What is meant by a *modulus of elasticity*?

Define Young's modulus and calculate its value for a wire of diameter 0·02 in. if a load of 10 lb. extends its length of 25 in. by 0·025 in. What energy is stored in the wire when this load is hung on the wire? (O.)

7. Explain the possible meanings of the term " modulus of elasticity " as applied to (*a*) solids, (*b*) fluids, and describe a method of measuring one particular modulus in one case.

A steel bar is heated to a temperature of $T°$ C. and its ends are secured in rigid clamps. If the bar is now allowed to cool, prove that the temperature at which it breaks is independent of the distance between the clamps. (C. S.)

8. Assuming Young's modulus for brass to be 10^{12} dyne/cm.^{2}, find the energy stored in a rod of 1 sq. cm. section and 10 cm. in length if compressed by a load of 10 kilograms. (C. S.)

9. Describe the changes which take place when a wire is stretched by a gradually increasing force until it breaks. How would you determine Young's modulus for a material in the form of a wire?

A uniform wire fixed at its upper end has a weight attached at its lower end. If the strain energy per unit volume of the wire is 2×10^4 ergs per c.c. and the increase in length per unit length is 2×10^{-4}, find (*a*) Young's modulus, (*b*) the stress in the wire. (C. S.)

10. Assuming that the bulk modulus of water is $2{\cdot}2 \times 10^4$ kg./cm.^{2}, calculate the approximate depth below the surface of an ocean of pure water at which the density is one per cent. greater than the density at the surface.

Compare the law of increase of pressure with depth upon which you base your conclusions with the law of decrease of pressure with increasing height in an atmosphere supposed to be at the same temperature throughout. (C. S.)

11. Describe an accurate method of measuring Young's modulus.

A uniform horizontal rigid bar of mass 100 kg. is supported horizontally by three equal vertical wires A, B and C, each of initial length 1 metre and cross-section 1 sq. mm., B is a copper wire passing through the centre of the bar ; A and C are steel and are arranged symmetrically one on each side of B. If Young's moduli for copper and steel are $1{\cdot}5 \times 10^{12}$ dynes per sq. cm. and 2×10^{12} dynes per sq. cm. respectively, find the tension in each wire and the extension. (O. & C.)

12. Define Young's modulus, and describe briefly how you would measure it for the material of a wire.

A steel wire of diameter 0·8 mm. is clamped firmly at its ends A and B so that it is horizontal and just taut, the distance AB being 1 metre. A mass of 100 gm. is then hung from the mid-point C of the wire ; calculate the depression of C below the line AB. (Assume that the portions AC, CB of the wire are straight. Young's modulus for steel $= 2 \times 10^{12}$ dynes/cm.^{2}.) (C. S.)

13. What is meant by Young's modulus?

How would you attempt to find its value for a cylindrical bar of steel having a circular cross-section of radius 2 cm.?

A bar of this kind is heated to 300° C. and its ends are then fixed in rigid clamps. What force does it exert on the clamps when it cools

to 0 °C.? (Young's modulus for steel $= 2{\cdot}1 \times 10^{12}$ dynes per sq. cm. Coefficient of expansion of steel $= 1{\cdot}1 \times 10^{-5}$ per °C.) (C. S.)

14. Explain what is meant by a modulus of elasticity. Describe a method of determining Young's modulus for a wire.

A wire is suspended by one end. If its length is l, its area of cross-section A, and the density and Young's modulus of the material are ρ and q respectively, find the potential energy of the wire due to stretching under its own weight. (C. S.)

15. A steel rod, 10 cm. in length, is held by abutments at the ends which prevent any change in the length of the rod. What pressure will be exerted on the abutments if the temperature of the rod is raised by 50 °C.? (Young's Modulus for steel $= 2{\cdot}2 \times 10^{12}$ dynes/cm.^{2}. Coefficient of linear expansion $= 1{\cdot}2 \times 10^{-5}$ cm./cm./ °C.) (C. S.)

16. Define *stress, strain, elastic limit, Young's modulus.*

A thin uniform wire of radius r is clamped horizontally between rigid supports A and B, so that it is just taut and the length of wire between A and B is $2l$. A mass m is hung from the mid-point C, and displaces C through a vertical distance y, small compared with l. Find expressions for the stress and strain in the wire, and for Young's modulus, in terms of m, l, r, and y, neglecting in each case squares and higher powers of y/l in comparison with unity. (O. & C.)

17. Explain what are meant by *elastic limit, modulus of elasticity, Young's modulus.*

Describe how you would determine Young's modulus for a material in the form of a uniform wire, and make an estimate of the accuracy of the determination. (O. & C.)

18. Define Young's modulus, and deduce an expression for the work done when a wire is stretched.

A light wire of diameter 1 mm. and length 2 metres is hung horizontally between two supports 2 metres apart. If Young's modulus of the material of the wire is 10^{12} dynes per sq. cm., find what weight hung at its middle point will depress it 10 cm., and the work done in so doing. (O. & C.)

19. State Boyle's law, and describe an experiment by which it could be verified.

A glass globe of volume V c.c., filled with gas at pressure p, is to be evacuated by repeated connection with a second vessel of volume v, the latter vessel being completely evacuated after each connection and at the start. Obtain an expression relating the pressure of gas remaining in the first vessel with the number of connections made.

If the first vessel has a volume of 500 c.c. and the second one a volume of 50 c.c., how many connections will be necessary to reduce the pressure in the larger vessel by one-half? (O. & C.)

20. A vertical barometer tube 100 cm. in length, and dipping into mercury, contains a small quantity of air. When the open end is 15 cm. below the surface of the mercury, the meniscus is 30 cm. from the upper

closed end. When the tube is pushed into the mercury so that the open end is 35 cm. below the surface, the meniscus is 20 cm. from the closed end. Calculate the atmospheric pressure. (O. & C.)

21. State Boyle's law and describe an experiment for verifying the law.

A narrow glass tube of uniform section and sealed at one end contains air enclosed by a mercury thread of length 7·5 cm. When the tube is held vertically with the open end uppermost, the enclosed air column is 30 cm. long and when the tube is held horizontally the length of the air column is 33 cm. Explain why this change in length occurs, and calculate the atmospheric pressure. (O. & C.)

22. Give a good diagram of a simple form of *air pump*, such as is used for *exhausting* apparatus in the laboratory, and explain how it works.

The barrel of an ordinary bicycle pump has a volume of 200 c.c. It is connected to a vessel of volume 5 litres, which contains air at atmospheric pressure. Find (*a*) the pressure in the vessel after ten strokes, (*b*) how many strokes are needed to bring the pressure in the vessel up to 3 atmospheres. (O.)

23. State Boyle's law, and describe an experiment to verify it.

A 250 c.c. flask, fitted with exit tube and stop-cock, is attached to a rough vacuum pump and partially exhausted. When the flask is removed and the stop-cock opened under water and the water levels inside and out are equalised, it is found that 242 c.c. of water enters the flask. The pressure of the atmosphere is 75·5 cm. of mercury. Find the pressure produced in the flask by the pump. (O.)

24. State Boyle's law, and describe an experiment to verify it.

A wide capillary tube of uniform bore is sealed at one end. Some air is trapped in the closed end by means of a mercury thread 25 cm. long. When the tube is held vertical, with the open end upwards, the air occupies 30 cm. of the tube ; when the tube is inverted so that the open end points vertically downwards, the length of the air column is 60 cm. Calculate the height of the mercury barometer. (O.)

25. State Boyle's law.

The following readings were taken with a Boyle's law apparatus of the usual kind.

Reading of mercury level in closed limb, in cm.	Reading of mercury level in open limb, in cm.	Volume of air in c.c.
76	126	30
66	85	40
56	56	50
46	33·5	60
36	14·5	70

Plot a graph of *head of mercury* against 1/*volume*, and use this graph to deduce the height of the barometer, explaining your calculation. (O.)

26. State Boyle's law, and show in a general way how it is explained on the kinetic theory.

A barometer, the closed end of which is kept at a height of 867 mm. above the surface of the mercury in the open vessel, contains a little air above the mercury. It reads 757 mm. when the true barometric height is 765 mm. If the tube is uniform, and the temperature remains constant, what is the true barometric height when the faulty barometer reads 742 mm.? (O. & C.)

27. Assuming the temperature of the atmosphere to be constant, calculate the height at which the barometer stands at 75 cm., if its reading at sea level is 76 cm.

Density of air $= 1{\cdot}3 \times 10^{-3}$ gm./cm.^{3}
Density of mercury $= 13{\cdot}6$ gm./cm.^{3} (C. S.)

28. Describe two forms of barometer and discuss their relative merits for two practical purposes.

A mercury barometer is known to be defective and to contain a small quantity of air in the space above the mercury. When an accurate barometer reads 770 mm. the defective one reads 760 mm. and when the accurate one reads 750 mm., the defective one reads 742 mm. What is the true atmospheric pressure when the defective barometer reads 750 mm.? (C. S.)

29. Describe, with diagrams, the construction and mode of action of a pump that will reduce the pressure of the air in a vessel to 0·001 mm. of mercury. How are pressures of this order of magnitude measured?

A piston pump of effective volume 200 c.c. is used to exhaust a vessel of volume 1 litre. How many complete strokes will be required to reduce the pressure of the air in the vessel to one-hundredth of its initial value? (Neglect the volume of the connecting tubes, etc., and assume that the temperature remains constant.) (O. & C.)

30. What is the relation between the pressure and the volume of a constant mass of gas at constant temperature?

Describe some form of air pump and explain its action.

The bell-jar and barrel of an exhaust pump of the piston type have volumes of 3 litres and 200 c.c. respectively. Find the ratio between the initial density of the air in the jar and the density after three strokes of the pump. (N.)

CHAPTER 9

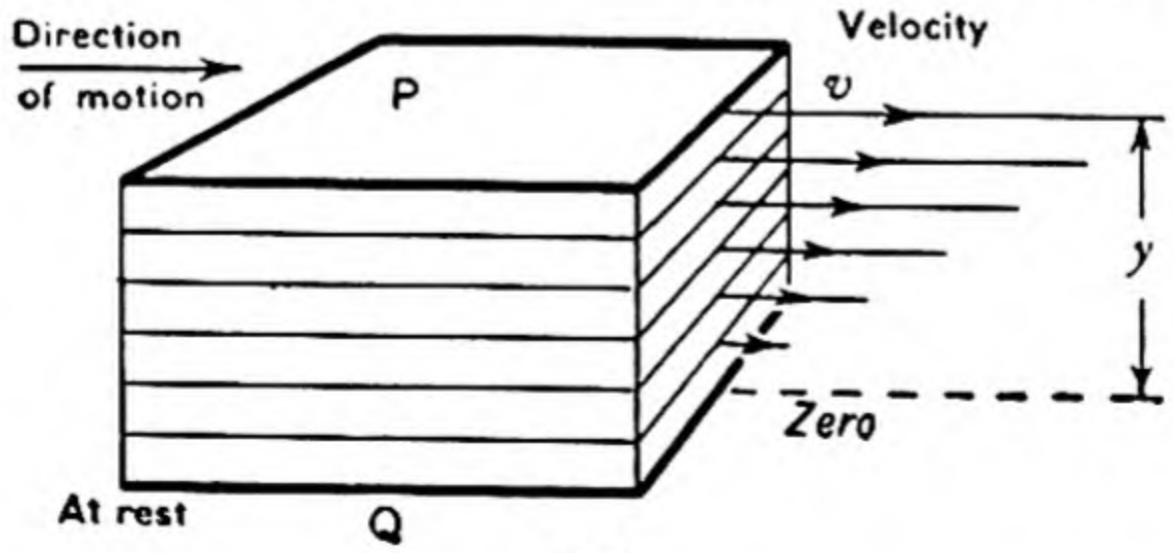
VISCOSITY AND SURFACE TENSION

Viscosity

Shear stress can be applied to, and transmitted through, a fluid, but shear strain cannot be permanently maintained therein. The response to an applied shear stress is simply a flow in the direction dictated by the stress. A fluid at rest has no "shape" in its own right, but takes that imposed by the containing vessel and other solid surfaces immersed in it; but while it is in motion from one vessel to another, or while solid bodies are moving through it, this shape is being *changed*.

Simple observations show that stress of some kind must be applied in order to set a fluid in motion, that a moving fluid exerts a stress on fixed solid bodies bounding it, and that a fluid resists the motion of a solid through it. It is also observed that the stress exerted by, and the resistance to motion through, liquids is very much greater than in the case of gases, while liquids vary considerably in this property among themselves. Liquids such as treacle and glycerin, for example, offer substantial resistance during any attempt to stir them; we say they are **viscous** liquids. Others, such as water, alcohol, ether, etc., are much less viscous. Gases, of course, show the effect; but their study is more difficult and will not be attempted here. We shall in this chapter consider the viscous behaviour of liquids.

Coefficient of viscosity. Consider a plane P of area A sq. cm.



moving parallel to itself with velocity v cm. per sec. as indicated in the diagram, while immersed in a liquid; below it, at distance y cm., is another parallel plane Q, which is at rest.

Under suitable conditions, it is found that each layer of the liquid parallel to P and Q moves steadily as a layer; that is, there is **laminar flow**, also called **streamline flow.**

The behaviour of a fluid under laminar flow was described by Newton (*Principia*, Book II, Prop. LI) with reference to the rotation of a long cylinder in a uniform fluid. He there showed that mechanical actions in an interplanetary fluid could not offer a satisfactory alternative to universal gravitation.

The velocity of the liquid in any one such layer is proportional to its distance from the fixed plane Q. There is thus a velocity gradient, v/y cm. per sec. per cm., at right angles to the planes P and Q, in the steadily flowing liquid.

It is found experimentally that, for a given liquid, provided this laminar flow is maintained, the *tangential stress* exerted on the liquid by P, transmitted through the liquid, and experienced by the plane Q, is *proportional to the velocity gradient.* The constant of proportionality is called the **coefficient of viscosity**, η, of the liquid. Thus,

$$\eta = \frac{\text{tangential stress}}{\text{velocity gradient}}.$$

The unit in which η is expressed is called the **poise** (in honour of Poiseuille, p. 249); 1 poise is the viscosity of a liquid in which a tangential stress of 1 dyne per sq. cm. maintains a velocity gradient of 1 cm. per sec. per cm., under laminar flow conditions. It should be emphasised that the coefficient of viscosity only has a meaning if laminar flow obtains.

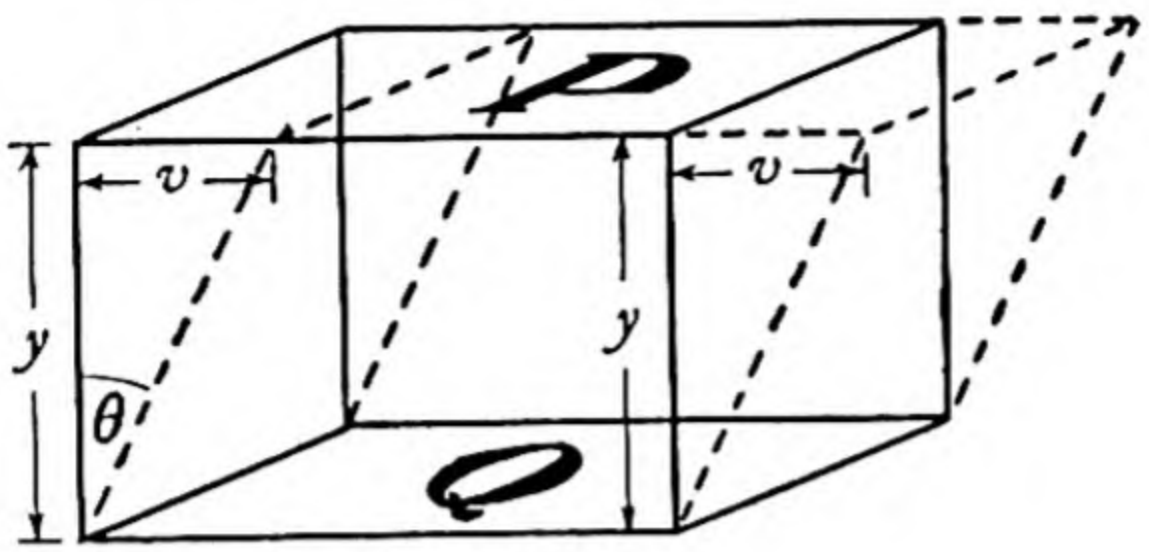
As the dimensions of tangential stress are $ML^{-1}T^{-2}$, and of velocity gradient LT^{-1}/L, the dimensions of η are

$$ML^{-1}T^{-2} \div (T^{-1}),$$

or,

$$ML^{-1}T^{-1}.$$

It is important to remember the dimensions of η, because direct analysis of the two chief problems ahead is difficult, and it is proposed to attack them by the method of dimensions.



Viscosity and rigidity. It will be seen that a fluid *in motion* behaves under shear rather like a solid *at rest*, and viscosity is closely analogous to rigidity.

If we imagine for a moment that the planes P and Q are bounding a slab of rigid material, and that in 1 sec. the upper plane P is displaced through a distance v cm. as indicated in the diagram, the shear strain produced in this second is v/y.

But if, in the actual liquid, P moves a distance v cm. in one second, liquid flows so as to destroy the shear strain that would have been produced in the absence of flow. Thus, in the liquid, v/y is the shear strain *destroyed* in one second, or the *rate of disappearance of shear strain*. Thus,

$$\text{velocity gradient} \equiv \text{rate of disappearance of shear strain};$$

so we can write

$$\eta = \frac{\text{tangential stress}}{\text{rate of disappearance of shear strain}}.$$

Comparing this with the expression for the rigidity modulus, n, of a solid (p. 200),

$$n = \frac{\text{tangential stress}}{\text{shear strain}},$$

it can be seen that the viscosity of a liquid *in motion* is a quantity essentially similar to the rigidity of a solid *at rest*.

The Method of Dimensions

An equation is necessarily a numerical statement, that is, a relation between numbers of things of the same kind. Each term in an equation must therefore be measured in the same units, and

therefore have the same dimensions. This **Principle of Homogeneous Dimensions** is useful in checking that an equation is not *physically* impossible. For example, the external work done by an isothermally expanding gas is given by

$$work = p_1 V_1 \log_e \frac{V_2}{V_1}.$$

The dimensions of work are $M^1L^2T^{-2}$,
of pressure, $M^1L^{-1}T^{-2}$,
of volume, L^3,

and of the logarithm zero, since all logarithms and exponents must necessarily be pure numbers and dimensionless. So, putting in dimensions of all the quantities,

$$M^1L^2T^{-2} = M^1L^{-1}T^{-2} \,.\, L^3.$$
$$= M^1L^2T^{-2},$$

hence the dimensions of both terms of the equation are the same. The equation is thus possible; but it may or may not be arithmetically correct, for we have no means of checking numerical factors, and the logarithm of any dimensionless ratio would fit.

Dimensions can be used to forecast the form of possible equations, as the following example shows.

The period τ of a simple pendulum is believed to be independent of the amplitude of swing and the mass of the bob, and to depend only on the length of the string, l, and the acceleration of gravity, g.

Let the relation be

$$\tau = k \,.\, l^x g^y.$$

The dimensions of τ are T^1; of l, L^1; of g, L^1T^{-2}; and of k, zero.

Making a dimensional equation,

$$T^1 = (L^1)^x \,.\, (L^1T^{-2})^y.$$

Collecting indices,

$$T^1 = L^{x+y} \,.\, T^{-2y}.$$

Now the dimensions of the left-hand and right-hand terms must be the same, so the powers of M, L, and T must individually be the same. Here M is not involved; L appears on the left-hand side as L^0 (not written down), and on the right-hand side as L^{x+y}, so

$$x + y = 0;$$

while T appears on the left-hand side as T^1 and on the right-hand side as T^{-2y}, so

$$-2y = 1.$$

Solving the two equations for x and y,

$$x = \tfrac{1}{2}\,; \quad y = -\tfrac{1}{2},$$

whence

$$\tau = k\,.\,l^{1/2}g^{-1/2}.$$

Three points should be noted :

(1) The equation gives no clue to the value of k, which may be a pure number (and is actually 2π if the swings are so small that the amplitude is as we believed irrelevant), or may involve a dimensionless ratio or product of factors we never considered. Thus if the amplitude or semi-arc of swing s is considerable, powers of the dimensionless ratio s/l do occur in k, which is found to be

$$2\pi\left(1 + \frac{s^2}{16l^2} + \ldots \text{ higher powers of } \frac{s}{l}\right).$$

But this could never be revealed by dimensions.

(2) We start by choosing only those factors already known or believed to be relevant, and discarding others. The justification for discarding the mass of the bob is experimental, but if it is included as m^z in the expression for τ, we soon see that $z = 0$. The amplitude of swing was discarded for the same reason, but as indicated in (1) it may for all we know be relevant.

(3) A simple functional relation, products of powers, is assumed.

Another example of similar type is : How does the pressure p due to a liquid column depend on the vertical height h, the density ρ and the acceleration of gravity g?

Let

$$p = k\,.\,h^x\rho^y g^z.$$

Putting in dimensions, these are :

for p, $ML^{-1}T^{-2}$; for h, L^1 ; for ρ, ML^{-3} ; for g, LT^{-2} ; for k, zero.

Making a dimensional equation :

$$ML^{-1}T^{-2} = L^x\,.\,(ML^{-3})^y\,.\,(LT^{-2})^z.$$

Collecting up indices :

$$ML^{-1}T^{-2} = M^y L^{x-3y+z} T^{-2z}.$$

Equating the indices of M, L, and T separately :

For M, $y = 1$; for L, $x - 3y + z = -1$; for T, $-2z = -2$,

so $x=1$, $y=1$, $z=1$, and

$$p = k \,.\, h\rho g.$$

As there are only three independent index equations available (for the powers of M, L, and T), relations involving more than three factors require rather more thought. Whatever the number of relevant factors, some useful information is always obtainable, but this does not emerge as a single explicit relation (see p. 252). But sometimes, as in the problem of flow through a capillary tube, a result is obtainable on the foregoing lines by reducing the number of independent factors, as by assuming that pressure difference p and length of tube l always occur together as the pressure gradient p/l.

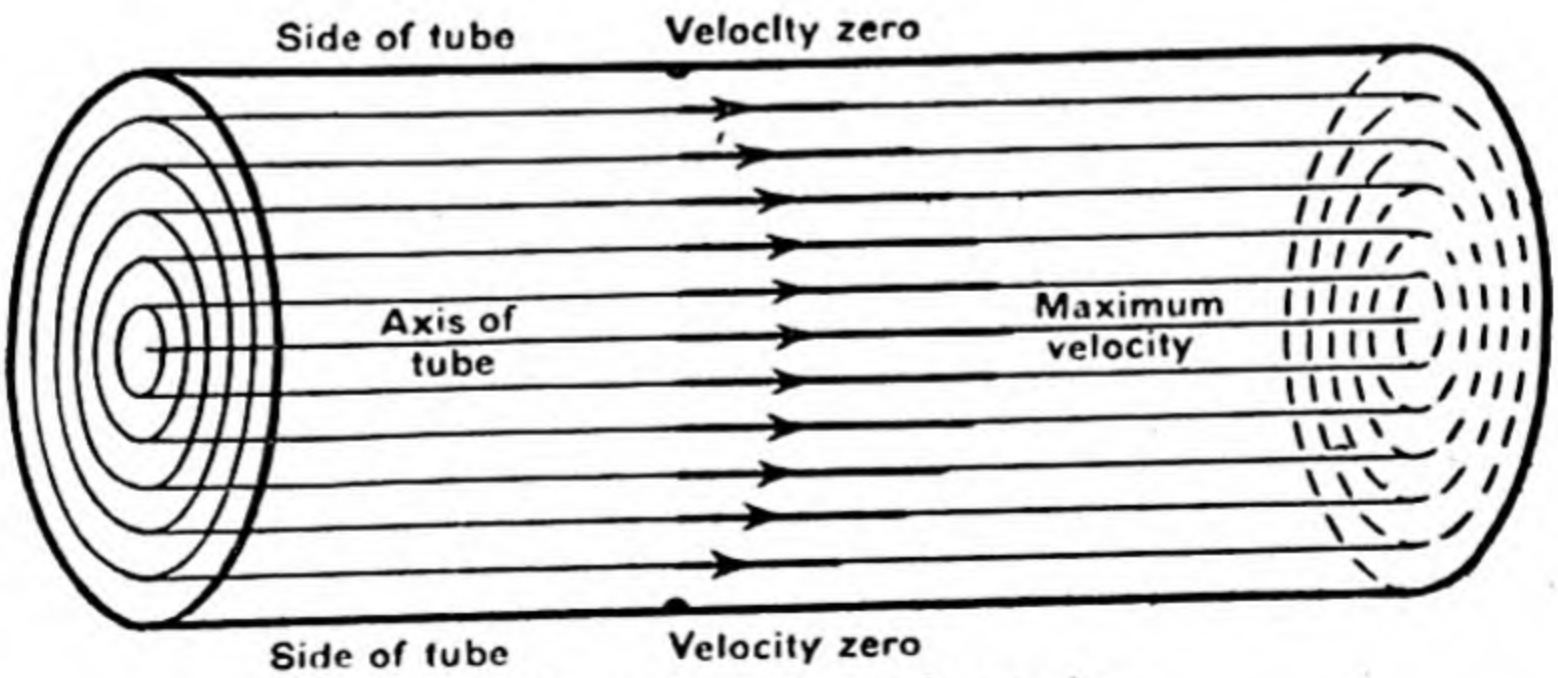
Laminar flow through a tube. Suppose that laminar flow is maintained in a liquid flowing through a cylindrical tube. The " laminae " are concentric cylinders, the outer of which is in contact with the walls of the tube and can be assumed at rest, while maximum velocity occurs along the axis of the tube, and the velocity distribution is as indicated in the diagram.

An expression can be found for the maximum velocity in terms of the radius of the tube (r cm.), the viscosity of the liquid (η poises), the pressure difference (p dynes per sq. cm.), and the length of the tube (l cm.).

First, suppose that p and l are together concerned as the *pressure gradient*, p/l.

Let u, the maximum velocity, be represented by the formula

$$u = k \,.\, \eta^x \,.\, r^y \,.\, (p/l)^z,$$



Laminar flow through a tube.

where k is a numerical constant, of zero dimensions, which we shall not be able to find by this method. The dimensions of the quantities concerned are :

for u, LT'^{-1}; η, $ML^{-1}T'^{-1}$;

r, L^1; and for p/l, $ML^{-1}T'^{-2}/L$, or $ML^{-2}T'^{-2}$.

Making a dimensional equation,

$$M^0L^1T'^{-1} = (ML^{-1}T'^{-1})^x \cdot L^y \cdot (ML^{-2}T'^{-2})^z$$

or, $$M^0L^1T'^{-1} = M^{x+z} \cdot L^{-x+y-2z}T'^{-x-2z}.$$

Equating separately the powers of M, L and T' :

$$0 = x + z ;$$

$$1 = -x + y - 2z ;$$

$$-1 = -x - 2z ;$$

giving $$x = -1 ; \ y = 2 ; \ z = 1 ;$$

whence $$u = k\frac{r^2p}{\eta l} \text{ cm. per sec.}$$

Detailed analysis of the problem shows that $k = 1/4$, so the true formula is

$$u = \frac{r^2p}{4\eta l} \text{ cm. per sec.}$$

It is important to know whether or not the flow through the tube is actually laminar. With laminar flow, it is easy to measure η by finding the volume transmitted per second by a tube of measured radius under a measured pressure gradient ; if the flow is turbulent, η plays an unimportant part in the matter. How can we tell what happens in any given case?

It was shown by Osborne Reynolds that streamline or laminar flow takes place for small values of u, up to a limiting critical velocity u_c. He found that there are really two critical velocities, a lower one at which laminar flow is unstable and turbulence possible, and a higher one above which turbulence is inevitable ; u_c stands for the lower critical velocity. Applying the method of dimensions, it can be shown that if any such critical velocity exists, and if it depends only on the density (ρ gm. per c.c.) and the viscosity η of the liquid

and the radius r of the tube, then

$$u_c = \frac{\kappa\eta}{\rho r},$$

where κ is a number, about 1150. The usual way in which this statement is put is to combine u, ρ, r, and η to give a dimensionless product, $\kappa = u\rho r/\eta$, which is here called the **Reynolds' Number** for the conditions discussed.

Reynolds' Number is really a name for the *form* of the dimensionless product $\frac{\rho}{\eta} \times a\ velocity \times a\ length$, rather than for any numerical value. Such products, involving the appropriate velocity and length, occur in many fluid-flow problems (cf. p. 252). In the present case the length can be either the radius r or the diameter d of the tube; the authorities listed in the bibliography vary in their choice, though d is preferred in technical literature.

If the Reynolds' Number, $u\rho r/\eta$, is less than about 1150 (or, if $u\rho d/\eta$ is less than about 2300), then streamline or laminar flow occurs; for higher Reynolds' Numbers flow is probably turbulent.

Substituting the value $r^2p/4\eta l$ for u in the expression

$$\kappa = \frac{u\rho r}{\eta},$$

we get for Reynolds' Number $\kappa = \frac{1}{4}\frac{r^3}{\eta^2} \cdot \frac{p}{l} \cdot \rho$;

so laminar flow occurs if

$$\frac{1}{4}\frac{r^3}{\eta^2} \cdot \frac{p}{l} \cdot \rho \text{ is less than } 1150.$$

Clearly, in designing apparatus, we must try to avoid tubes of large radius, and the use of big pressure gradients.

The dangers of setting up apparatus at random may be seen from the following figures, (*a*) actually used in an experiment, and (*b*) differing very little from these, and made up for an example.

(*a*) $r = 0{\cdot}075$ cm., η (at 12° C.) $= 0{\cdot}013$ poise, $p/l = 1{\cdot}2 \times 10^3$ dynes per sq. cm. per cm., gives a Reynolds' Number of

$$\frac{1}{4} \times \frac{4{\cdot}2 \times 10^{-4} \times 1{\cdot}2 \times 10^3}{1{\cdot}69 \times 10^{-4}} = \text{about } 740.$$

(*b*) $r = 0{\cdot}1$ cm., $p/l = 10^3$ dynes per sq. cm. per cm., η (at 20° C.) $= 0{\cdot}01$ poise, gives a Reynolds' Number of

$$\frac{1}{4} \times \frac{10^{-3} \times 10^3}{10^{-4}} = 2500.$$

In the first case, flow is laminar, in the second case turbulent.

It is obviously necessary, to be on the safe side, to work in all experiments of this kind with a long narrow tube and a small pressure difference, so far as possible.

Volume transmitted per second ; Poiseuille's formula. The form of the expression relating the volume of liquid transmitted per second, V c.c. per sec., the coefficient of viscosity, η poises, the radius, r cm., of the tube, and the pressure gradient, p/l dynes per sq. cm. per cm., can be found by the method of dimensions.

Let $$V = k.\eta^x \,.\, r^y \,.\, (p/l)^z,$$

where k is some numerical constant with no dimensions.

The dimensions of V are L^3T^{-1}; of η, $ML^{-1}T^{-1}$; of r, L^1; and of (p/l), $ML^{-2}T^{-2}$.

Making a dimensional equation,

$$M^0L^3T^{-1} = (ML^{-1}T^{-1})^x \,.\, L^y \,.\, (ML^{-2}T^{-2})^z$$

or, $$M^0L^3T^{-1} = M^{x+z} \,.\, L^{-x+y-2z} \,.\, T^{-x-2z},$$

and, equating the indices of M, L and T separately,

$$0 = x + z,$$
$$3 = -x + y - 2z,$$
$$-1 = -x - 2z,$$

whence $x = -1$, $y = 4$, $z = 1$.

Thus $$V = k\frac{r^4p}{\eta l} \text{ c.c. per sec.},$$

a result in agreement with the empirical formula of Poiseuille (1842), who showed that the volume issuing per second was proportional to the pressure gradient and to the fourth power of the radius of the tube.

Proper analysis shows that the numerical value of the constant k is $\pi/8$;

$$\therefore \; V = \frac{\pi r^4 p}{8\eta l} \text{ c.c. per sec.}$$

This equation is known as **Poiseuille's formula.** It may be arranged to give η explicitly,

$$\eta = \frac{\pi r^4 p}{8Vl} \text{ poises.}$$

Measurement of η by the capillary-flow method. The radius r cm. of the capillary is measured by, for example, finding the mass of a mercury thread which occupies a measured length. A narrow capillary should be selected, for reasons explained on p. 248, to ensure laminar flow.

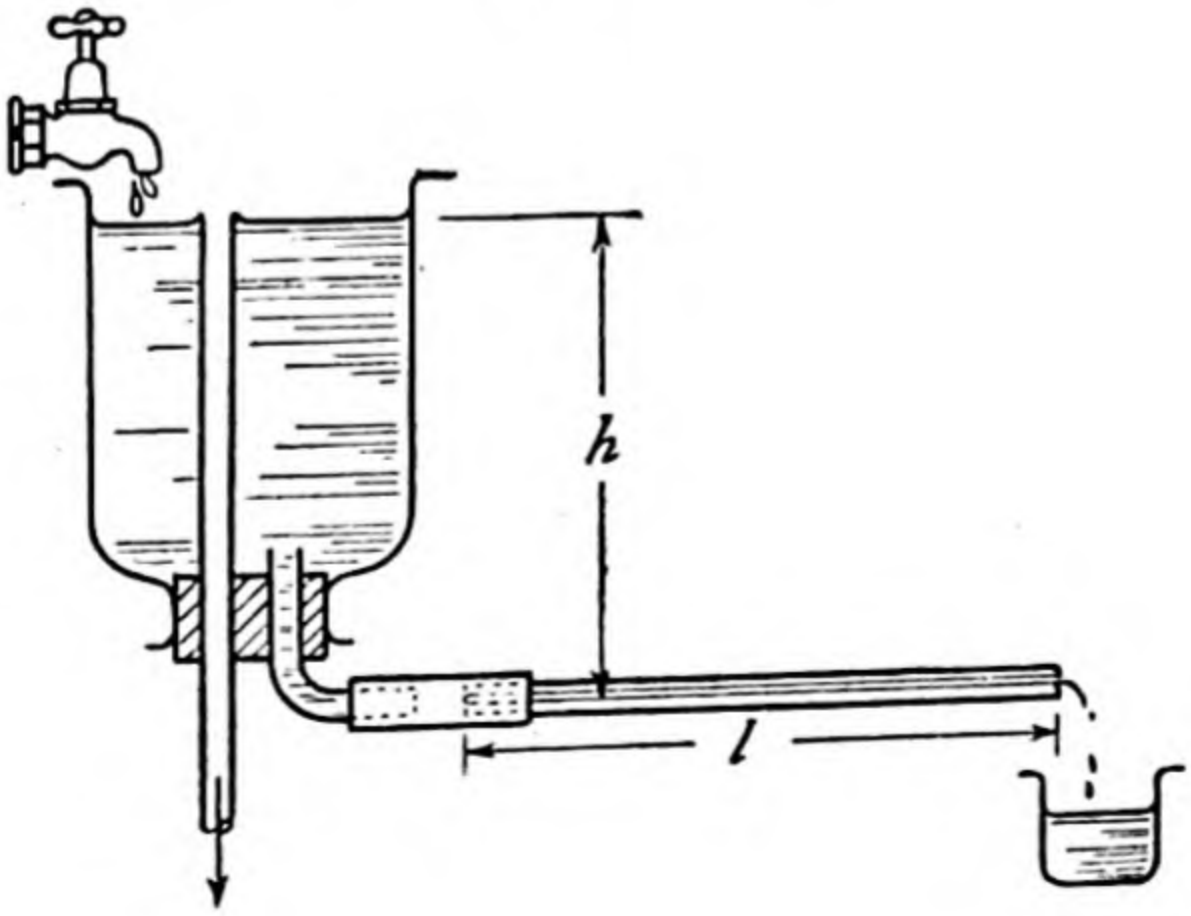
The density ρ gm./c.c. of the liquid is found by any convenient means. The length l cm. of the tube is measured.

The apparatus illustrated is then assembled, with the capillary mounted horizontally at a distance h cm. below the free surface of the liquid in the reservoir, which should be fed from above rather faster than the liquid can escape through the capillary, so that this head is constant. Tap-water is the liquid indicated in the diagram.

The volume of liquid flowing through the tube in a measured time is collected and measured, and thus V, the volume flowing per second, is determined.

The pressure difference across the ends of the tube is $hg\rho$ dynes per sq. cm., and the pressure gradient p/l is $hg\rho/l$ dynes per sq. cm. per cm., so η is obtained by substituting in the Poiseuille formula,

$$\eta = \frac{\pi r^4 p}{8Vl} \text{ poises.}$$



The work done per second by the potential energy sacrificed by the liquid flowing is $hg\rho V$ ergs. Some part of this is used in providing the issuing jet with kinetic energy. The work actually done in overcoming the viscous resistance of the liquid is thus less than $hg\rho V$, say $h'g\rho V$ ergs, where h' is slightly less than h. It follows that a correction should be made for the kinetic energy of the issuing jet, since the measured h is really an over-estimate of the effective pressure head. Details of this correction are given in the standard practical text-books.

Resistance opposing motion of a solid sphere. A solid sphere moving through a fluid imparts a shear stress to the fluid, with resulting transient shear strain, as it cleaves its way through. The sphere itself experiences a viscous drag tending to oppose motion. The size of the force on the sphere depends in the first place on whether the motion of the fluid relative to the sphere is "laminar", or whether it is turbulent. Assuming laminar conditions, the resistance F dynes must depend on the velocity, v cm. per sec., and viscosity, η poises, of the fluid, and the radius r cm., of the sphere. In any ordinary experiment the walls of the container will not be so remote that their presence can be neglected, and the dimensions of the containing vessel may affect conditions. But it can be supposed to start with that there is a large expanse of fluid.

Using the dimensional method again, let

$$F = k.\eta^x \,.\, v^y \,.\, r^z,$$

where k is some numerical constant.

The dimensions of F are MLT^{-2}; of η are $ML^{-1}T^{-1}$; of v are LT^{-1}; and of r are L^1.

Making a dimensional equation,

$$MLT^{-2} = (ML^{-1}T^{-1})^x \,.\, (LT^{-1})^y \,.\, (L)^z$$

or,

$$MLT^{-2} = M^x L^{-x+y+z} T^{-x-y}.$$

Equating the indices of M, L, and T separately,

$$1 = x; \quad 1 = -x + y + z; \quad -2 = -x - y.$$

Hence $x = 1$, $y = 1$, $z = 1$, and the form of the equation is

$$F = k\eta rv.$$

The true value of k, obtained by analysis, is 6π, so

$$F = 6\pi\eta rv \text{ dynes.}$$

This is known as **Stokes' formula.** It is true only for small values of v (so small that powers of v higher than the first can be neglected) in a large expanse of fluid, and if there is no slipping between fluid and sphere.

If the sphere is moving along the axis of a tube of radius R cm., the only addition to the formula which is dimensionally suitable and leads to the above equation when R is infinite, is a series involving powers of r/R, and the usual form of the corrected equation is

$$F = 6\pi\eta rv\left(1 + a\frac{r}{R}\right) \text{ dynes},$$

where a is a numerical coefficient (about 2·4), usually determinable experimentally. A further correction of the same type is made for finite depth of the fluid.

The Reynold's Number relevant in this connection is the product $\frac{r\rho v}{\eta}$, where r is the radius of the sphere. If r is such that $\frac{r\rho v}{\eta} < 1$, laminar flow conditions hold, and Stokes' formula applies.

Dimensional problems with more than three factors. We have conveniently supposed in deriving the form of the Poiseuille and Stokes' formulae that the density of the liquid was not involved, and then discussed the important part played by the Reynolds' Number $rv\rho/\eta$ which includes the density ρ.

We cannot have it both ways, and it is only honest to admit that ρ was omitted because we knew beforehand that it would not appear in the answer we happened to want, and because its inclusion would give us four unknown indiçes to solve by means of three independent equations.

Let us begin again, this time assuming that ρ is involved.

(a) *Resistance to a moving sphere*

With the same symbols as before, let the viscous resistance to a sphere be given by

$$F = k \,.\, \rho^w \eta^x v^y r^z,$$

and the dimensional equation

$$MLT^{-2} = (ML^{-3})^w \,.\, (ML^{-1}T^{-1})^x \,.\, (LT^{-1})^y \,.\, L^z,$$
$$= M^{w+x} \,.\, L^{-3w-x+y+z} \,.\, T^{-x-y}.$$

Giving $x+y=2$; $w+x=1$; $-3w-x+y+z=1$.

Expressing all in terms of w,

$$x=1-w\ ;\ \ y=1+w\ ;\ \ z=1+w.$$

So
$$F=k\rho^{w}\eta^{1-w}v^{1+w}r^{1+w},$$

or
$$F=k\ .\ \eta vr\left(\frac{\rho vr}{\eta}\right)^{w}.$$

The dimensions of the product $\rho vr/\eta$, which is Reynolds' Number, are zero, and so w may have any value ; though, of course, it does not follow that every value of w gives an equation with a physical meaning.

If
$$w=0,\ \ F=k\eta vr,$$

as holds for small values of v.

If
$$w=1,\ \ F=k\rho v^{2}r^{2},$$

or the resistance is proportional to the square of the velocity and independent of the viscosity.

If $w=2$, the resistance is proportional to the cube of the velocity, and both ρ and η are involved.

The equation may be rewritten

$$\left(\frac{F}{\eta vr}\right)\left(\frac{\rho vr}{\eta}\right)^{-w}=k,$$

each of the brackets being dimensionless. Any products or arrangement of products of these brackets will still be dimensionless, so

$$\left(\frac{F}{\eta vr}\right)^{q}.\left(\frac{\rho vr}{\eta}\right)^{-w}=k,$$

where q and w may be any numbers of either sign, gives the general result. The left-hand side might, of course, include a number of such terms with different q and w.

Here we started with four unknown indices, and obtained useful relations with one undetermined. Or, starting with the five independent quantities F, η, v, r, ρ, we have found relations involving two independent dimensionless products. We can always extract some useful information in this way ; if we are given n independent quantities, the three index equations lead to relations involving $(n-3)$ independent dimensionless products. Often it is better to try to find the dimensionless products at the start, instead of using the index equations, in the following way :

(b) *Flow through capillary tube*

Let the velocity u of axial flow in a capillary tube depend on

$$r,\ p/l,\ \eta, \text{ and } \rho.$$

One dimensionless product available from these five quantities is the appropriate Reynolds' Number

$$\rho ur/\eta\ ;$$

another is $\dfrac{r^2}{u\eta} \cdot \dfrac{p}{l}$, as is seen from p. 247 ;

another is $\dfrac{r}{\rho u^2} \cdot \dfrac{p}{l}$, which is the quotient of the first two and not independent of them.

So all possible relations are of the form

$$\left(\frac{r^2}{u\eta} \cdot \frac{p}{l}\right)^a \cdot \left(\frac{\rho ur}{\eta}\right)^b = \text{constant.}$$

If $a=1$ and $b=0$, the result of p. 249 follows.

If $a=1$ and $b=1$,

$$u = k\sqrt{\frac{r}{\rho} \cdot \frac{p}{l}},$$

or the velocity is proportional to the square root of the pressure gradient and independent of η, as is actually found for turbulent flow through wide tubes.

Terminal velocity of a falling sphere. A sphere falling vertically through a viscous fluid (see diagram) is acted on by:

(a) its weight, W dynes, acting downwards,

(b) the Archimedes' upthrust of buoyancy, U dynes, equal to the weight of fluid displaced, acting upwards, and

(c) the viscous resistance F dynes, acting upwards.

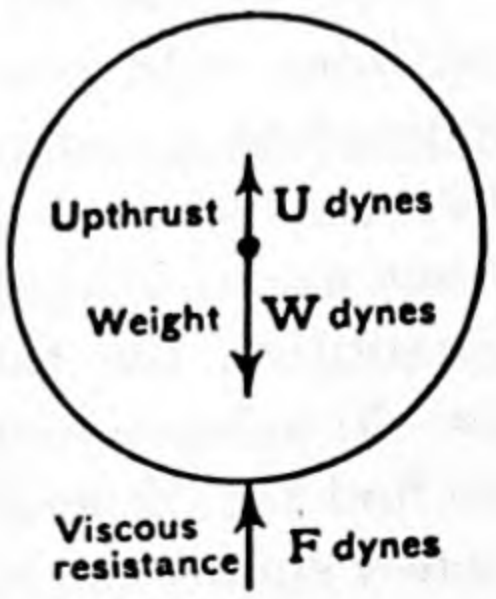
The resultant downward force is then

$$W - U - F \text{ dynes ;}$$

and the sphere is accelerated until F has reached such a value that

$$W - U - F = 0.$$

The sphere has then attained its terminal velocity, and will travel uniformly with this velocity.



Let ρ gm. per c.c. be the density of the fluid, and σ gm. per c.c. the density of the material of the sphere, the other symbols being as already used in this chapter. Suppose the fluid is extensive.

The mass of the sphere is $\frac{4}{3}\pi r^3\sigma$ gm., and its weight W is $\frac{4}{3}\pi r^3\sigma g$ dynes.

The mass of fluid displaced is $\frac{4}{3}\pi r^3\rho$ gm., and its weight U, equal to $\frac{4}{3}\pi r^3\rho g$ dynes, is the upthrust.

If v cm. per sec. now represents the terminal velocity, and F the maximum viscous resistance,

$$F = 6\pi\eta rv \text{ dynes.}$$

Since $F = W - U$,

$$6\pi\eta rv = \tfrac{4}{3}\pi r^3 g(\sigma - \rho) :$$

whence

$$v = \frac{2}{9}\frac{r^2 g(\sigma - \rho)}{\eta} \text{ cm. per sec., so } \eta = \frac{2}{9}\frac{r^2 g}{v}(\sigma - \rho) \text{ poises.}$$

Therefore by measuring v, r, σ, and ρ, the viscosity η can be calculated.

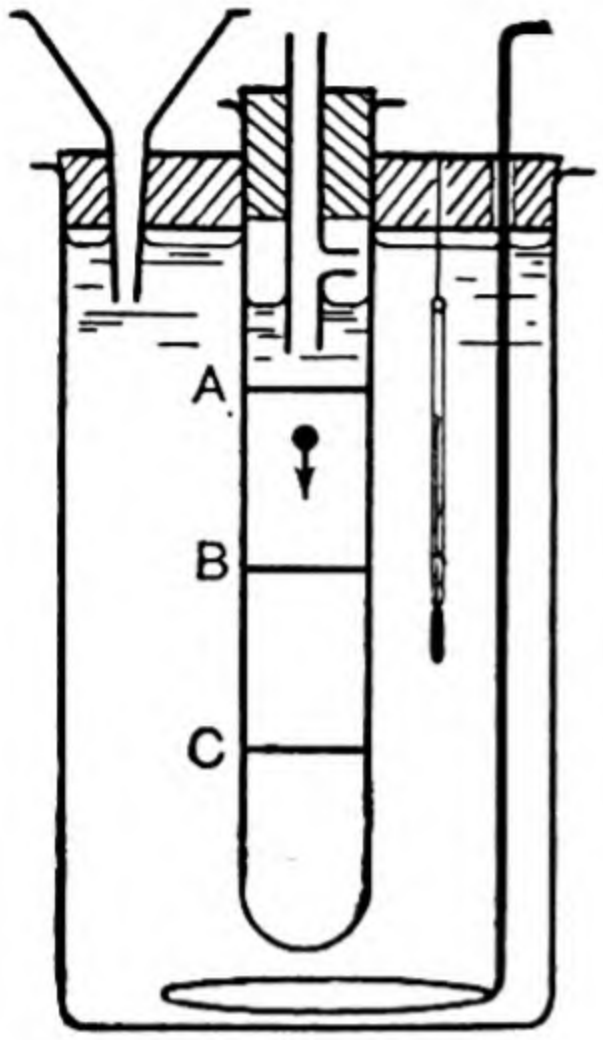
The term " fluid " has been used in place of liquid throughout this section, as the argument and findings are approximately true if the fluid is a gas—and even hold if the sphere is a small drop of liquid falling through air or other gas. Thus, if η is known and v measured, the radius of very small drops falling through the air can be measured, since

$$r = \sqrt{\frac{9\eta v}{2g(\sigma - \rho)}}$$

This method was used by Townsend, and in a corrected form by Millikan, to determine the size of small charge-bearing droplets as part of their respective pioneer experiments on the determination of the electronic charge.

Measurement of η by the falling sphere method. This is an excellent method for the measurement of η for a viscous liquid such as glycerin at room temperatures.

The liquid is contained in a tube, carefully set up vertically (see diagram). This is by no means the infinite ocean of fluid demanded by the simple theory, but with small ball-bearings of about 2 mm. diameter falling down the axis of a tube of about 2 cm. diameter, the error is not considerable. (It could be evaluated, and the factor



a, which is about 2·4, of the formula of p. 252 found, varying the conditions by using spheres of different small diameters.)

The density of the liquid, ρ gm. per c.c., and that of the material of the spheres, σ gm. per c.c., are first determined by the specific gravity bottle or other suitable means.

Three reference marks A, B, and C are made on the tube, say by fine rubber rings; A should be several centimetres below the free surface of the liquid, and the distances AB and BC equal.

The diameter of a ball-bearing is carefully measured using a micrometer screw gauge. It is moistened with the liquid and then gently set on the surface; these are necessary precautions, as a dry sphere sent in with a splash usually carries with it a small parachute of air for at least part of the journey. The times t_1 and t_2 sec. taken to cover the distances AB and BC are measured. These should be very nearly equal; if they are not, the terminal velocity had certainly not been reached at A and we have no means of knowing whether or not it was reached even at B, so the whole experiment fails. If they are nearly equal, then $(AB+BC)/(t_1+t_2)$ gives the terminal velocity v, and η is found by substituting in the formula

$$\eta = \frac{2r^2g(\sigma-\rho)}{9v} \text{ poises.}$$

Variation of η with temperature. The viscosity of every liquid decreases as the temperature rises, and the change with temperature is very considerable. Thus, for water, η is $1{\cdot}79\times10^{-2}$ poise at 0° C., and $6{\cdot}6\times10^{-3}$ poise at 40° C.; for glycerin, 46·0 poises at 0° C., and 3·5 poises at 30° C.; for pitch (a true liquid) $5{\cdot}1\times10^{11}$ poises at 0° C., and $1{\cdot}19\times10^{2}$ poises at 100° C.

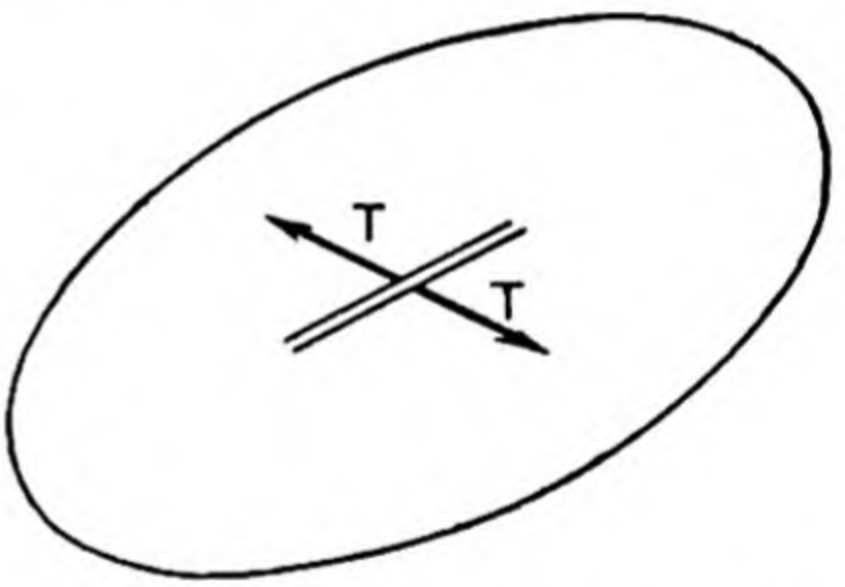
Surface Tension

Several effects, all grouped together as aspects of the same general phenomenon, are observed at the surface of a liquid.

The free upper surface of a liquid at rest in a vessel is able to

support small objects, themselves denser than the liquid; small quantities of liquid gather themselves up into drops, which approach a spherical shape more nearly if the drops are very small; ripples, distinguishable from the "gravity waves" which involve the body of the liquid, are propagated along the surface of a liquid in a trough. All these observations suggest that the surface of the liquid behaves rather as if it were a stretched skin or membrane. This is intelligible if we suppose that the attraction between neighbouring molecules of liquid greatly exceeds the attraction between the surface layer of liquid molecules and the molecules of air or vapour above it; any increase in surface area, as by the small depression of the surface by a solid placed on it, the distortion of a drop, or the displacement of the surface in wave propagation, thus does work against this molecular attraction, or increases the potential energy of the system.

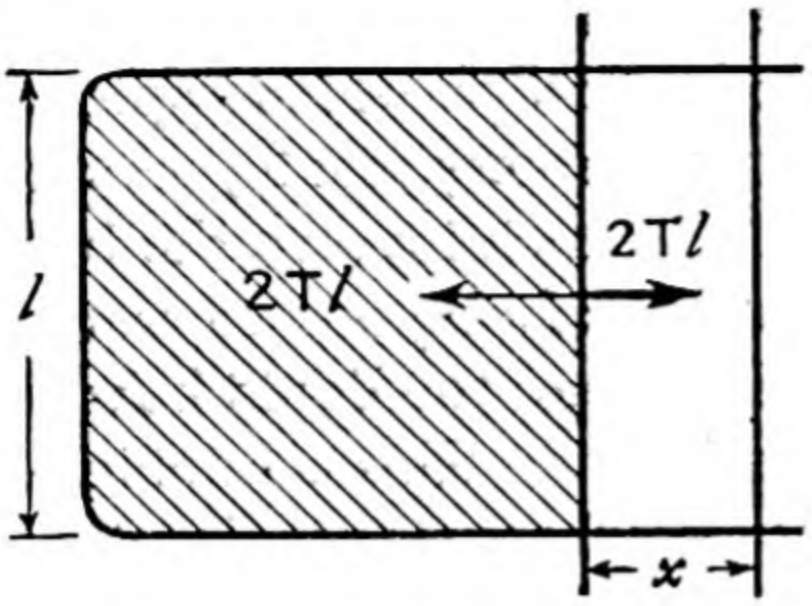
Definition of surface tension. Imagine a line described in the free surface of the liquid. Elements of surface on this line are in equilibrium, just as are elements of string in a stretched string, because the forces acting on them from either side are equal and opposite. Considering one side of this line, experiments suggest that the force acting at right angles to it is proportional to the length of the line.



The force per unit length acting at right angles to a line described in any direction in the surface of a liquid (and acting in, or tangentially to, the surface) is called the **surface tension**, or coefficient of surface tension, of the liquid. The usual symbol is T; the units are dynes per centimetre; and the dimensions of T are those of force divided by length, or $M \times L \times (Time)^{-2}/L$, which is $M \times (Time)^{-2}$.

There is one very important distinction between the surface of a liquid and the stretched skin to which we liken it. For a solid membrane, any extension of the area increases the tension in the membrane. The surface tension of a liquid is independent of the area of the surface, and is not altered by any change in the area of the surface.

Surface energy. Consider a wire frame, with two parallel sides at distance l cm. apart, completed as in the diagram by a thin wire which

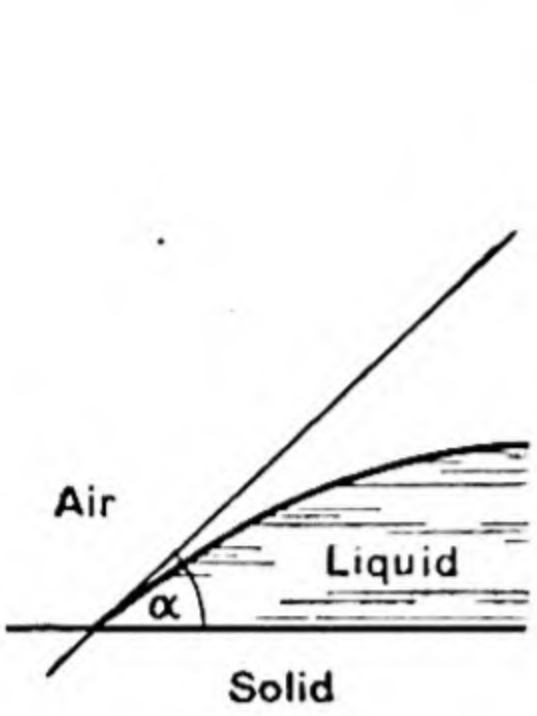


is free to move. Suppose that we have a soap-film (which is simply a layer of liquid bounded by two surfaces) on the frame, and that the surface tension of the liquid soap-solution is T dynes per cm. The wire experiences a force of Tl dynes due to each face of the film, in the direction shown, and in order to balance this an external force of $2Tl$ dynes must be applied. Now let the wire move outwards a distance x cm. The work done on the system by the external force is $2Tlx$ ergs; the film has increased in area by $2lx$ sq. cm. Hence, the work which must be done on the film to produce unit area of fresh surface is $2Tlx/2lx$, or T ergs. If, during the operation, there has been no other means by which the film can gain energy, then the gain in potential energy of the film is T ergs per square centimetre. Starting with the wire adjacent to the parallel fixed end of the frame, we can imagine that the whole of the film was formed in this way, and that each square centimetre of it is thus a reservoir of T ergs of potential energy.

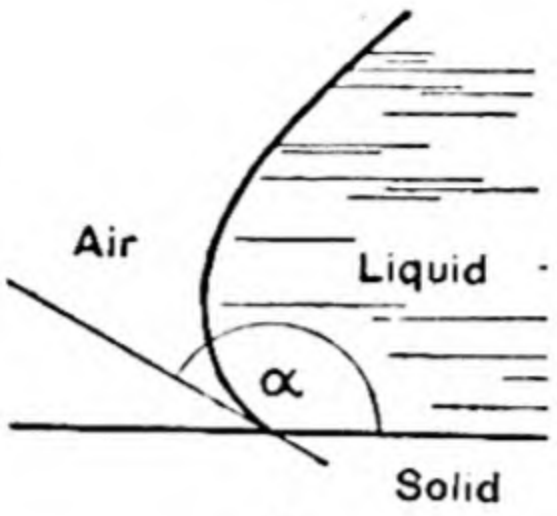
The work which must be done on the surface to produce 1 sq. cm. of fresh surface is, with the assumptions we have made, the same as the potential energy per unit area, called the free energy per unit area, or more simply the **surface energy** of the film. Its value is T ergs per sq. cm. Actually, extending such a film causes a fall in temperature, so that there is a flow of heat from the surroundings to the film, and the total energy of unit area is greater than T ergs.

Boundary of liquid and solid. Where the free surface of a liquid meets a solid boundary, the two intersect at a definite angle, α, which depends on the nature of the liquid and of the solid. The **angle of contact** between a liquid and a solid is the angle, measured within the liquid, between the solid surface and the tangent plane to the liquid surface at the point of intersection.

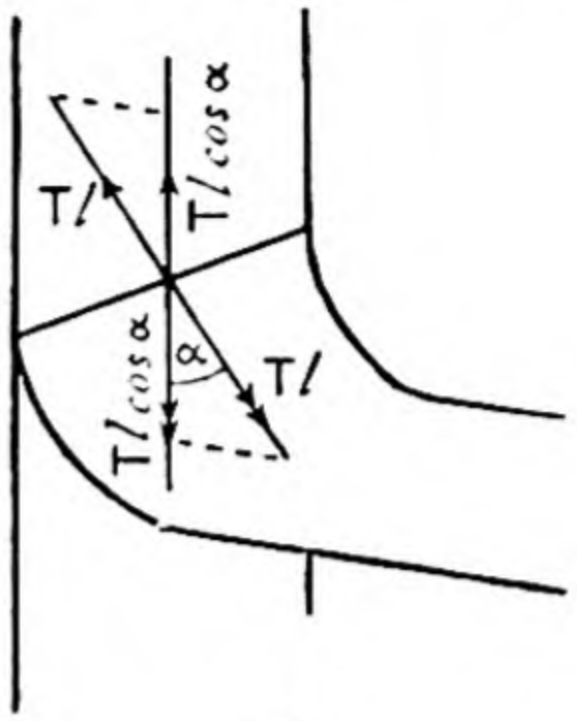
If α is zero, the liquid wets the solid; both in this case, and also if α is less than 90°, the liquid surface at the junction is shaped as Fig. *a*. If α is greater than 90°, for example, with mercury on clean



(*a*)



(*b*)



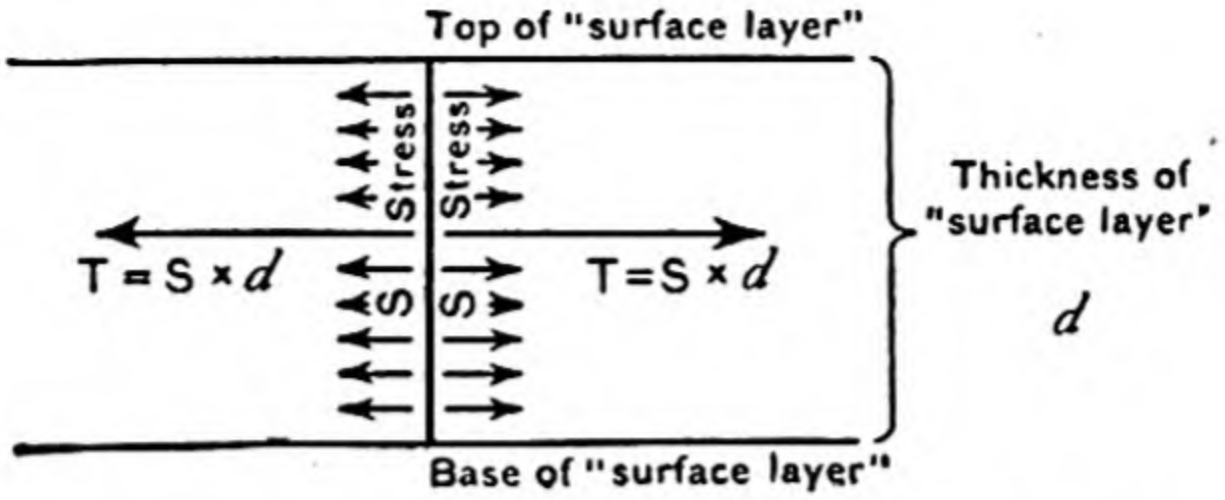
(*c*)

glass, where the angle of contact is about 136°, the liquid surface at the junction is shaped as Fig. *b*. The angle of contact may be measured roughly by inclining a flat glass plate in the surface of the liquid until the surface appears to be continuous right up to the glass.

Fig. *c* represents a portion of a liquid - solid boundary, l cm. long; the angle of contact is α, and the surface tension T dynes per centimetre. At the boundary the liquid exerts on the wall a force Tl dynes in the direction shown, and the wall exerts an equal and opposite force Tl dynes on the liquid. If α is less than 90°, the vertical component of this force, $Tl \cos \alpha$, is so directed as to pull down on the wall, and the reaction pulls the liquid upwards; if α is greater than 90°, the solid wall is pulled upwards, and the reaction on the liquid pushes it downwards.

The rise (or fall) of liquids in narrow tubes (**capillarity**) can be explained in this way, as is done on p. 275. Liquids for which the angle of contact is less than 90° "climb up" narrow tubes, while mercury and other liquids for which this angle is obtuse "climb downwards". The converse of this effect is shown when a small object is placed on the surface of a liquid. A clean steel needle, for example, placed on water is drawn through the surface and sinks, for as α is zero the resultant boundary forces on the solid act downwards; if the needle is greased, so that α is greater than 90°, the resultant force on the needle is directed upwards, so that its weight can be supported by the surface.

Further discussion. Students are often reluctant to accept the idea of a tension acting at right angles to a "line in the surface",



and even some standard advanced books regard this as a fiction, useful or otherwise. There is a real difficulty here, which is worth examination in terms of *stress*.

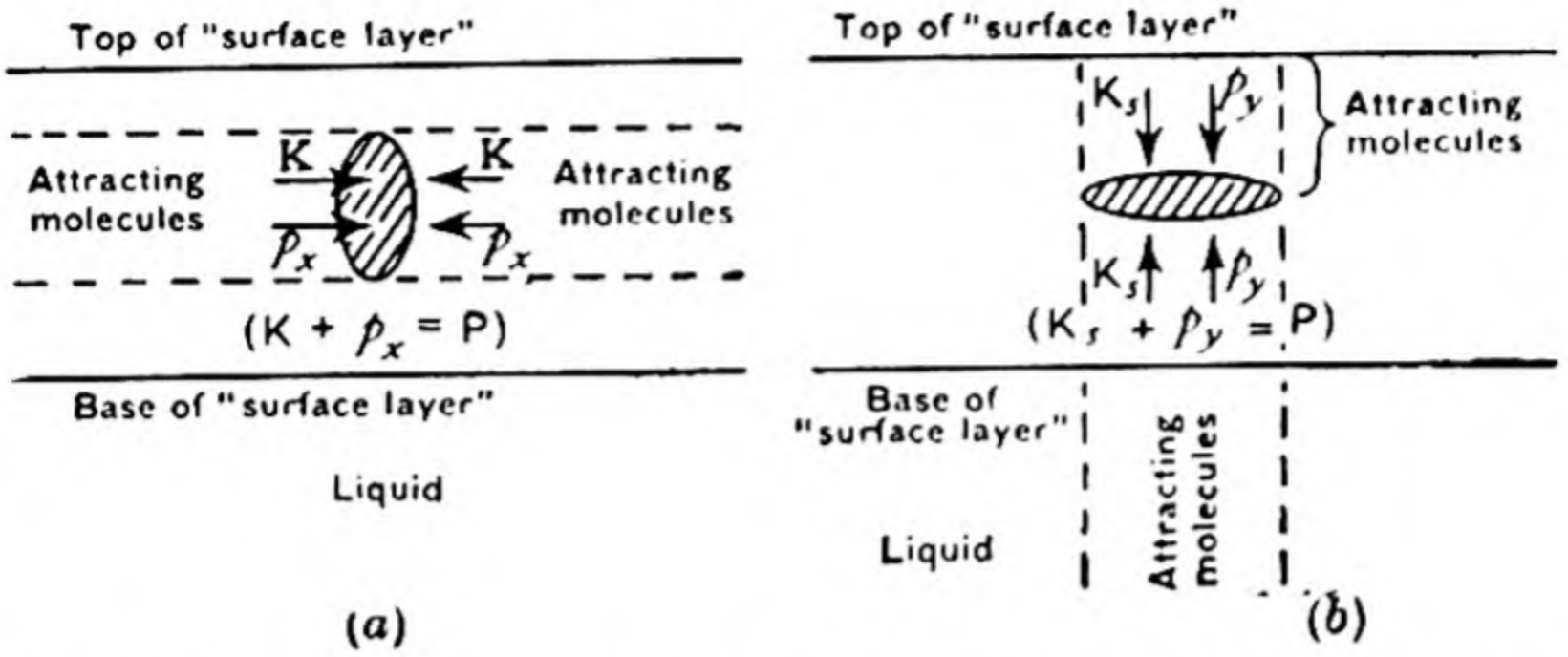
It is agreed that a liquid surface does behave as a stressed membrane. Imagine that the "surface-layer" has a thickness d cm., and that there is a uniform stress S dynes per sq. cm., then the force acting at right angles to an element 1 cm. long and d cm. deep is $S \times 1 \times d$ dynes, whence $T = S \times d$. The existence of this stress can be explained, and guesses at the value of d can be made, but neither can be measured; the surface tension T is the observed quantity. For water ($T = 73$ dynes per cm.), if we imagine d to be of the order 10^{-5} cm., which is about the order of the thinnest soap films on wire frames, S works out to $7{\cdot}3 \times 10^6$ dynes per sq. cm., which is well below the measured tensile strength (p. 216); while if d is 10^{-8} cm., of the order of molecular diameters, S is $7{\cdot}3 \times 10^9$ dynes per sq. cm. still below the stress calculable from the van der Waals's intrinsic pressure (p. 222).

Dr. R. C. Brown (*Proceedings of the Physical Society*, May 1947) describes Bakker's stress explanation of surface tension. The total pressure P at any point in a liquid at rest is the sum of the hydrostatic pressure p and the intrinsic pressure K; the total pressure P, and not p, is the quantity that is the same in all directions at a given point—that is, whatever the values of p and K individually,

$$P = p + K \quad \text{is the same in all directions.}$$

Within the surface layer (which is supposed of small but unknown thickness), take a small element of area lying at right angles to the surface (Fig. *a*). On both sides of it there is a long column of liquid parallel to the surface, and the effects of molecular attraction in this direction are the same as if it were in the body of the liquid, so that

$$P = p_x + K,$$



where p_x is the hydrostatic pressure parallel to the surface, and K the intrinsic pressure within the liquid.

Now take another element at the same place, lying parallel to the surface (Fig. *b*). Above this there are few molecules to exert attractions, and the intrinsic pressure must be the same on both sides; but as before,

$$P = p_y + K_s,$$

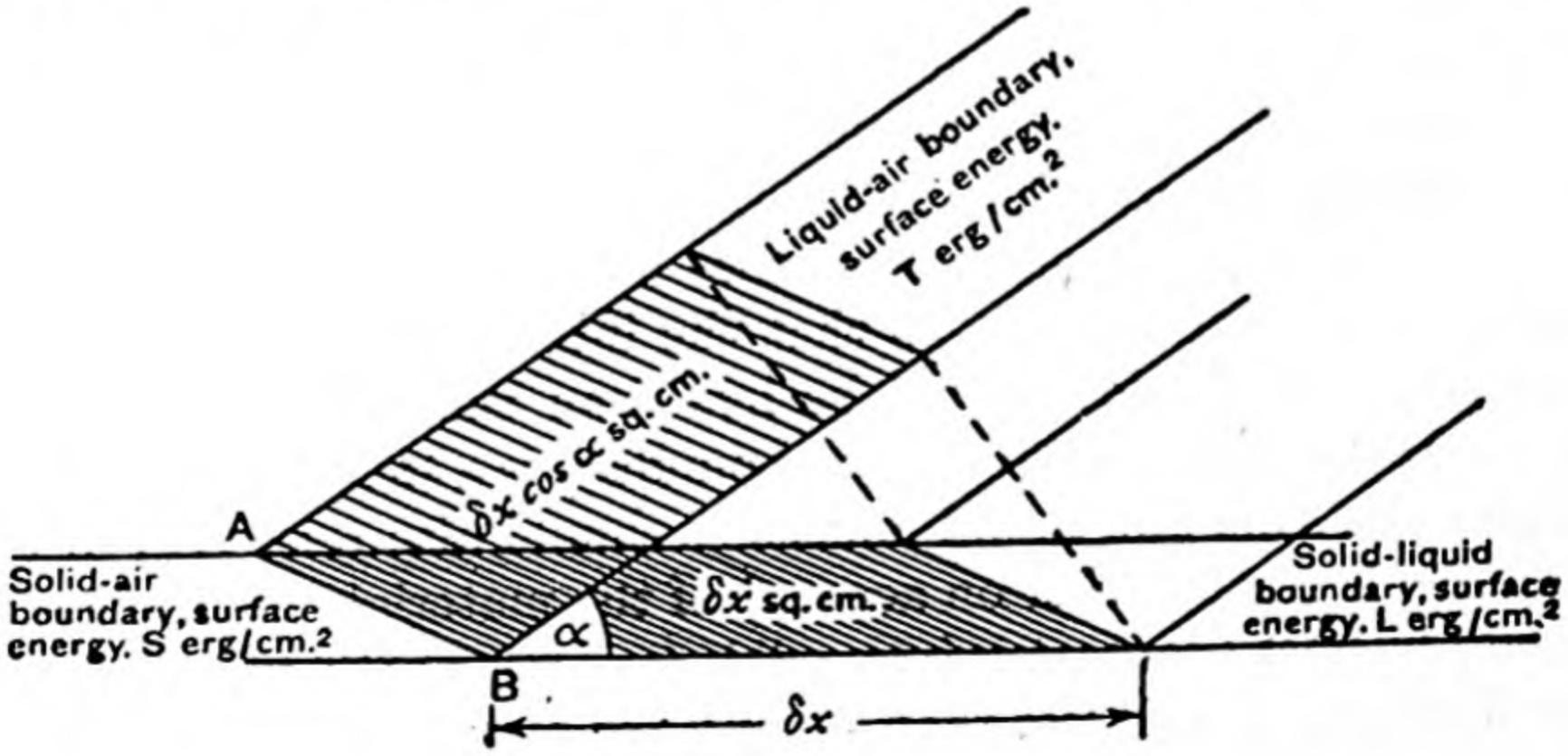
where p_y is the hydrostatic pressure at right angles to the surface, and K_s the intrinsic pressure in that direction. As P is the same in both cases, $p_y - p_x = K - K_s$, and as $K_s < K$, then $p_x < p_y$, or, there is a defect of hydrostatic pressure in a direction parallel to the surface. This is the surface stress which causes the surface to tend to contract, and if only the thickness of the surface layer and the distribution of stress over it were known, the value of the surface tension could be calculated from it.

The range of molecular attraction is very limited, so that the widely-spaced molecules of vapour, or of air, in the space above the free surface can have only a very small effect on K_s within the surface. The value of the surface tension should thus be affected only very slightly by the nature of the substance beyond the surface, if this is gaseous. But the presence of another liquid would be expected to raise K_s very considerably, so that the interfacial tension between two immiscible liquids should be less than the surface tension of either. If the material beyond the boundary is a solid, the attraction between the solid and liquid molecules should also raise K_s; if this attraction is small, either on account of the materials themselves, or because intimate contact is impossible (as with a solid surface fouled with dust), K_s is still less than K, and the surface of contact tends,

like a free surface, to contract. If the attraction is large, it may happen that K_s exceeds K ; then p_x exceeds p_y, and the surface of contact tends to expand, so that the liquid spreads over the solid if this is possible. It is often said that a liquid spreads over a solid if the " adhesion "—that is, the attraction between solid and liquid molecules—exceeds the " cohesion ", or the attraction between the liquid molecules that gives rise to K.

The conception of intrinsic pressure, or internal stress, applies equally well to a solid ; but the foregoing hydrostatic argument can only apply to a liquid, and there is no question of any tension in the surface of a solid. What happens here is that the internal stresses cause the surface layers to be strained, and the surface energy of a solid is best regarded as energy due to strain (p. 203). This strain energy is usually reduced when a layer of liquid covers the solid surface, since the liquid molecules exert attractions which reduce K_s within the solid boundary. So, a liquid possesses surface energy, and also exhibits surface tension because of its fluid nature ; a solid possesses surface energy, but can have no surface tension ; and when the surface of a solid is covered by liquid, its surface energy is altered.

Angle of contact and surface energy. If the surface tension is T dynes per cm., the potential energy of each square centimetre of an air - liquid surface is T ergs. When the liquid is in contact with a solid, suppose the surface energy of the solid - air boundary is S ergs per sq. cm., and that of the liquid - solid boundary L ergs per sq. cm.



Consider the element shown in the diagram, 1 cm. thick, at rest, the angle of contact being α. Let the boundary AB move a distance δx to the right.

The reduction in area of the liquid - air surface is $\delta x \cos \alpha$ sq. cm., so the work done by surface tension is $T'\delta x \cos \alpha$ ergs. The gain in energy of the solid - air face, which is increased by δx sq. cm., is $S\delta x$ ergs, and the loss of energy of the solid - liquid boundary, which is decreased by δx sq. cm., is $L\delta x$ ergs.

Thus, $$T'\delta x \cos \alpha = L\delta x - S\delta x,$$

or, $$\cos \alpha = (L - S)/T'.$$

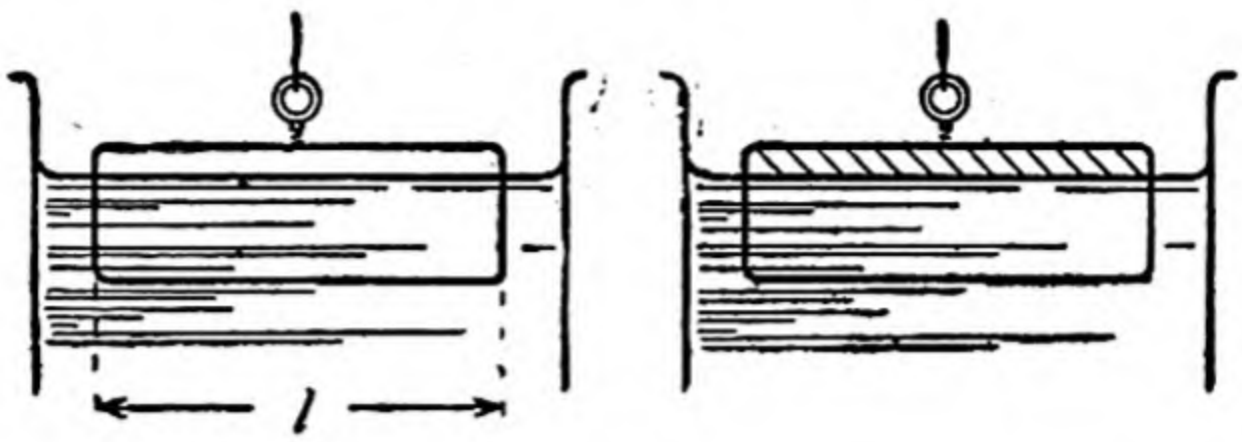
If S is very small, and L is very nearly equal to T', the angle of contact is very nearly zero. If S exceeds L, as happens with mercury on glass, $\cos \alpha$ is negative, and α obtuse.

Measurement of surface tension ; direct methods. It is impossible here to give in full the experimental details of various direct methods of measuring surface tension. For these, one of the standard practical text-books should be consulted. Here an outline only of the simpler methods, together with an appropriately simple account of the theory, is given. That this is, in fact, a deliberately simplified approach may not be evident ; but the reader who seeks a fuller treatment should consult Champion and Davy's *Properties of Matter* (Blackie).

One important general precaution must be mentioned. All apparatus must be scrupulously clean, as very small contaminations have a great effect on the surfaces involved.

Simple wire-frame method. A carefully-cleaned wire frame, in the form of a rectangle with horizontal sides l cm. long (see diagram), is suspended from one arm of a sensitive balance, and counterpoised with the upper side just clear of the surface of the liquid concerned. The beaker of liquid is raised a little, so that the side of the frame engages the surface, and then lowered again. On raising the balance beam, it is found that a further m gm. must be added in order to lift the upper side of the frame clear of the liquid.

The downward force on the frame, assuming it to act tangentially to the wire, and neglecting the weight of the film of liquid raised, is $2Tl$ dynes ; this is balanced by the added weight, mg dynes.



Thus $$2Tl = mg,$$

or $$T = mg/2l \text{ dynes per cm.}$$

With a really clean wire, consistent results are usually obtained by this method. The value of m is quite large enough to be measured fairly accurately; for example, with water, for which T is about 73 dynes per cm. at ordinary temperatures, a frame 5 cm. long will need 730 dynes, or about three-quarters of a gram weight, to lift it clear. The disadvantage of an ordinary balance for this purpose is the need for continual adjustment of weights, with a rupture of the film every time the correct weight is exceeded. The torsion balance, as described in Searle's *Experimental Physics*, is better, since this allows a steadily increasing pull to be applied to the frame until it is lifted out once for all.

Weighing a solid cylinder immersed to different depths. A vertical cylinder of radius r cm. is weighed in air, and then weighed again while partially immersed in a liquid; its base is a vertical distance h cm. below the free surface (see diagram). The difference between these two weighings gives the upthrust, U, which is brought to dynes.

By Archimedes' principle, as the cylinder displaces $\pi r^2 h$ c.c. of liquid, the upward force due to buoyancy is $\pi r^2 h\rho$ gm wt., where ρ gm. per c.c. is the density of the liquid, or $\pi r^2 h\rho g$ dynes.

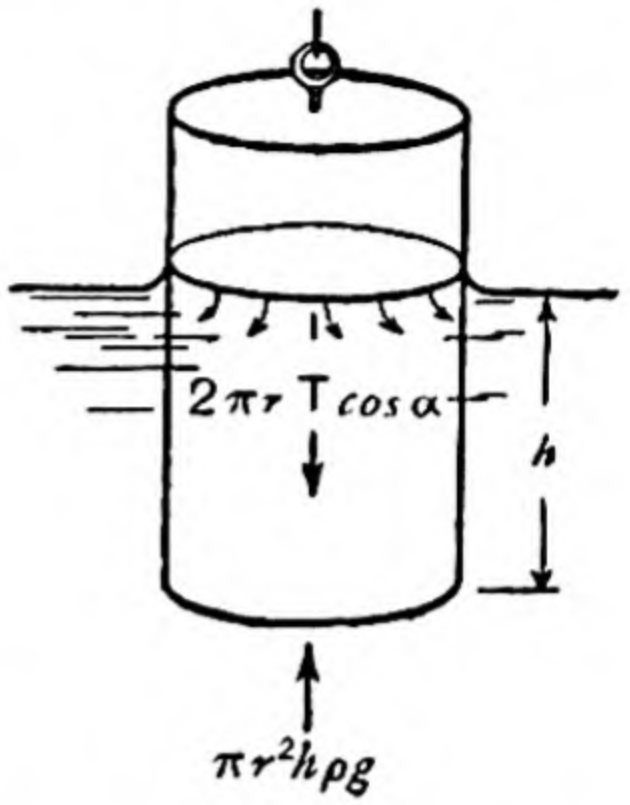
Acting round the perimeter, and pulling at an angle α to the vertical, is the surface tension force. The horizontal component of the force on any element of the perimeter is balanced by the force on the corresponding element at the opposite side of a diameter, so that only the vertical component is involved. This is $2\pi rT \cos \alpha$ dynes.

The resultant upthrust on the cylinder is then

$$U = \pi r^2 h\rho g - 2\pi rT \cos \alpha \text{ dynes.}$$

Values of U for different values of h are determined, and a graph of U against h drawn. This is a straight line, of which the gradient gives $\pi r^2 \rho g$, and the intercept on the axis of U gives $2\pi r T' \cos \alpha$.

The radius r of the cylinder is found by using callipers or any other convenient method.



Limiting thickness of a large drop. The shape taken by a drop standing on a horizontal surface is determined by the condition that the potential energy shall be a minimum. If there were only surface tension to consider, then this shape would be that giving minimum area for a given volume—that is, spherical ; if only that due to the weight of the drop were involved, then the drop would spread out until the centre of gravity was as low as possible—that is, it would settle into a uniform very thin sheet. As both effects are present together, the resulting shape is a compromise such that the sum of the two contributions to the potential energy is a minimum. Small drops are very nearly spherical as a rule ; larger ones are appreciably flattened, and very large drops may take the form of a flat disc with rounded edges. Once this last stage has been reached, the thickness of the drop is independent of its size (see diagram).

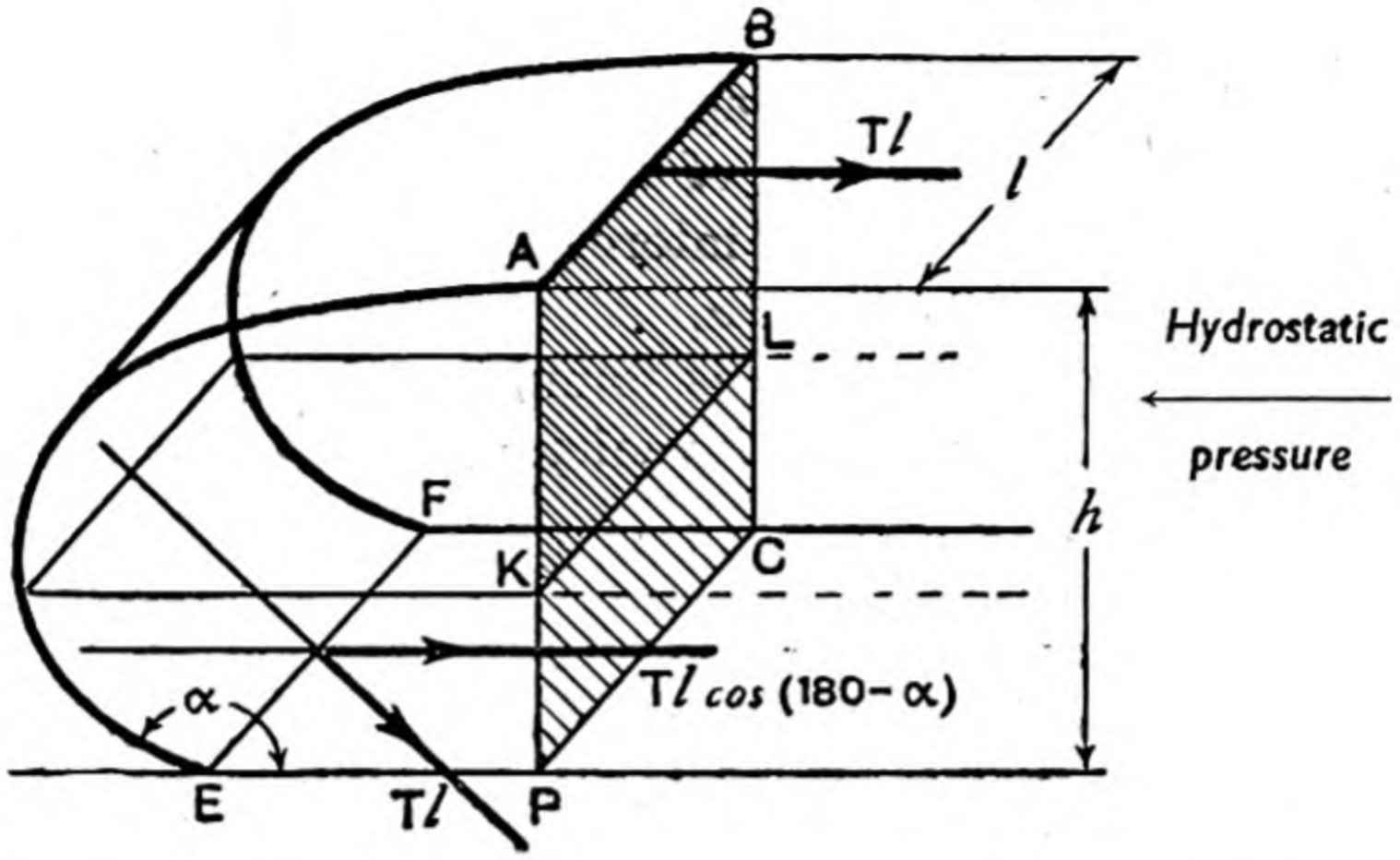
Consider a section cut out of the edge of such a large drop, at rest on a horizontal plate, the angle of contact between liquid and plate being α. Let the density of the liquid be ρ gm. per c.c., and the surface tension T dynes per cm. Let the vertical thickness of the drop be h cm., and the horizontal thickness AB of the section l cm.

This section is in equilibrium under three horizontal forces, namely :

Tl dynes to the right, at AB ;

$Tl \cos(180° - \alpha)$ dynes to the right, at EF ;

a force to the left acting over the area $ABCD$,



which is the average hydrostatic pressure $\frac{1}{2}g\rho h$, multiplied by the area $l \times h$, or $\frac{1}{2}g\rho h^2 l$ dynes.

Thus $$Tl + Tl\cos(180° - \alpha) = \tfrac{1}{2}g\rho h^2 l,$$

or, $$T(1 - \cos\alpha) = \tfrac{1}{2}g\rho h^2,$$

as $\cos(180° - \alpha) = -\cos\alpha$, so

$$T = \tfrac{1}{2}g\rho h^2/(1 - \cos\alpha) \text{ dynes per cm.}$$

Note that if the liquid wets the plate, α is 0, and $\cos\alpha = 1$; so h is always zero—in other words, the liquid will spread over the plate as a film.

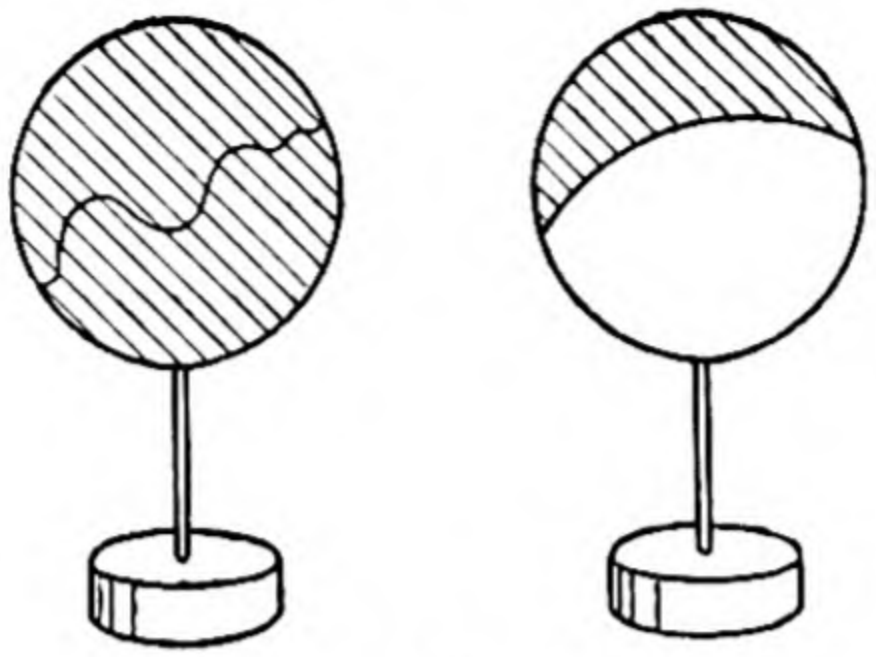
At the level KL, y cm. below the upper surface, the tangent to the edge of the drop is vertical. Applying the same argument as above, the part of the drop above this level is in equilibrium under forces Tl dynes at AB, to the right, and $\frac{1}{2}g\rho y^2 l$ dynes over $ABKL$ to the left.

Hence $$Tl = \tfrac{1}{2}g\rho y^2 l,$$

or $$T = \tfrac{1}{2}g\rho y^2 \text{ dynes per cm.}$$

Measurement of y and h thus enables both T and α to be determined.

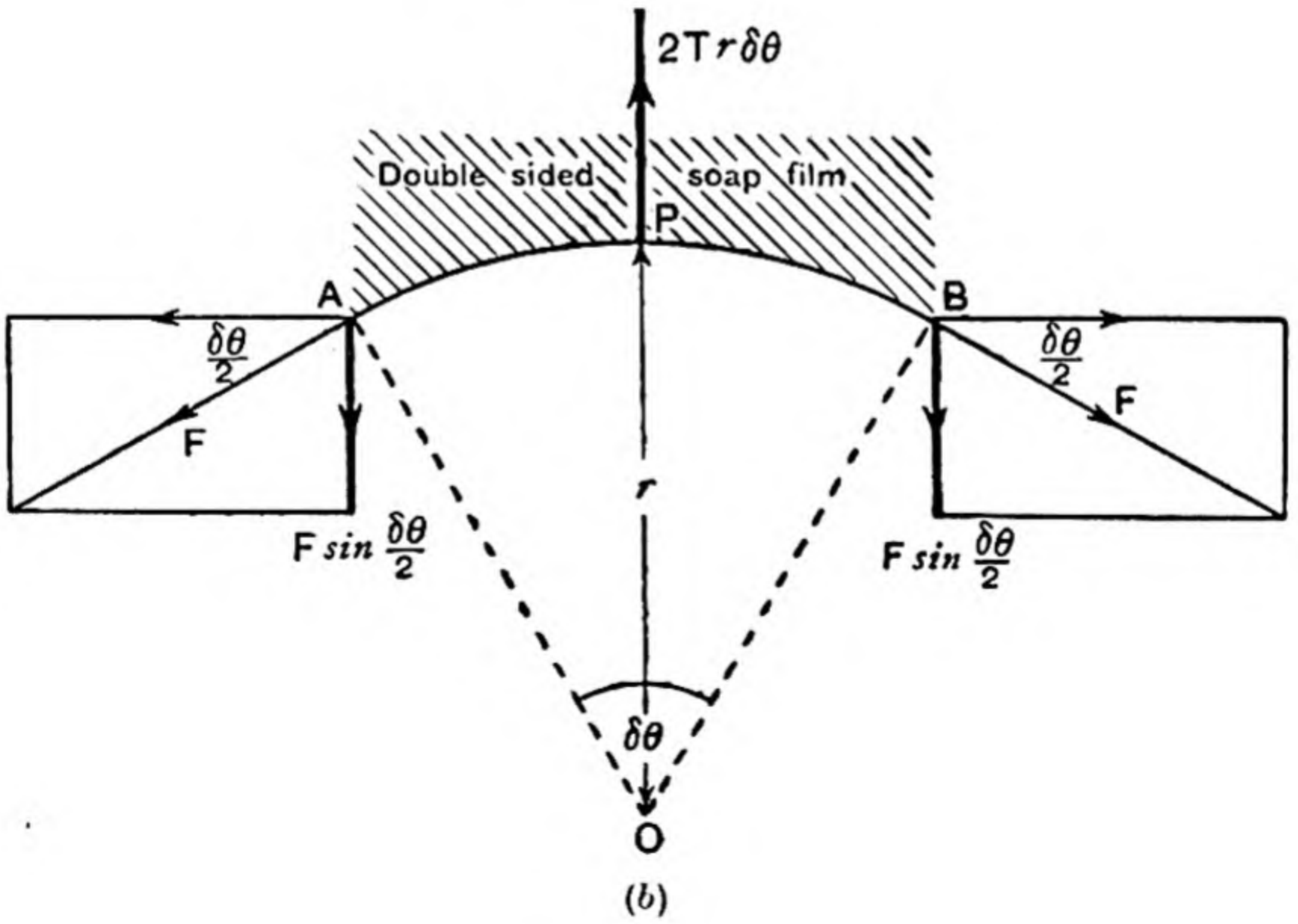
A thread subjected to pull by a soap film on one side only. A soap film is formed on a wire ring which has a thread tied across the wire as shown in the diagram (*a*). When the film is complete, the thread lies loosely in it. If the film on one side is broken, the thread is pulled into an arc, as shown by the surface tension of the surviving film. What will be the tension in the thread, and what shape will it take?



(*a*)

Consider an element of the thread APB (diagram *b*); let the radius of curvature at P be r cm., and let the arc subtend at the centre of curvature O an angle $\delta\theta$. The length of AB is $r\delta\theta$, and the arc is in equilibrium under three forces, the tensions F, acting tangentially at A and B, and the force $2Tr\delta\theta$ acting normally outwards at P (the factor 2 is included because the soap film has two surfaces). If we resolve in the direction of OP and at right angles to it, the components of F at right angles to OP cancel out, and along OP we have

$$2Tr\delta\theta = 2F \sin \delta\theta/2 ;$$



(*b*)

as $\delta\theta$ is very small, we can write $\delta\theta$ for sin $\delta\theta$; hence

$$2Tr\delta\theta = 2F\delta\theta/2,$$

or,

$$F = 2Tr \text{ dynes.}$$

As the surface tension T is the same wherever P is taken, and as the tension F in the string must be the same throughout its length, it follows that r must be the same wherever P is taken, so that the thread sets in an arc of a circle.

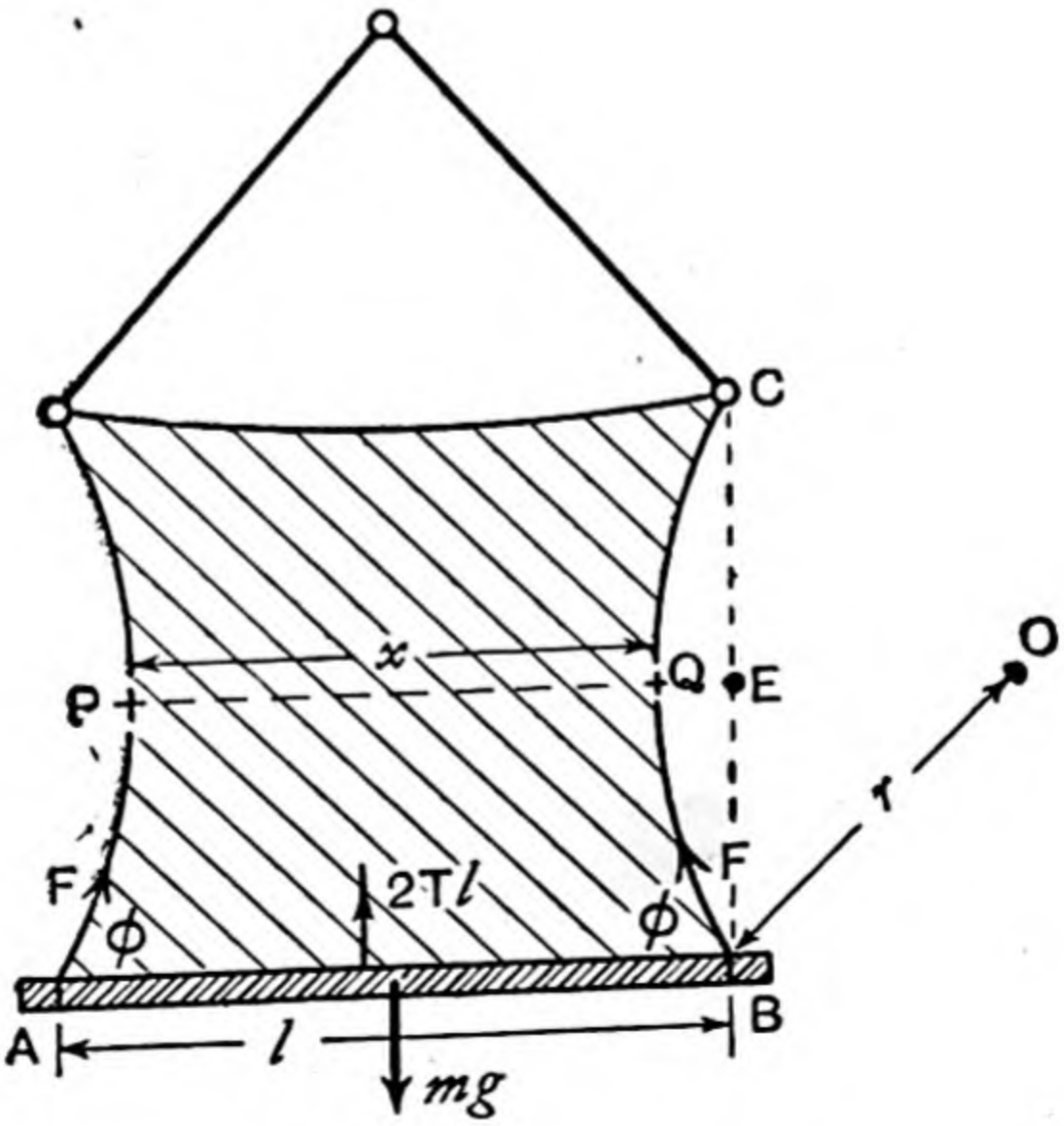
This result may be used to measure T for a soap solution by observing the shape of the curve taken up by two initially parallel threads attached at their lower ends to two points AB which are l cm. apart, on a horizontal glass rod of mass m gm., when the space between the threads, rod, and support is filled with a thin soap film (see diagram).

Let ϕ be the angle made by the threads at A and B with the rod, and F dynes the tension in the thread. The rod is supported by the vertical pull of the double-sided soap film along AB, and by the vertical components of the tensions acting at A and B ; thus

$$mg = 2Tl + 2F \sin \phi.$$

Now

$$F = 2Tr,$$



so $$mg = 2Tl + 4Tr \sin \phi,$$

and $$T = \frac{mg}{2l + 4r \sin \phi} \text{ dynes per cm.}$$

Alternatively, considering the narrowest portion PQ of the film, of width x cm., the tension in the string here acts vertically and, neglecting the weight of the film below PQ,

$$mg = 2Tx + 2F = 2Tx + 4Tr,$$

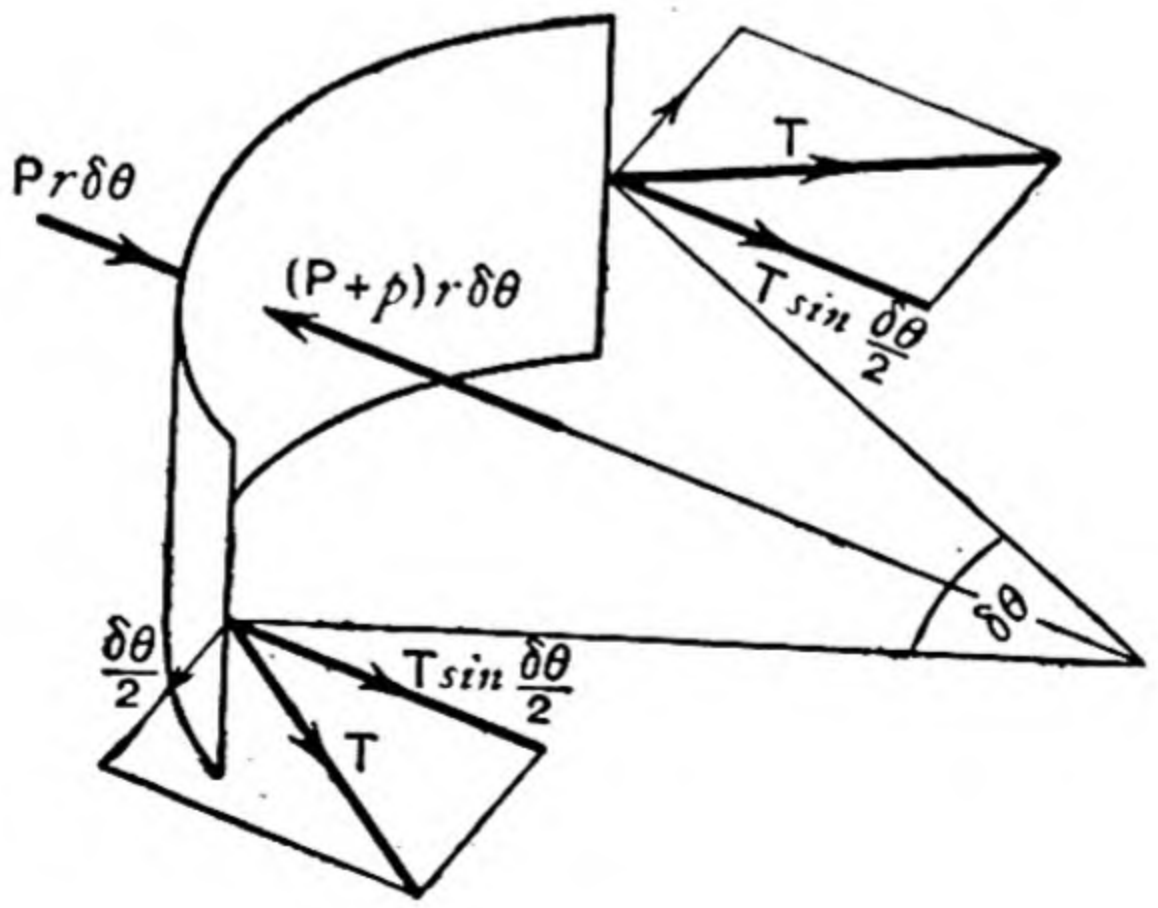
so $$T = \frac{mg}{2x + 4r} \text{ dynes per cm.}$$

To find r, and also ϕ, the distances BC and QE are measured. Then $QE\,(2r - QE) = (\frac{1}{2}BC)^2$; and $\frac{1}{2}BC = r \cos \phi$.

Pressure difference across a curved surface. If the only forces acting on the surface of a liquid lie in the surface itself, we should expect the surface to lie in one plane ; and if the surface is not plane, there must be some force acting at right angles to the surface. This will be provided if, for any curved liquid surface, the pressure is greater over the concave side than over the convex side.

Consider first a cylindrical liquid surface, the radius of the curved face of the cylinder being r cm. Take an element of this surface, 1 cm. wide and of length $r\delta\theta$, subtending an angle $\delta\theta$ at the axis of the cylinder. Let the pressure over the convex side be P dynes per sq. cm., and that over the concave side $P + p$ dynes per sq. cm.

The element is in equilibrium under the forces T dynes at the ends,



directed as shown in the diagram, and the forces due to the two pressures P and $P+p$, acting on opposite sides of the element of area $r\delta\theta$ sq. cm.

Resolving along the normal to the element,

$$Pr\delta\theta + 2T \sin \delta\theta/2 = (P+p)r\delta\theta.$$

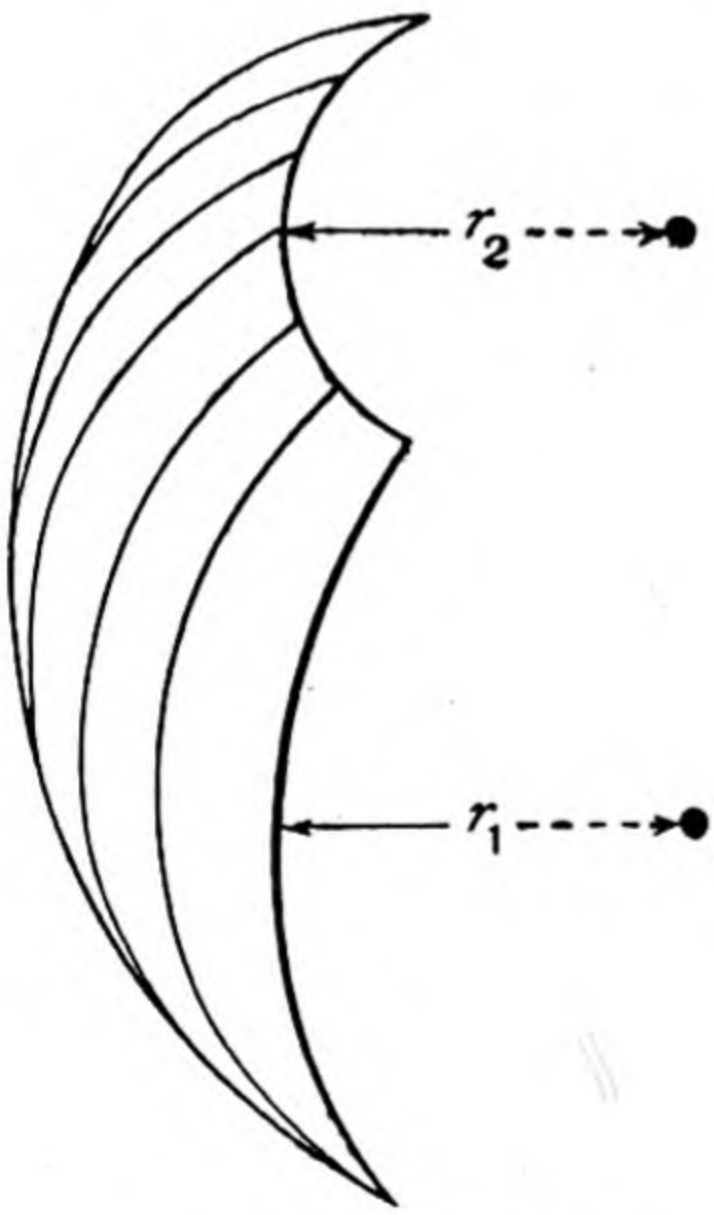
As $\delta\theta$ is so small that $\sin \delta\theta/2$ can be taken as $\delta\theta/2$,

$$2T\delta\theta/2 = pr\delta\theta,$$

so
$$T = pr,$$

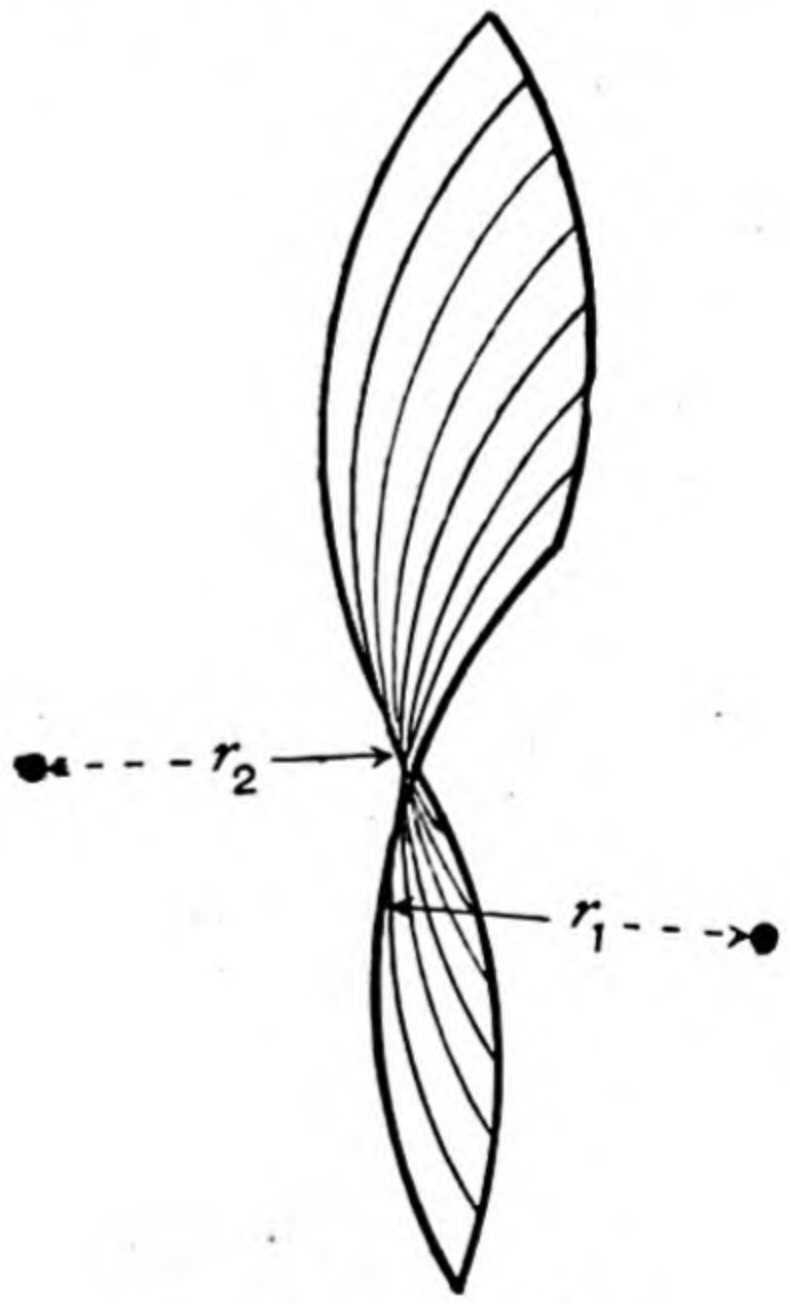
or,
$$p = \frac{T}{r} \text{ dynes per sq. cm.}$$

Now, an element of any curved surface can be constructed, as in the diagram below, by taking a number of cylindrical elements of radius of curvature r_1, and arranging them along an element of arc of radius of curvature r_2. If both centres of curvature are on the same side of the surface, it is said to be *synclastic*; if they are on opposite sides, it is said to be *anticlastic*. The radii r_1 and r_2 are called the *principal radii of curvature of the surface* at the point where the element lies.



Synclastic surface

(*a*)



Anticlastic surface

(*b*)

Repeating the preceding argument, it can be seen that for the r_1 curvature, there will be a pressure excess

$$p_1 = \frac{T}{r_1},$$

and for the r_2 curvature a pressure excess

$$p_2 = \frac{T}{r_2}.$$

The resulting pressure excess p equals $p_1 + p_2$, or

$$p = \frac{T}{r_1} + \frac{T}{r_2} = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

For a synclastic surface (Fig. *a*), r_1 and r_2 have the same sign, and the pressure is greater on the concave side. For an anticlastic surface (Fig. *b*), r_1 and r_2 have opposite signs and the pressure difference is numerically

$$T\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

the pressure being greater on the side of smaller radius of curvature.

As T is constant, it follows that any liquid surface subjected only to surface tension and pressure difference between the faces will have the form represented by the equation

$$\left(\frac{1}{r_1} \pm \frac{1}{r_2}\right) = \text{constant},$$

and it is possible for an anticlastic surface to be stable in the absence of any pressure difference at all if

$$\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 0.$$

For a spherical surface, $r_1 = r_2 = r$, and

$$p = \frac{2T}{r} \text{ dynes per sq. cm.}$$

Soap films are double-sided, whence the appropriate expressions for the pressure difference between the faces of a soap film are :

$p = \frac{2T}{r}$ dynes per sq. cm. for a cylindrical film,

$p = 2T\left(\frac{1}{r_1} \pm \frac{1}{r_2}\right)$ dynes per sq. cm. in general, and

$p = \frac{4T}{r}$ dynes per sq. cm. for a spherical soap bubble.

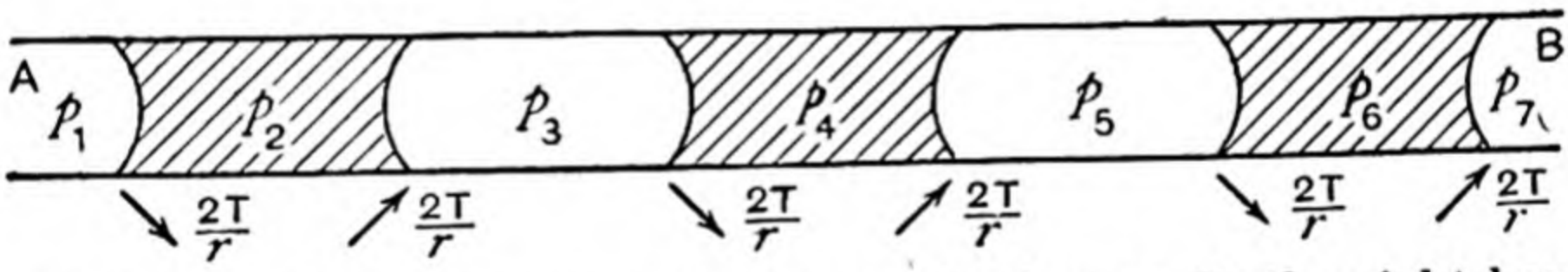
Adhesion of moistened planes. Two plane surfaces separated by a thin film of liquid require considerable force to separate them. Suppose the film is 0·01 cm. thick ; at the edge, if the liquid wets the plates, the surface will be curved, of radius of curvature 0·005 cm. The pressure within the liquid will then be less than that of the atmosphere outside by $T/0{\cdot}005 = 200T$ dynes per sq. cm. If the liquid is water, for which T is about 73 dynes per cm., this is 14,600 dynes per sq. cm. Suppose the area of the plates covering the liquid is 100 sq. cm. The resulting force holding the plates together is 1,460,000 dynes, or about 1·5 kgm. weight.

Pressure difference across a broken liquid column. Again, suppose that the liquid wets the solid, so that the radius of curvature of the surfaces of the broken thread is initially the radius of the tube. If there is no pressure difference between the ends A and B (see diagram), then

$$p_2 = p_1 - 2T/r,$$
$$p_2 = p_3 - 2T/r,$$

so $$p_1 = p_3,$$

and similarly $$p_5 = p_3, \text{ etc.}$$



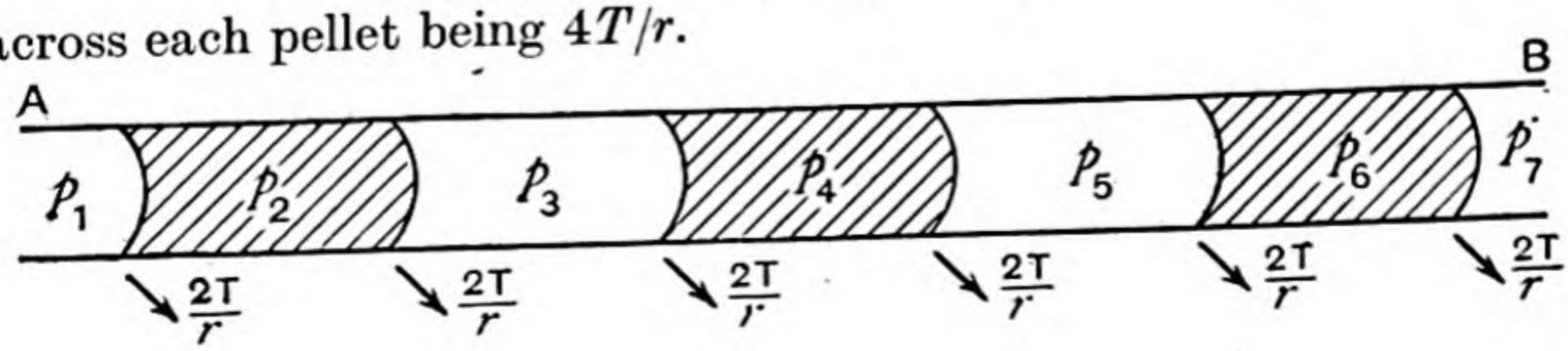
If, however, the pressure at A exceeds that at B, the right-hand surface of the first pellet accommodates itself to take up part of the pressure difference, or all of it if it is small. In the extreme case, the surface will be turned inside out completely, when

$$p_2 = p_1 - 2T/r,$$
$$p_3 = p_2 - 2T/r = p_1 - 4T/r ;$$

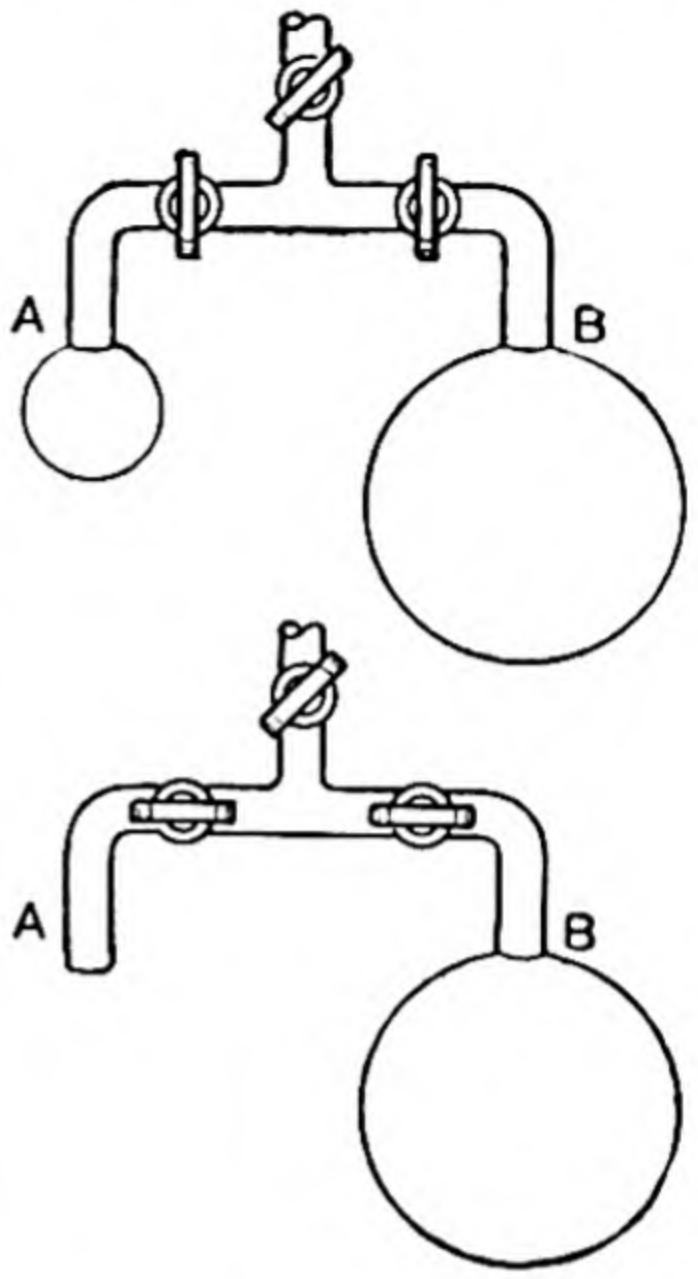
and similarly

$$p_5 = p_3 - 4T/r = p_1 - 8T/r \ldots, \text{ the pressure drop}$$

across each pellet being $4T/r$.

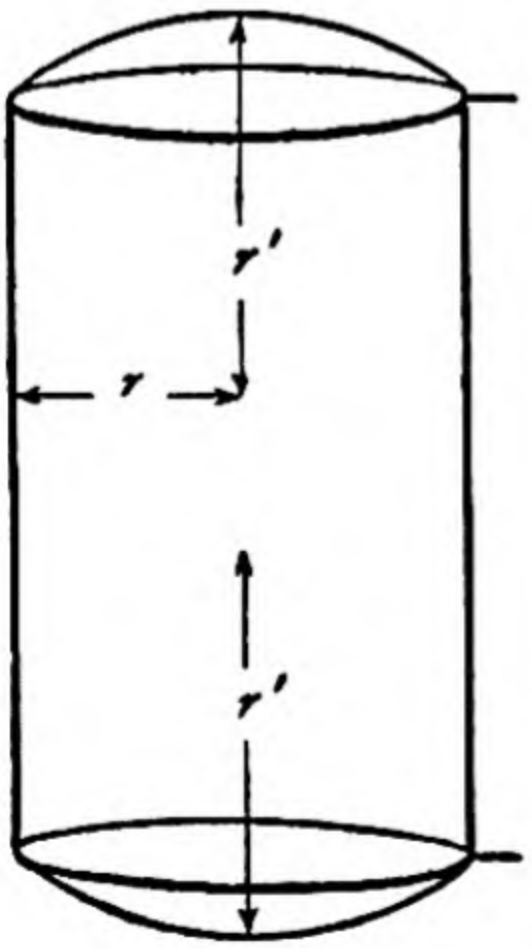


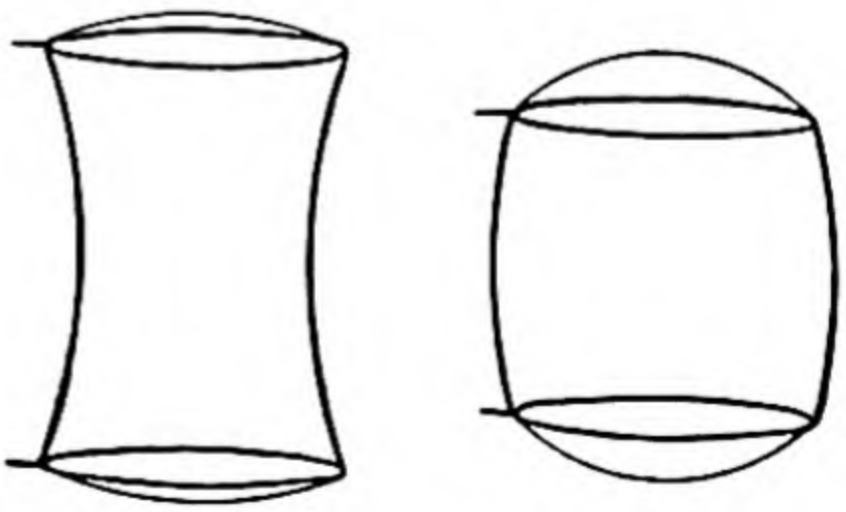
Communicating soap bubbles of different radii. Two bubbles, A and B, of radii r_a and r_b (r_b being the greater) are separately blown on the two jets of the apparatus shown. On opening the two taps on the horizontal limb, so that the bubbles communicate, A is seen to collapse and B increase in size, until the remaining spherical cap of A has the same radius of curvature as B. This is because the pressure excess in A, $4T/r_a$, is greater than that in B, which is $4T/r_b$; the final common pressure excess is $4T/r$, where r is the radius of each bubble finally. The complete calculation of r is a rather difficult problem for which the actual pressures must be known, as of course it involves Boyle's law.



Soap films on parallel circular rings. By introducing circular rings of thin wire, previously wetted with soap solution, into the surfaces of a soap bubble, and pulling them apart, a soap film as in the diagram can be formed. If r is the radius of each ring, then for the cylindrical sides, the pressure excess p is equal to $2T/r$; if r' is the radius of the two spherical caps at the end, then $p=4T/r'$. As the pressure excess is the same throughout the bubble,

$$2T/r=4T/r', \quad \text{or} \quad r'=2r.$$





If the sides are not cylindrical, but have principal radii of curvature r_1 and r_2, then

$$p = 2T(1/r_1 + 1/r_2) = 4T/r' ;$$

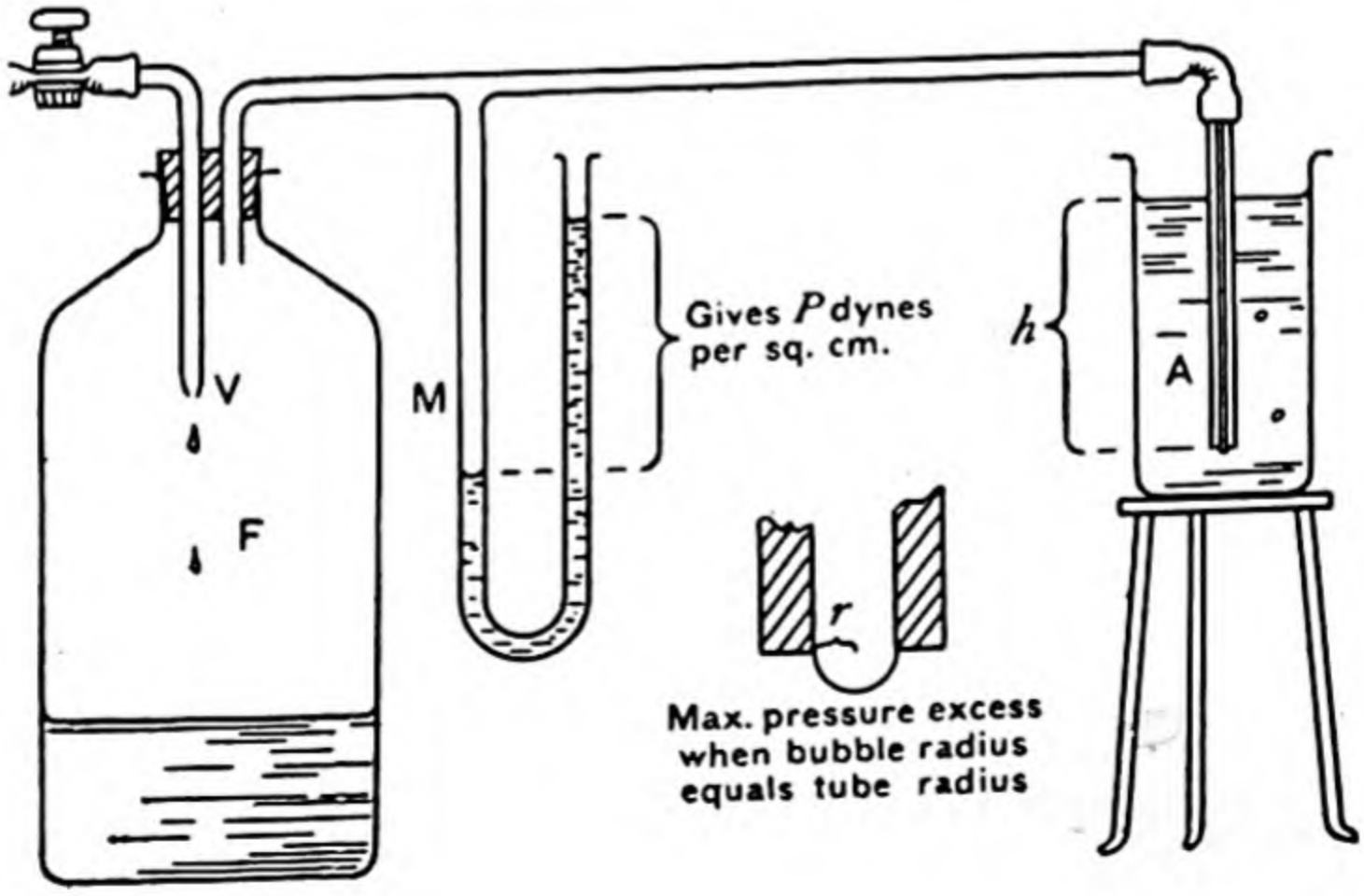
and shapes such as those shown are obtained.

If either of the spherical caps is broken the pressure excess at once becomes zero. The only shape for the other end is a plane, for which r is infinite and $1/r$ zero. The cylindrical side film, however, adjusts itself to a curved surface (anticlastic), for which the two principal radii of curvature satisfy the equation

$$(1/r_1 - 1/r_2) = 0.$$

This is called a *catenoid*.

Measurement of surface tension by spherical bubbles (Jaeger's Method). A fine-bore tube A is held vertically with its lower end at depth h cm. below the free surface of the liquid (see diagram), and is connected with a liquid manometer M, and a Winchester bottle F from which a slow steady stream of air can be sent by allowing water to drip slowly in through the dropping tube V. As the pressure in A slowly increases, liquid is forced down it by the air, until eventually a hemispherical bubble of air, of radius the same as that of the bore of the tube, is formed. Before this stage is reached, the pressure excess in the bubble will be less, as the radius is greater; the pressure excess will also be less as the bubble grows, prior to detaching itself. This *maximum pressure* excess is observed on the



manometer, and expressed as P dynes per square centimetre; it represents the pressure needed to overcome the head of liquid of density ρ gm. per c.c. in the beaker, plus the pressure excess within the drop.

Thus $$P = g\rho h + 2T'/r.$$

Measurement of ρ and r, together with P and h, enables T' to be determined.

This method is convenient for measuring the value of T at different temperatures.

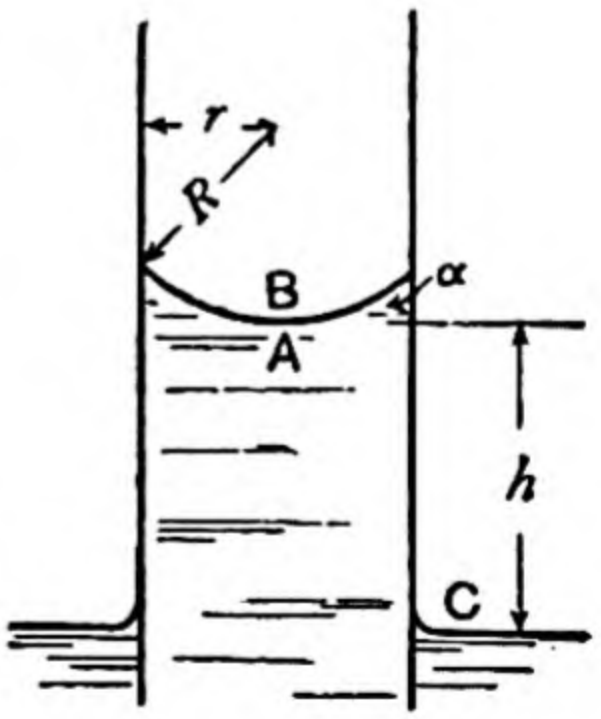
Rise of a liquid in a capillary tube. A column of liquid of vertical height h cm. above the free surface of the liquid in a vessel (see diagram) stands in a capillary tube. The density of the liquid is ρ gm. per c.c., and the radius of the tube at the place where the meniscus stands is r cm. The angle of contact is α.

Assume that the meniscus is spherical in shape. (This cannot actually be true, as the following reasoning shows. If the pressure is the same at all points at the level A within the liquid, and the external air pressure is the same over the upper surface, different parts of the meniscus are at different vertical heights above A; hence there must be different pressure excesses and therefore different radii of curvature at different points on the meniscus surface.)

Suppose that the radius of curvature of the meniscus is R cm. Then

$$r = R\cos\alpha.$$

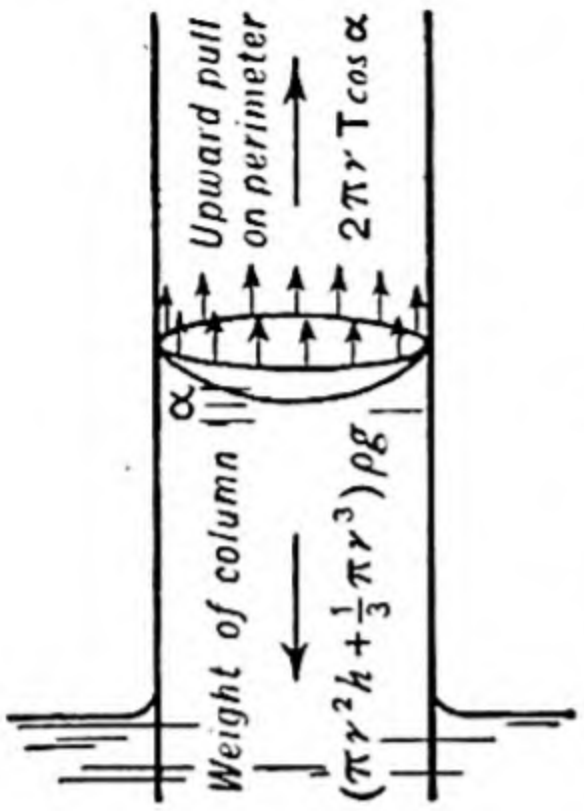
The pressures at B and C outside the liquid are very nearly the same, so the pressure defect in going from B to A must be balanced by the pressure due to the vertical column h cm. of liquid.



Hence $$\frac{2T}{R} = \frac{2T\cos\alpha}{r} = h\rho g,$$

or $$T = \frac{hrg\rho}{2\cos\alpha} \text{ dynes per sq. cm.}$$

As T must always be positive, if α is greater than 90°, h must be negative; in that case, the liquid would "climb down" the tube instead of up it.



To measure T by observing the rise in a capillary tube, we must thus determine the density, find the radius of the capillary at the actual point where the meniscus stands, measure the vertical height h, and determine the angle of contact. It is of the utmost importance that the capillary tube, and the vessel containing the liquid, should be clean and free from grease.

This formula can also be derived by another method. Considering the perimeter of the meniscus, there is a force T dynes for each centimetre of perimeter, directed as shown in the diagram, acting on the tube. An equal and opposite reaction acts on the liquid. Resolving the forces acting on each element of the perimeter on the liquid, the horizontal components cancel by symmetry, and the vertical components reinforce each other to give a total upward pull on the liquid of

$$2\pi rT\cos\alpha \text{ dynes.}$$

This supports the weight of the vertical column below the meniscus, which is a cylinder of height h and radius r, capped by a small "concave lens" of liquid, the volume of which is $1/3\pi r^3$ if the meniscus is hemispherical, and very nearly this in most other practical cases.

The weight supported is thus

$$(\pi r^2 h+\tfrac{1}{3}\pi r^3)\rho g \text{ dynes.}$$

Hence,

$$2\pi rT\cos\alpha=(\pi r^2 h+\tfrac{1}{3}\pi r^3)\rho g,$$

or,

$$T=\frac{(h+\frac{1}{3}r)r\rho g}{2\cos\alpha} \text{ dynes per cm.}$$

Some interesting points arise in comparing the two methods. No correction for the volume of the meniscus appears in the first method, which assumes the meniscus to be part of a sphere, which it cannot be. The second method also makes this assumption but this time in order to make the correction that a spherical surface, according to the first method, does not require. The second method assumes that the tube is uniform ; the first method only requires that

the value of r be known at the place where the meniscus stands, and this appears to be correct experimentally. Both are simplified versions of the correct theory.

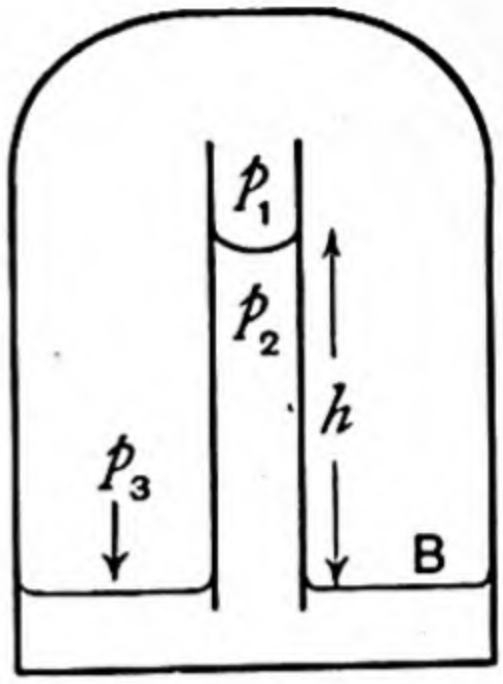
Some further points. The surface tension of all liquids decreases as the temperature rises. This is to be expected, since at higher temperatures the molecules themselves possess greater average kinetic energy, and are moving about more rapidly, reducing the effect of molecular attractions ; so that less work is needed to bring a molecule from the interior of the liquid to the surface.

Impurities, contaminations, and dissolved substances all lower the surface tension of a liquid. Simple experiments to show this are : pour a shallow layer of water in a large flat dish, and touch the surface with a rod moistened with alcohol ; the surface will part leaving the dish dry below. The surface has broken at the " weak spot " contaminated with alcohol. A wooden spill is laid on the surface of water in a large dish ; at one end of the spill, just touching the water, a small piece of camphor is placed. The spill moves, with the camphor to the rear, for the surface contaminated with camphor exerts a pull less than the pure water round the rest of the spill, which thus moves away from the contaminated part.

All experiments carried out in the ordinary school laboratory to measure surface tension usually give low values, on account of the great difficulty of securing really clean apparatus and maintaining it uncontaminated during an experiment.

Surface tension, curvature, and vapour pressure. Molecules evaporate from the free surface of a liquid, and exert a pressure which depends on the number of molecules per unit volume in the space above it ; molecules also return to the liquid from this space, and the pressure reaches a maximum steady value, called the saturated vapour pressure, or simply the vapour pressure, when the number of molecules returning to the surface per second is equal to the number leaving it per second. A molecule will leave the surface if it possesses sufficient kinetic energy to do the necessary work against the attractions of its neighbours, and any factor which expedites departure will raise the vapour pressure. Temperature is the chief of these factors, for if the temperature is raised the average kinetic energy is increased. The shape of the surface is another

factor, for if a molecule is escaping from the peak of a convex surface, there are fewer molecules in its immediate neighbourhood than if the surface were plane, so the effect of their attraction is reduced; similarly, a molecule escaping from the trough of a concave surface experiences a greater attraction than it would if the surface were plane. It would then be expected that the vapour pressure would be greater over a convex surface than over a plane, and less over a concave surface than over a plane.



The effect of surface curvature on the vapour pressure of a liquid can be calculated in terms of the surface tension, remembering that the pressure difference between the opposite faces of a surface of radius of curvature r cm., and surface tension T dynes per cm., is $2T/r$ dynes per sq. cm.

Consider a liquid in an enclosure containing only its saturated vapour, and let a vertical capillary tube of radius r stand upright in the liquid. Let p_3 be the S.V.P. at the plane surface, p_1 that over the concave surface, and p_2 that just inside the liquid in the tube. Let ρ be the density of the liquid in gm. per c.c., σ the density of the saturated vapour, in the same units, and let h cm. be the vertical height of the column.

Then
$$p_1 - p_2 = \frac{2T}{r},$$

and, writing down the two expressions for the pressure at B inside and outside the tube, which must be the same,

$$\left.\begin{aligned} p_1 + g\sigma h &= p_3 \\ p_2 + g\sigma h &= p_3 \end{aligned}\right\}.$$

Eliminating h and p_2,

$$p_3 - p_1 = \frac{2T}{r} \cdot \frac{\sigma}{\rho - \sigma} \text{ dynes per sq. cm.}$$

Thus the vapour pressure over a concave surface of radius r is less than that over a plane surface by an amount $\frac{2T}{r} \cdot \frac{\sigma}{\rho - \sigma}$;

similarly, the vapour pressure over a convex surface of radius r exceeds that over a plane surface by $\frac{2T}{r} \cdot \frac{\sigma}{\rho - \sigma}$. Some idea of the size of this difference may be gained by substituting rough values and making an approximate calculation.

For a water drop of radius 0·0001 cm. at 20° C., $\rho = 1$ gm. per c.c., $\sigma = 0{\cdot}000017$ gm. per c.c., $T = 73$ dynes per cm. The pressure excess is $\frac{2 \times 73}{0{\cdot}0001} \times \frac{0{\cdot}000017}{1}$ dynes per sq. cm., $\simeq 25$ dynes per sq. cm., or about 2×10^{-2} mm. mercury.

EXAMPLES ON CHAPTER 9

1. Show that the critical velocity for the breakdown of streamline flow in a capillary tube is given by

$$u_c = k \cdot \frac{\eta}{\rho r},$$

where k is a numerical constant and the other symbols have their usual meaning. Show also that the critical radius for the breakdown of Stokes' formula for a sphere moving with velocity v is

$$r = k \cdot \frac{\eta}{\rho v}.$$

2. Prove that the force F resisting the motion of a sphere of radius r moving with velocity v through a fluid of density ρ and viscosity η may be represented by

$$F = k \cdot r^2 v^2 \rho \left(\frac{\eta}{\rho v r}\right)^n.$$

3. Show that the pressure difference between the ends of a tube of radius r, through which a liquid of density ρ and viscosity η is moving with axial velocity u, is given by

$$p = k \cdot \rho u^2 \cdot \left(\frac{\eta}{\rho u r}\right)^n.$$

4. A non-viscous liquid of density ρ is discharged through a pipe of radius r under pressure difference p. Find an expression for the volume V discharged per second.

5. According to Svedberg, the maximum safe angular velocity at which a solid disc can be spun depends only on the radius R of the disc and the breaking stress S and density ρ of the material. Find the form of the relation between these quantities.

6. Find the form of the relations expressing the pressure p of a gas in terms of its density ρ and the root-mean-square velocity c, and giving

the mean-free-path l in terms of the density ρ, the molecular diameter σ, and the mass m of a molecule.

7. Show that the expressions for the rise of a liquid in a capillary tube (p. 275), for the excess pressure in a soap-bubble (p. 271) and for the limiting thickness of a mercury drop (p. 265) are dimensionally correct.

8. Obtain an expression for the velocity of propagation of ripples on the surface of a liquid in terms of the surface tension and density of the liquid, and the wave-length of the ripples.

9. Find an expression for the period of oscillation of a small drop about its mean spherical shape in terms of the radius of the drop and the density and surface tension of the liquid.

10. Find a formula for the frequency of vibration, n, of a stretched string in terms of its length, the stress, and the density of the material.

11. The couple C which rotates the free end of a wire of length l and radius of cross-section r through an angle ϕ is known to be proportional to ϕ/l and to the modulus of rigidity n of the material. Find an expression for C in terms of all these quantities.

12. Find an expression for the couple C required to bend a cylindrical rod of radius of cross-section r and Young's modulus Y into an arc of radius R, assuming that C is inversely proportional to R.

13. Find an expression for the energy per unit volume in a strained material in terms of Young's modulus Y and the stress S.

14. The maximum height h for stability of a vertical cylindrical column depends on the radius r of the column, Young's modulus Y for the material, the density ρ and the acceleration of gravity, g. Find an expression relating h, r, ρ, Y and g.

15. Assuming that the volume of liquid flowing per sec. through a cylindrical tube depends on (*a*) the pressure gradient, (*b*) the radius of the tube, (*c*) the coefficient of viscosity of the liquid, find by the method of dimensions how it depends on these quantities. (C. S.)

16. Define coefficient of viscosity, and explain how it can be measured for a liquid.

Assuming that for a liquid flowing through a tube there is a critical velocity above which streamline motion breaks down, show, by the method of dimensions, how this velocity must depend on (*a*) the radius of the tube, (*b*) the density of the liquid, (*c*) the viscosity of the liquid. (C. S.)

17. Define the coefficient of viscosity of a fluid and describe a method of measuring it for a liquid.

Investigate by the method of dimensions how the force opposing the motion of a sphere through a viscous fluid depends on the radius of the sphere, its velocity, and the viscosity of the fluid.

Two equal drops of water are falling through the air with a steady velocity of 10 cm./sec. If the drops coalesce, what will be the new terminal velocity? (C. S.)

18. What do you understand by the viscosity of a fluid? The viscosity η of a liquid may be obtained by measuring the volume V of the liquid flowing through a tube of radius r and length l in time t from the equation

$$\frac{V}{t} = \frac{\pi p r^4}{8 l \eta},$$

where p is the pressure difference between the ends of the tube. Describe such an experiment, pointing out what precautions you would take to obtain an accurate result.

Assuming that any direct measurement of p, t and V can be made exactly, discuss the errors which will arise in the determination of η when the radius of the tube is measured (*a*) by means of a travelling microscope, (*b*) by filling the tube with mercury and weighing the mercury. (C. S.)

19. Define coefficient of viscosity. Explain how it is that small bodies free to fall through a viscous fluid eventually move with a uniform velocity. Show, without detailed calculation, that, for different spheres of the same material, this final velocity of fall is proportional to the square of the radii of the spheres. (C. S.)

20. According to Stokes, the resistance experienced by a small sphere of radius a moving through a medium of viscosity η with velocity v, is given by the expression $6\pi\eta a v$. Use this to find the terminal velocity of an oil drop of density 0·95 gm. per c.c. and radius 10^{-4} cm., falling through air of density 0·0013 gm. cm.$^{-3}$, if the viscosity of air is 181×10^{-6} gm. cm.$^{-1}$ sec.$^{-1}$. (N.)

21. Explain what is meant by the dimensions of a physical quantity in mass, length, and time. Find the dimensions of weight, velocity gradient, coefficient of viscosity.

A small sphere of radius r, falling under gravity in a liquid of viscosity η, ultimately attains a steady velocity v. Apply the method of dimensions to find how v depends on r, ρ, and w, where w is the effective weight of the sphere in the fluid, i.e. the difference between the true weight and the upward thrust due to the displaced fluid. (O. & C.)

22. What do you understand by the surface tension of a liquid? Describe one method of measuring the surface tension of water.

A film is formed by dipping a plain ring of wire into a soap solution and removing it. A loop of cotton, irregular in shape and wetted with solution, lies with its plane in the film, but does not touch the wire ring. If the film is broken inside the loop, what shape does the loop take, and why? The total length of thread is less than the circumference of the wire ring. (C. S.)

23. Deduce an expression for the pressure difference between the inside and the outside of a non-spherical soap bubble.

A spherical soap bubble is blown in a space where the pressure is negligible. The bubble suffers a small isothermal radial expansion or contraction. Discuss the stability of the bubble (*a*) if it is isolated, (*b*) if it is connected to a source of gas at constant pressure. (C. S.)

24. What is meant by the surface tension of a liquid? Deduce an expression for the pressure difference between the inside and outside of a cylindrical soap bubble.

A spherical soap bubble of radius 2 cm. becomes attached to the outside of a spherical bubble of radius 4 cm. Find the radius of curvature of the common surface. (C. S.)

25. Obtain an expression for the pressure difference between the inside and the outside of a spherical bubble. Describe and explain a method for determining the surface tension of water which makes use of the expression.

Two equal soap bubbles are blown and connected by a capillary tube. Discuss the stability of the system. (C. S.)

26. The end of a straight tube of circular cross-section is placed in the focal plane of a converging lens and has a soap film stretched across it. The pressure of the air in the tube is adjusted so that the film assumes the form of either a convex or concave mirror. A pin is adjusted on the far side of the lens so that it is self-conjugate with its image formed by light which has passed through the lens twice and been reflected from the soap film once. Show how the position of the pin depends on the pressure in the tube. (C. S.)

27. What is meant by the coefficient of viscosity of a fluid?

Assuming that the flow is not turbulent, show that the volume v cm. of a liquid of viscosity η C.G.S. units, that passes per second through a uniform capillary tube of radius r cm. and length l cm., under a pressure difference of p dyne cm.$^{-2}$ is given by the formula

$$v = \frac{\pi p r^4}{8 l \eta}.$$

Two capillary tubes, AB and BC, are joined end to end at B. AB is 16 cm. long, of diameter 0·4 cm.; BC is 4 cm. long, and of diameter 0·2 cm. The composite tube is held horizontally, as in Poiseuille's experiment, with A connected to a vessel of water giving a constant head of 3 cm., and C open to the air. Calculate the pressure difference between B and C. (O.)

28. Describe and give the theory of some method of determining surface tension which depends on the measurement of the excess pressure within a bubble.

The limbs of an open glass U-tube have internal diameters 5 mm. and 1 mm. respectively. Calculate the difference in level in the mercury

surfaces in the two limbs. The surface tension of mercury is 480 dynes per cm., its specific gravity is 13·6, and the angle of contact between mercury and glass is 140°. (N.)

29. Describe in detail the experiments you would make, using a capillary tube method, to find the surface tension of a liquid which wets glass.

A circular ring of thin wire of mean radius 2 cm., suspended horizontally by a thread passing through the 5 cm. mark on a metre rule pivoted at its centre, is balanced by a 5 gm. weight suspended from the 70 cm. mark. A beaker of liquid is then arranged so that the ring is just attached to the liquid surface when the ring is horizontal. It is found necessary to move the 5 gm. to the 80 cm. mark in order to detach the ring from the liquid surface. Find the surface tension of the liquid. (N.)

30. Define surface tension. Derive an expression for the excess pressure inside a spherical soap bubble in terms of the radius r and surface tension T. Describe how the value of T may be found from measurements on a bubble.

The small space between two parallel circular glass plates is filled with water. Explain why it is much easier to separate the plates by sliding one over the other than by a direct pull. (N.)

31. Describe how you would measure the surface tension of a liquid which has zero angle of contact with glass, and explain how you would calculate the results from the measurements made.

A uniform rod AB, of mass 1·59 gm., is suspended horizontally by light flexible vertical threads AD and BC from a horizontal bar DC. When a soap film is stretched on $ABCD$ the surface tension causes the threads to assume the form of arcs of circles of radii 13·0 cm., and the horizontal distance between the mid-points of the threads is then 3·0 cm. Calculate (a) the surface tension of the soap film, (b) the tension in the threads. (O. & C.)

32. What is meant by (a) the surface tension, (b) the surface energy of a liquid?

A solid cylinder, of length L, radius r, and weight mg, is placed vertically with its lower end submerged to a depth h ($<L$) in a liquid of density ρ, surface tension T, and zero angle of contact. Find an expression for its apparent weight $m'g$, and sketch the form of the graph that would be obtained by plotting m' against h. Explain how values of ρ and T could be deduced from the graph. (O. & C.)

33. Give a brief account of the phenomenon of surface tension, and explain its quantitative definition.

Show that the pressure inside a spherical soap bubble will be greater than the pressure of the surrounding atmosphere, and deduce an expression for the excess.

A vertical tube, communicating with the atmosphere through a small orifice at the bottom, and closed at the top by a spherical soap film, is kept filled with hydrogen. If the vertical height of the tube between the

orifice at the bottom and the soap film at the top is 100 cm., the atmospheric pressure 760 mm. of mercury, and the temperature 17° C., calculate the radius of curvature of the soap film, and indicate whether it is convex or concave towards the outside. (Density of air at N.T.P., 0·00129 gm. per c.c.; of hydrogen, 0·00009 gm. per c.c.; surface tension of soap solution, 74 dynes per cm.) (O. & C.)

34. Describe a method of determining the surface tension of a liquid such as mercury which does not wet glass.

A glass U-tube has its two limbs of radius a and b respectively. It is fixed vertically with the open ends upwards and partially filled with a liquid, which wets glass and is of density d and surface tension T. Find the difference in level of the liquid in the two arms of the U-tube. (O. S.)

35. Define surface tension and surface energy. Find an expression for the excess pressure inside a spherical air bubble of radius r in a liquid of surface tension T.

Calculate the total pressure (in C.G.S. units) inside a spherical bubble of radius 0·1 mm. at a distance of 10 cm. below the surface of a liquid of density 1·1 gm. per c.c., and surface tension 50 dynes per cm. (Height of mercury barometer, 760 mm. Density of mercury, 13·6 gm. per c.c.) (O. & C.)

36. What is meant by the surface tension of a liquid? Describe a method of measuring the surface tension of a liquid, and deduce any formulae you use.

Find the work done in blowing a spherical soap bubble of radius 2 cm. with soap solution of surface tension 25 dynes per cm. (O. & C.)

37. Find the difference in height between the water levels in two arms of a vertical U-tube if the internal radii of the two arms are 1 mm. and 2 mm. respectively. The surface tension of water is 72 dynes per cm., and the meniscus in each limb may be assumed hemispherical.

38. Calculate the radius of the largest drop of water that might evaporate at 0° C., without heat being supplied to it. The surface energy of water is 117 ergs per sq. cm., and the latent heat of evaporation of water 606 calories per gram at 0° C. One calorie equals $4{\cdot}2 \times 10^7$ ergs. (C. S.)

39. Two water drops of radius 0·00002 cm. coalesce. What rise of temperature will result? The surface tension of water is 74 dynes per cm. (C. S.)

40. Describe one method of determining the surface tension of a soap film.

A thin circular rubber band, the radius of which when unstretched is 2·0 cm., is placed on a soap film held in a frame. The film inside the band is then broken, and the band is stretched outwards by the pull of the film outside. If the surface tension of the soap film is 30 dynes per cm., and the radius of the band becomes 2·1 cm., what force would be

required to double the length of the rubber band, assuming it to be perfectly elastic? (C. S.)

41. Describe in detail a method of determining the surface tension of a liquid, and indicate the advantages and disadvantages of the method you describe.

A glass rod of mass m is suspended horizontally from a support by two vertical threads x cm. apart. The rod and suspending threads are now immersed in soap solution and removed with a film between the threads. The distance apart of the threads at the centre is now y cm., and the vertical distance between the support and the glass rod is h cm. Find the surface tension. (C. S.)

42. Derive an expression connecting the curvature of a film with the surface tension and the difference of pressure on the two sides.

A bubble is blown between two equal rings, which are then drawn apart to form a cylindrical bubble with bulging ends. What is the ratio of the radius of curvature of the ends to the diameter of the cylinder? If the film over one end of the cylindrical bubble is broken, comment on the geometrical properties of the remaining films. (C. S.)

43. Describe the capillary tube method of measuring the surface tension of a liquid.

An inverted U-tube (Hare's apparatus) for measuring the specific gravity of a liquid was constructed of glass tubing of internal diameter about 2 mm. The following observations of the heights of balanced columns of water and another liquid were obtained :

Height of water (cm.)	2·8	4·2	5·4	6·9	8·5	9·8	11·6
Height of liquid (cm.)	2·0	3·8	5·3	7·0	9·1	10·7	13·0

Plot the above results, explain why the graph does not pass through the origin, and deduce from the graph an accurate value for the specific gravity of the liquid. (N.)

44. Derive an expression showing the change in the saturation vapour pressure which is produced by curvature of a liquid surface. Discuss the effects of this change on meteorological phenomena.

Calculate the change in the saturated vapour pressure due to curvature in the case of a water droplet of diameter 0·002 mm. at 15° C., given the following data :

Surface tension of water at 15° C., 74 dynes per cm.
S.V.P. of water at 15° C., 1·3 cm. of mercury.
Density of air under standard pressure at 15° C., 1·23 gm. per litre.
Relative density of water vapour to air, 0·625. (N.)

CHAPTER 10

SIMPLE HARMONIC MOTION

WHEN a body, subject to elastic constraints of any kind, is displaced from its equilibrium position against any such constraint, it suffers a restoring force tending to return it towards the equilibrium position. If also Hooke's law holds, then the *restoring force* acting on the body is proportional to the displacement from the equilibrium position.

As, by the Second Law of Motion,

$$\text{force} = \text{mass} \times \text{acceleration},$$

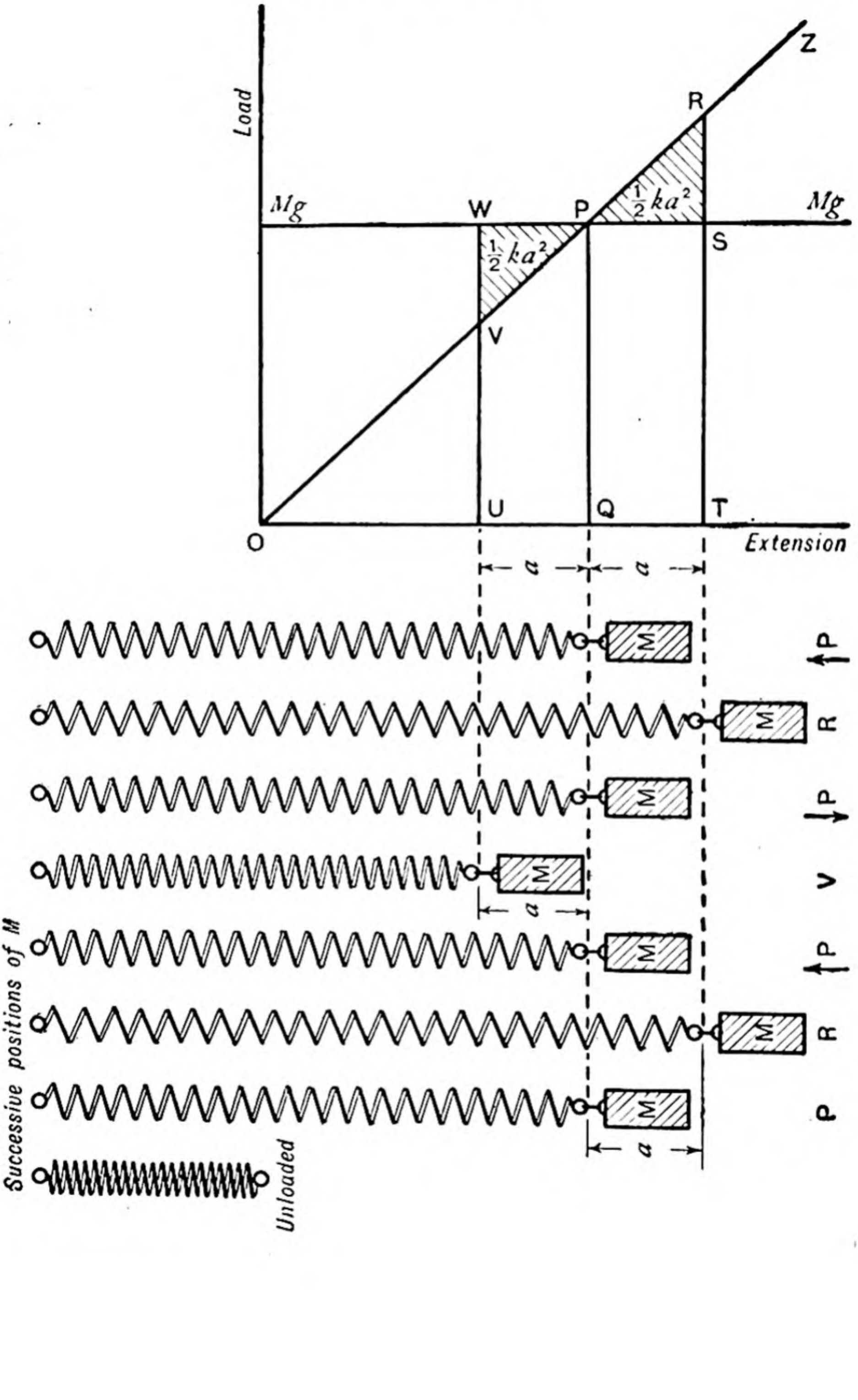
this means that the *acceleration* of the body is proportional to its displacement from the equilibrium position and is directed always towards it.

The resulting motion is a type of great importance in all branches of Physics. It is called **Simple Harmonic Motion** (often written **S.H.M.**).

If a particle moves in a straight line so that, at any instant, its acceleration is proportional to its displacement at that instant from a fixed point in the line, and is always directed towards that point, the particle executes a linear simple harmonic motion.

Examination of a familiar case will show what kind of motion this is. While simple harmonic motion and motions that are nearly simple harmonic can arise in many ways, *elastically constrained* motion is the most important type of true S.H.M.

Oscillations. A mass of M gm. hangs vertically from a spiral spring, fixed at its upper end. The spring obeys Hooke's law, and the force required to extend it by 1 cm. is k dynes. The load-extension graph is the straight line OZ (see graph); the horizontal line represents the weight Mg dynes of the mass. The two lines intersect at P, and the steady extension when M is at equilibrium is represented by OQ, while PQ (which is Mg) is the tension in the spring.


Load
Mg
W
P
R
Z
$\frac{1}{2}ka^2$
$\frac{1}{2}ka^2$
Mg
S
V
U
Q
T
O
Extension
a
a
M
P
R
V
P
R
P
a
a
Successive positions of M
Unloaded

Areas beneath the graph and the extension-axis represent quantities of work.

Let the mass M be pulled downwards a distance a cm., so that its state is represented by the point R on the graph ; the work done in producing this displacement is represented by the area $PRSTQ$; this is the energy stored in the spring ready to restore matters when M is released. But to raise M a distance a cm. needs only Mga ergs of energy, represented by the area $PSTQ$. So the balance, represented by PRS, is still possessed by M when it has returned to the zero position ; and it must be in the form of kinetic energy. So M overshoots the zero position. How far past this point will it go?

The work done by the spring in contracting a distance a cm. from the equilibrium position is represented by the area $PQUV$; but the work done in raising the *weight* through this distance is Mga ergs, represented by the area $PWUQ$, and this rectangle is completed by adding to $PQUV$ the area of WPV, which is equal to that of PRS. In other words, the kinetic energy of the mass just suffices, together with the potential energy released by the spring, to raise the load a distance a cm. *above* the mean position, to the state represented by the point V on the graph.

Here, the restoring force is downwards ; the potential energy released by M in falling back to P is represented by the area $WPQU$, the work done in extending the spring to Q by the area $VPQU$, so M shoots through the zero position with kinetic energy represented by the area WPV, which suffices (together with the potential energy given up by the fall of M) to extend the spring to R again, whence the whole cycle of events is repeated.

So the mass M oscillates about the mean position, travelling backwards and forwards with a maximum displacement a cm. on either side of the mean. This maximum displacement is called the **amplitude** of the oscillation.

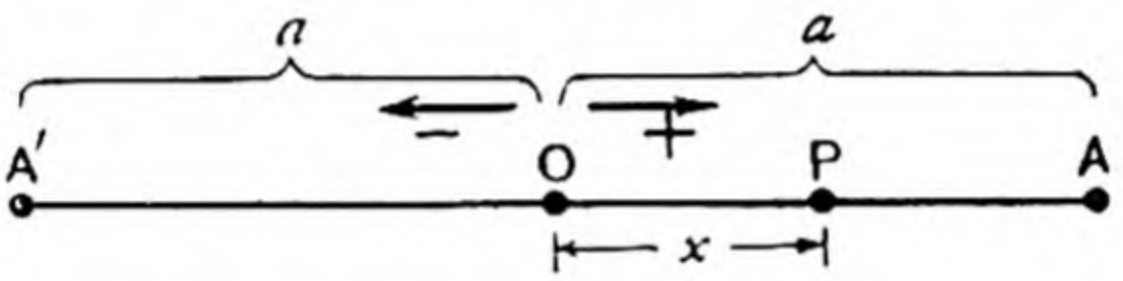
The **period of oscillation** does not depend on the amplitude, but only on the value of k and of M. This period, T, is defined as **the interval between two successive passages of the same point in the same direction.**

It will be seen that the oscillations are really a continuous conversion of energy from one form to another. The initial displacement a cm. furnishes a store of potential energy, $\frac{1}{2}ka^2$ ergs, which is

converted to kinetic energy as M travels towards the centre of its path, is all kinetic energy as M passes through the centre, and is all potential energy again when the upper turning point is reached. If the system described had no means of dissipating this energy, the oscillations would go on for ever. In practice, of course, work is done in overcoming air resistance and overcoming the internal forces in the spring, and this energy is gradually removed. The oscillations actually observed are **damped harmonic motion.** The extent of the damping, or means of dissipating energy from the vibrating system, is an important matter in actual problems.

Equations of Linear Simple Harmonic Motion

Let x cm. be the displacement at time t sec. from the mean position ; a cm. be the amplitude ; m gm. be the mass of the particle and let the velocity at displacement x cm. be v cm. per sec.



Acceleration at a given displacement. From the definition of S.H.M., the acceleration is proportional to the displacement from the mean position and is directed towards it, so

$$\text{acceleration} = -\mu x,$$

where the minus sign indicates that the acceleration is always towards the centre, while x is measured outwards away from the centre ; μ is the constant of proportionality and is always positive, a point often emphasised by writing μ as a *square* (ω^2), so that the expression for the acceleration is $-\omega^2 x$.

Velocity at a given displacement. The velocity, v cm./sec., at a given displacement x cm. is given by

$$v = \pm\sqrt{\mu(a^2 - x^2)},$$

the double sign being used because each value of x is passed through twice during one complete oscillation—once on the outward and once on the return journey. The velocity is zero when $x = a$, at the extreme limit of the path, and the velocity is a maximum, $\sqrt{\mu a}$ cm. per sec., when $x = 0$ at the centre.

Energy at a given displacement. The total energy,

kinetic energy + potential energy,

must be the same at all stages of the motion.

At displacement x,

$$\text{K.E.} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\mu(a^2 - x^2) \text{ ergs.}$$

Now, when $x = 0$, all the energy is kinetic, so the total energy is the kinetic energy when $x = 0$, namely $\tfrac{1}{2}m\mu a^2$ ergs.

Thus, at displacement x cm.,

$$\text{P.E.} = (\text{total} - \text{K.E.}) = \tfrac{1}{2}m\mu a^2 - \tfrac{1}{2}m\mu(a^2 - x^2) = \tfrac{1}{2}m\mu x^2 \text{ ergs.}$$

When $x = a$, all the energy is P.E., and is equal to $\tfrac{1}{2}m\mu a^2$ ergs.

The following table illustrates the changes of energy during an oscillation :

Displacement (x cm.)	K.E. (ergs)	P.E. (ergs)	Total energy (ergs)
0	$\tfrac{1}{2}m\mu a^2$	0	$\tfrac{1}{2}m\mu a^2$
Between 0 and $\pm a$	$\tfrac{1}{2}m\mu(a^2 - x^2)$	$\tfrac{1}{2}m\mu x^2$	$\tfrac{1}{2}m\mu a^2$
$\pm a$	0	$\tfrac{1}{2}m\mu a^2$	$\tfrac{1}{2}m\mu a^2$

Displacement at a given time. The displacement x cm. at time t sec. is

$$x = a \sin(\sqrt{\mu}\, t + \epsilon),$$

where $(\sqrt{\mu}\, t + \epsilon)$ is called the **phase**, or **phase angle**, and ϵ, variously known as the **phase difference** or **epoch**, depends solely on the displacement at the instant when timing is started. The value of ϵ can be found from the initial displacement, for when $t = 0$, then $x = a \sin \epsilon$.

If $x = 0$ when $t = 0$, then

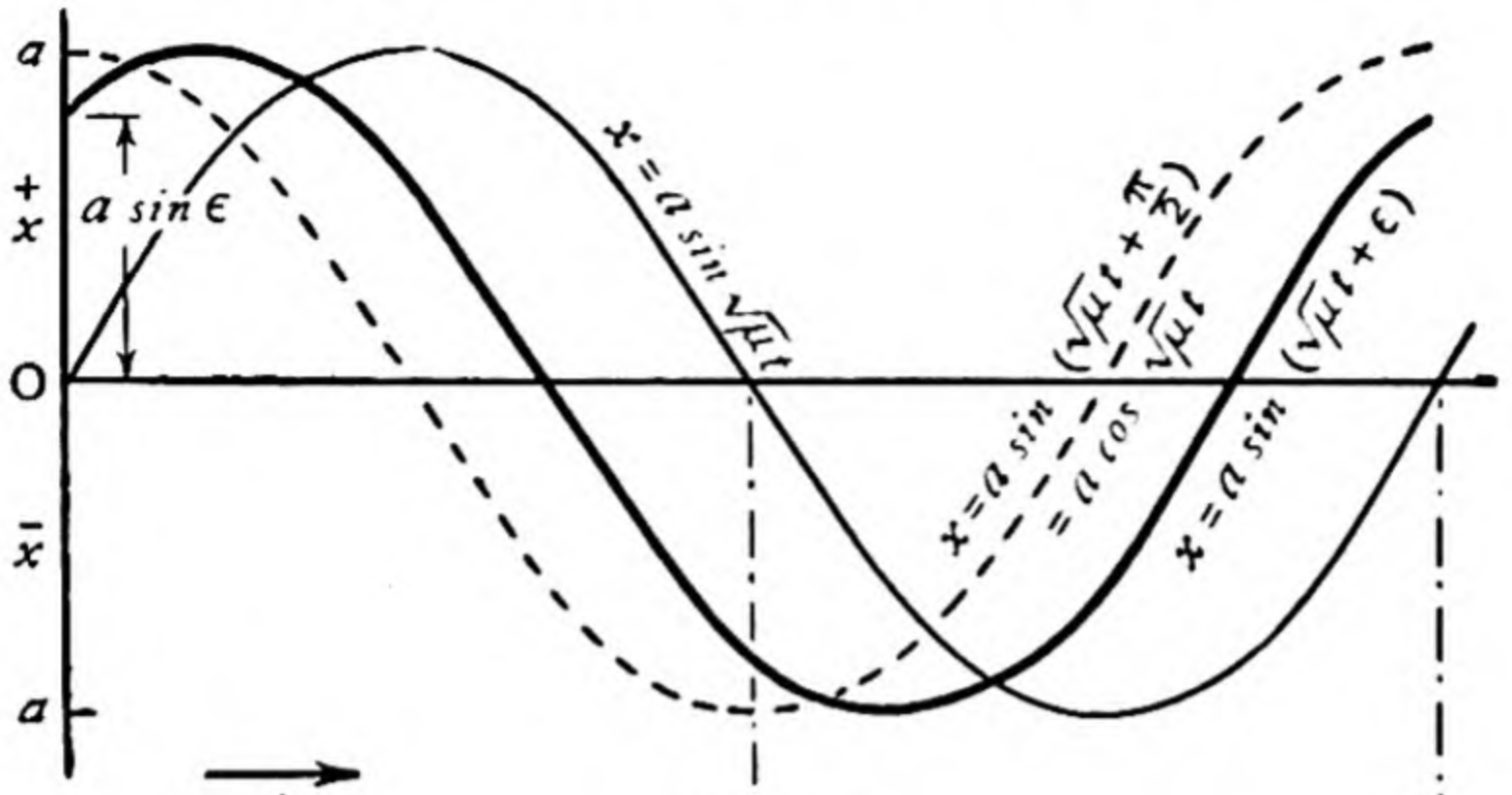
$$\epsilon = 0, \text{ and } x = a \sin \sqrt{\mu}\, t.$$

If $x = a$ when $t = 0$, then

$$\epsilon = \pi/2, \text{ and } x = a \sin(\sqrt{\mu}\, t + \pi/2), \text{ or}$$

$$x = a \cos \sqrt{\mu}\, t.$$

The diagram on p. 291 shows the displacement-time graphs for each of the forms in which the S.H.M. equation may be met.



The thick line represents the general form $x = a \sin (\sqrt{\mu}\, t + \epsilon)$; the thin line $x = a \sin \sqrt{\mu}\, t$; and the dotted line $x = a \cos \sqrt{\mu}\, t$.

Period and frequency. In one complete oscillation, the phase $(\sqrt{\mu}\, t + \epsilon)$ increases by 2π, for the sine of an angle goes through all possible values as the angle goes through one complete revolution, or 2π radians.

As $\sqrt{\mu}\, t + \epsilon$ increases by 2π, and ϵ is constant, $\sqrt{\mu}\, t$ must have increased by 2π, and t increased by $2\pi/\sqrt{\mu}$; thus the period, T sec., which is the interval between successive passings of the same point on the path in the same direction, is given by

$$T = 2\pi/\sqrt{\mu} \text{ sec.}$$

But, μ = displacement/acceleration (p. 289), so

$$T = 2\pi \sqrt{\frac{displacement}{acceleration}} \text{ sec.}$$

This may be variously written as

$$T = 2\pi/\sqrt{(acceleration\ at\ unit\ displacement)},$$

or

$$T = 2\pi \sqrt{\frac{mass\ of\ particle}{force\ at\ unit\ displacement}}.$$

The frequency, n per sec., is the number of oscillations per sec.; thus

$$n = 1/T = \sqrt{\mu}/2\pi, \quad \text{or} \quad \sqrt{\mu} = 2\pi n.$$

Substitution for $\sqrt{\mu}$ in the displacement-time relation gives other forms of the equation which may be useful:

$$x = a \sin (2\pi n t + \epsilon)$$

and

$$x = a \sin \left(\frac{2\pi}{T} t + \epsilon\right).$$

The deduction of the velocity-displacement and displacement-time equations from the defining equation will be found in text-books of applied mathematics, where they should be sought by the student who has reached the stage of solving elementary differential equations. They will be obtained below by considering a simple particular case of S.H.M. It is recommended that an effort be made to memorise the equations listed above, and to understand what they represent, without troubling at this stage about the most elegant proof. The basic minimum that must certainly be known is

(i) any equation of the form

$$acceleration = -\mu\ (displacement)$$

represents a simple harmonic motion.

(ii) the period of this simple harmonic motion is

$$T = 2\pi \sqrt{\frac{displacement}{acceleration}}.$$

Uniform circular motion as two S.H.M's. at right angles. A particle describes a circle of radius a cm. with uniform angular velocity $\sqrt{\mu}$ radians per second. If we start timing when it passes A, then at time t the angle $\theta = \sqrt{\mu}\, t$; BOB' and AOA' are two diameters at right angles, which we will take as co-ordinate axes.

OP and OQ represent the displacements x and y along these axes at time t.

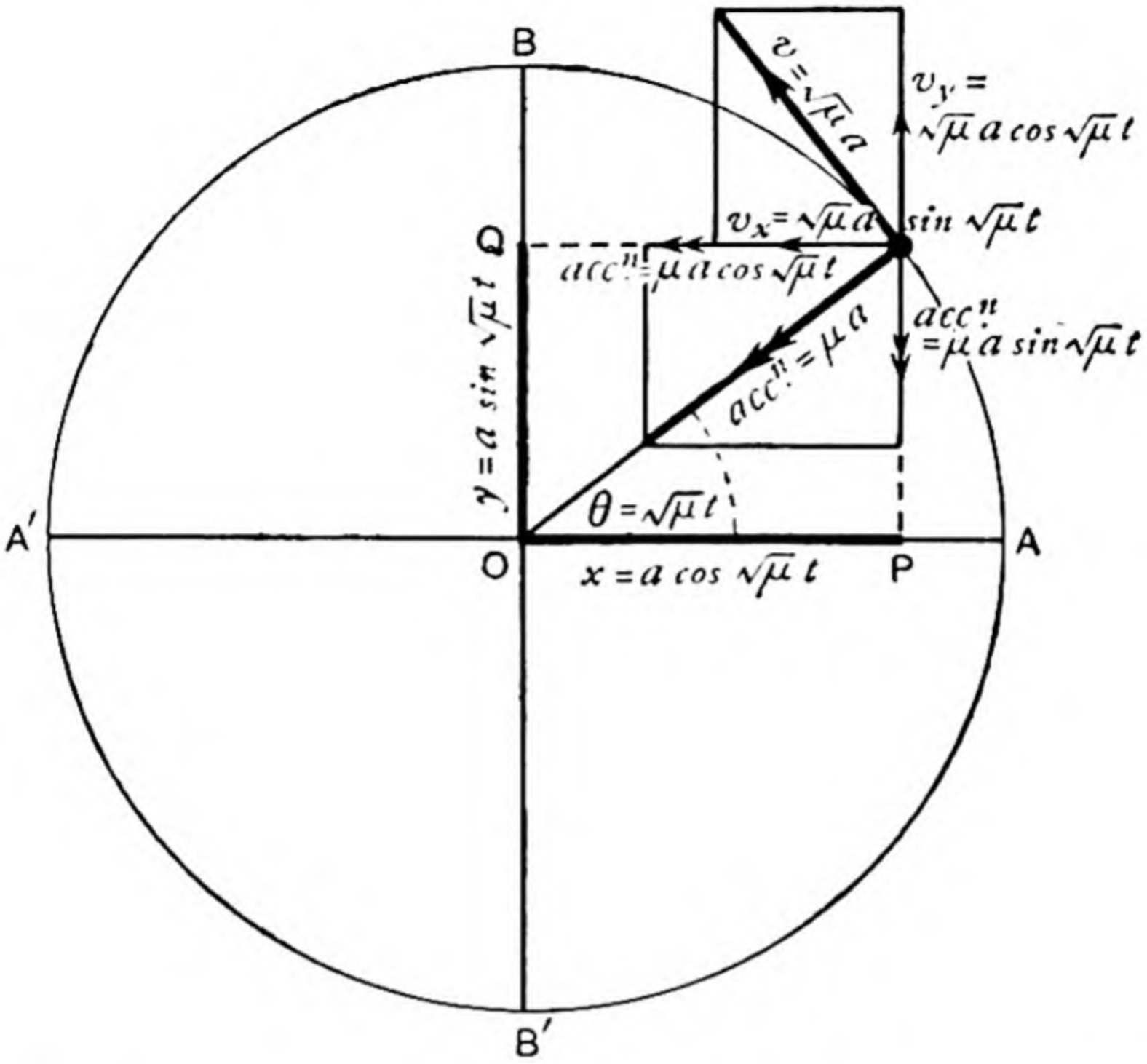
Then,

$$x = a \cos \theta = a \cos \sqrt{\mu}\, t,$$

and

$$y = a \sin \theta = a \sin \sqrt{\mu}\, t = a \cos (\sqrt{\mu}\, t - \pi/2).$$

Thus P moves along AOA' with S.H.M., and Q moves along BOB' with S.H.M. of the same amplitude and period, but differing in phase by $\pi/2$ (or a quarter of a period "out of step").



Uniform motion in a circle can thus be regarded as a motion compounded of two simple harmonic motions of equal amplitude and period, with a phase difference of $\pi/2$.

Proof of formulae. This discussion can be reworked from first principles, showing that the points P and Q must necessarily execute S.H.M., and using the information emerging to prove the equations listed earlier.

The *velocity* of the particle at time t (since $v = r\omega$) is $\sqrt{\mu}\, a$ cm. per sec., at $\left(\frac{\pi}{2} - \theta\right)$ to OA and θ to OB.

The *acceleration* of the particle at time t (since accel. $= r\omega^2$) is μa towards O at an angle θ to AO and $\left(\frac{\pi}{2} - \theta\right)$ to BO.

The resolved parts of each of these quantities in the directions OA and OB give the values appropriate to P and Q respectively, thus :

For the displacements at time t,

$$x = OP = a\cos\theta = a\cos\sqrt{\mu}\,t$$
$$y = OQ = a\sin\theta = a\sin\sqrt{\mu}\,t.$$

For the velocities,

$$v_x = \sqrt{\mu}\,a\sin\sqrt{\mu}\,t = -\sqrt{\mu(a^2-x^2)}$$
$$v_y = \sqrt{\mu}\,a\cos\sqrt{\mu}\,t = \sqrt{\mu(a^2-y^2)}.$$

For the accelerations,

$$accel._x = \mu a\cos\theta \text{ towards } O$$
$$= -\mu a\cos\theta \text{ outwards from } O,$$
$$= -\mu x,$$

so the motion of P is simple harmonic, as defined on p. 286.

$$accel._y = \mu a\sin\theta \text{ towards } O,$$
$$= -\mu a\sin\theta \text{ away from } O,$$
$$= -\mu y,$$

and again, the motion of Q is simple harmonic, as defined on p. 286.

The period of each S.H.M. is the time taken to describe one revolution (2π radians) at $\sqrt{\mu}$ radians per second, so

$$T = 2\pi/\sqrt{\mu} \text{ sec.}$$

Thus we have shown that if the equations listed on pp. 290-292 are applicable to the motion of a particle, then this motion is simple harmonic, as defined on p. 286.

Examples of simple harmonic motion.

(*a*) *Mass on spiral spring.* A mass of M gm. is hung from a spiral spring, for which it is found that Hooke's law is obeyed and that k dynes produce an extension of 1 cm.

It rests in equilibrium with the extension such that the tension just balances the weight of M. Consider a small downwards displacement x cm. ; the additional tension is kx dynes, so the resultant force (upwards) is $-kx$ dynes ; and, using

$$force = mass \times acceleration,$$

the acceleration is k/Mx cm. per sec. per sec. upwards ; or, using the algebraic sign to indicate direction,

$$accel. = \frac{-k}{M}\,.\,x.$$

This is a simple harmonic motion, of period

$$T = 2\pi\sqrt{\frac{M}{k}} \text{ sec.}$$

It is here supposed that the mass of the spring is so small compared with M that it can be neglected. If this is not so, the effective addition to M is one-third the actual mass S gm. of the spring itself; so the usual formula becomes

$$T = 2\pi\sqrt{\frac{M + \frac{1}{3}S}{k}} \text{ sec.}$$

(*b*) *Floating cylinder.* A cylinder of mass M gm. and area of cross-section A sq. cm., floats upright immersed to a depth l cm. in a liquid of density ρ gm. per c.c. (see diagram).

In equilibrium the weight is balanced by the Archimedes' upthrust. Consider a small displacement depressing it a distance x below the rest position. A further Ax c.c. of liquid are displaced, of mass $Ax\rho$ gm., giving an additional unbalanced upthrust of $Ax\rho g$ dynes.

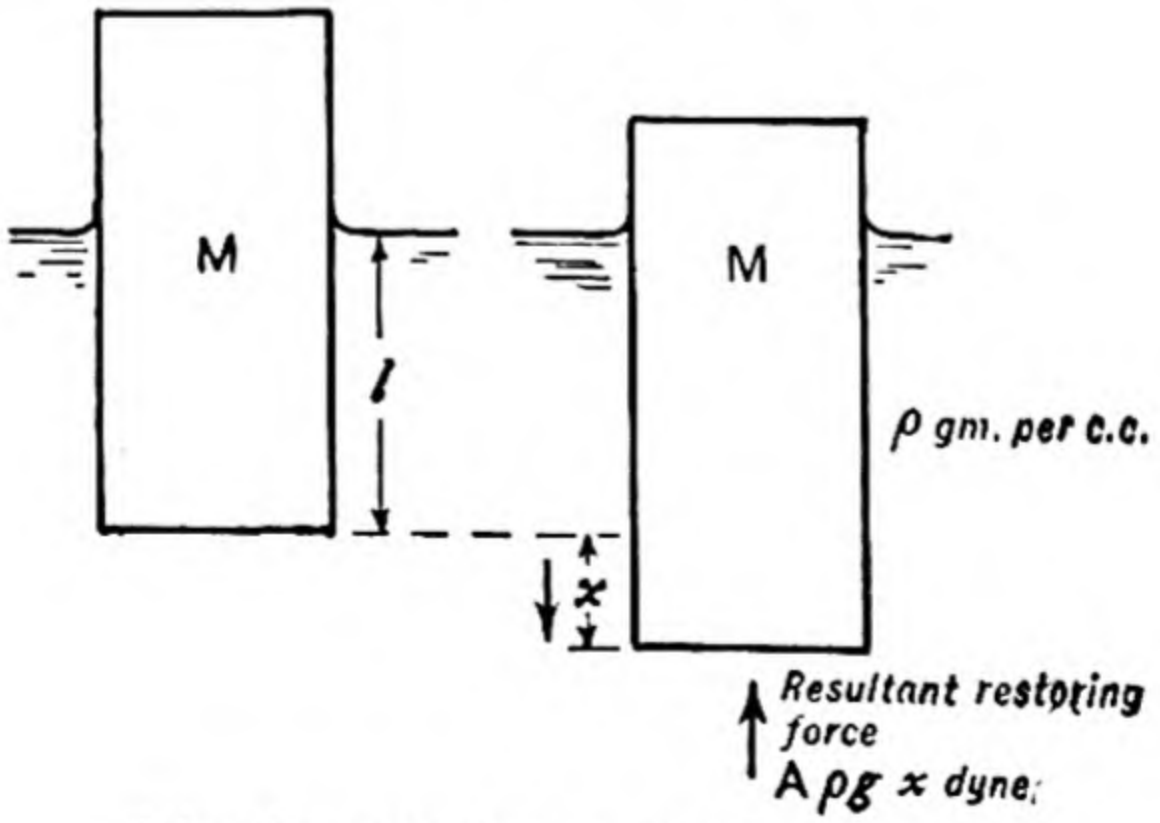
So, the restoring force is $-Ag\rho x$ dynes, and, using

$$force = mass \times acceleration, \quad acceleration = -\frac{A\rho g}{M} \cdot x.$$

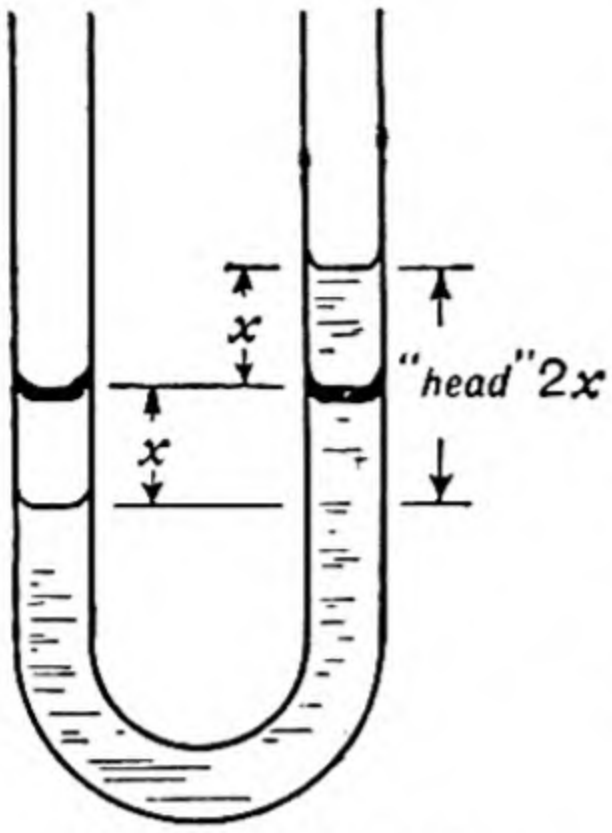
The resulting motion is S.H.M., of period

$$T = 2\pi\sqrt{\frac{M}{A\rho g}}.$$

As $M = Al\rho$, this may be written $T = 2\pi\sqrt{l/g}$.



Floating cylinder oscillating in liquid



Liquid oscillating in a U-tube

(c) *Liquid column in a U-tube.* A liquid column of total length l cm. and density ρ gm. per c.c. is at rest in a U-tube of uniform cross-section area A sq. cm..

Consider a small displacement x cm., as shown in the diagram; this causes a "head" of $2x$ cm., and a restoring force of $-2xA\rho g$ dynes.

The total mass of liquid is $lA\rho$ gm.; so, using

$$force = mass \times acceleration,$$

$$acceleration = -\frac{2A\rho g}{l\rho A}x = -\frac{2g}{l}x.$$

Thus the motion is S.H.M., of period

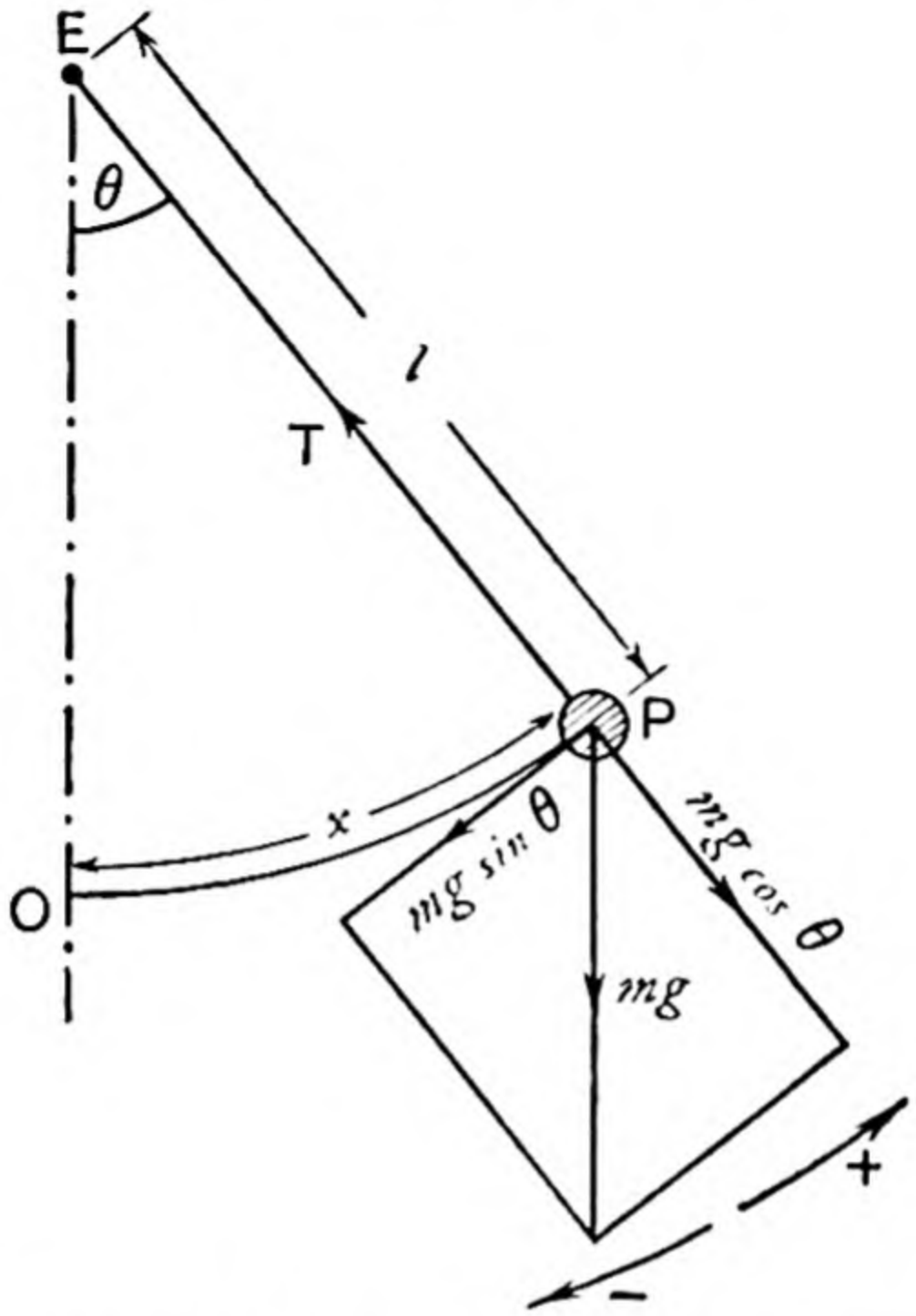
$$T = 2\pi\sqrt{l/2g} \text{ sec.}$$

In both the last two examples, very heavy damping prevents the maintenance of the calculated motion and the results would only be true for a liquid of negligible viscosity. But in all three cases, it is not necessary that the displacement x should be extremely small, provided only that the limit of proportionality between restoring force and x is not exceeded.

Approximate simple harmonic motion; the simple pendulum. The simple pendulum consists of a small heavy bob, of mass m gm., at the end of a light inextensible string of length l cm., firmly held at the upper end E. When pulled aside and released, it swings to and fro in one plane.

Consider a small displacement x cm. along the arc from O to P, and let the angle OEP be θ radians. Then $\theta = x/l$ (see diagram).

The external forces acting on the bob are the tension T dynes in the string, and the weight mg dynes acting vertically downwards. Resolve mg parallel and perpendicular to the tangent at P. The component $mg\cos\theta$ acts radially and is balanced by part of the tension T; the component $mg\sin\theta$, acting towards O, is the only unbalanced force.



Thus, the restoring force is $-mg \sin\theta$, the acceleration is $-g \sin\theta$. Provided that θ is so small that $\sin\theta/\theta$ is nearly equal to unity, then

$$\text{acceleration} = -g\theta \text{ cm. per sec. per sec.}$$

Now, $\theta = x/l$, so

$$\text{acceleration} = -\frac{g}{l}x \text{ cm. per sec. per sec.}$$

Hence the motion is simple harmonic, of period

$$T = 2\pi\sqrt{l/g} \text{ sec.}$$

Note that an approximation has here been made; the motion is only truly simple harmonic if the arc of swing θ is infinitely small. But, for all practical purposes in ordinary work, provided the arc of swing does not exceed a few degrees, the approximation is satisfactory enough.

Angular simple harmonic motion.

If a rigid body rotates about a fixed axis so that at any instant its angular acceleration is proportional to its angular displacement from a fixed line through that axis, the body executes angular simple harmonic motion.

The same calculations as for linear S.H.M. are applicable, with the substitution of the appropriate angular quantities for their linear counterparts.

Thus, angular S.H.M. occurs when

$$\textit{angular acceleration} = -\mu \times \textit{angular displacement};$$

and the period is then

$$T = 2\pi\sqrt{\frac{\textit{angular displacement}}{\textit{angular acceleration}}}.$$

True angular S.H.M. Consider a body of moment of inertia I gm. cm.^{2} about the axis of rotation, suspended from a torsion wire obeying Hooke's law, and such that the couple required to give one radian twist is k dyne cm.

For a small displacement θ radians from the mean position (see diagram),

restoring couple $= -k\theta$ dyne cm.

Using $\textit{couple} = \textit{moment of inertia} \times \textit{angular acceleration}$,

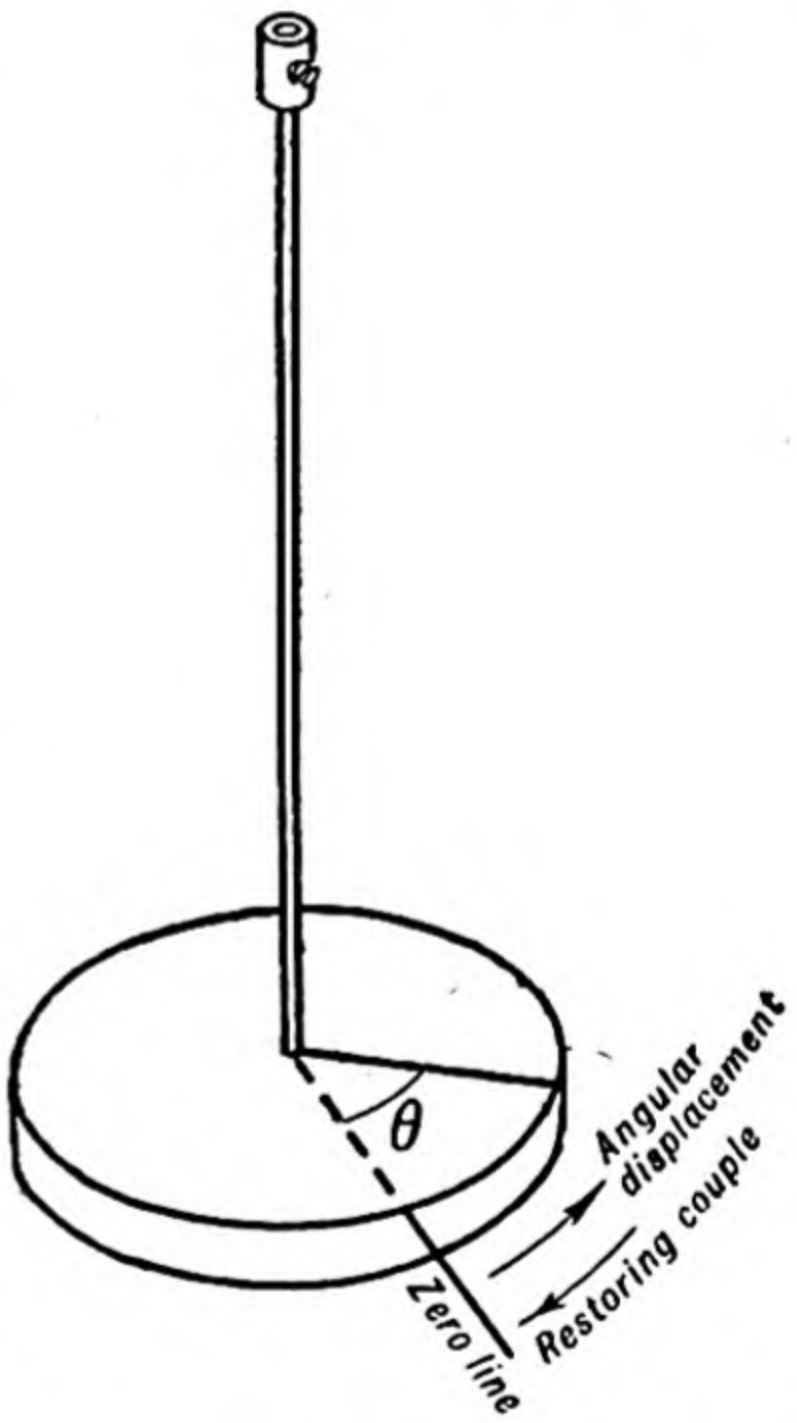
the angular acceleration

$$= -\frac{k}{I}\theta \text{ rad. sec.}^2;$$

this is simple harmonic motion, of

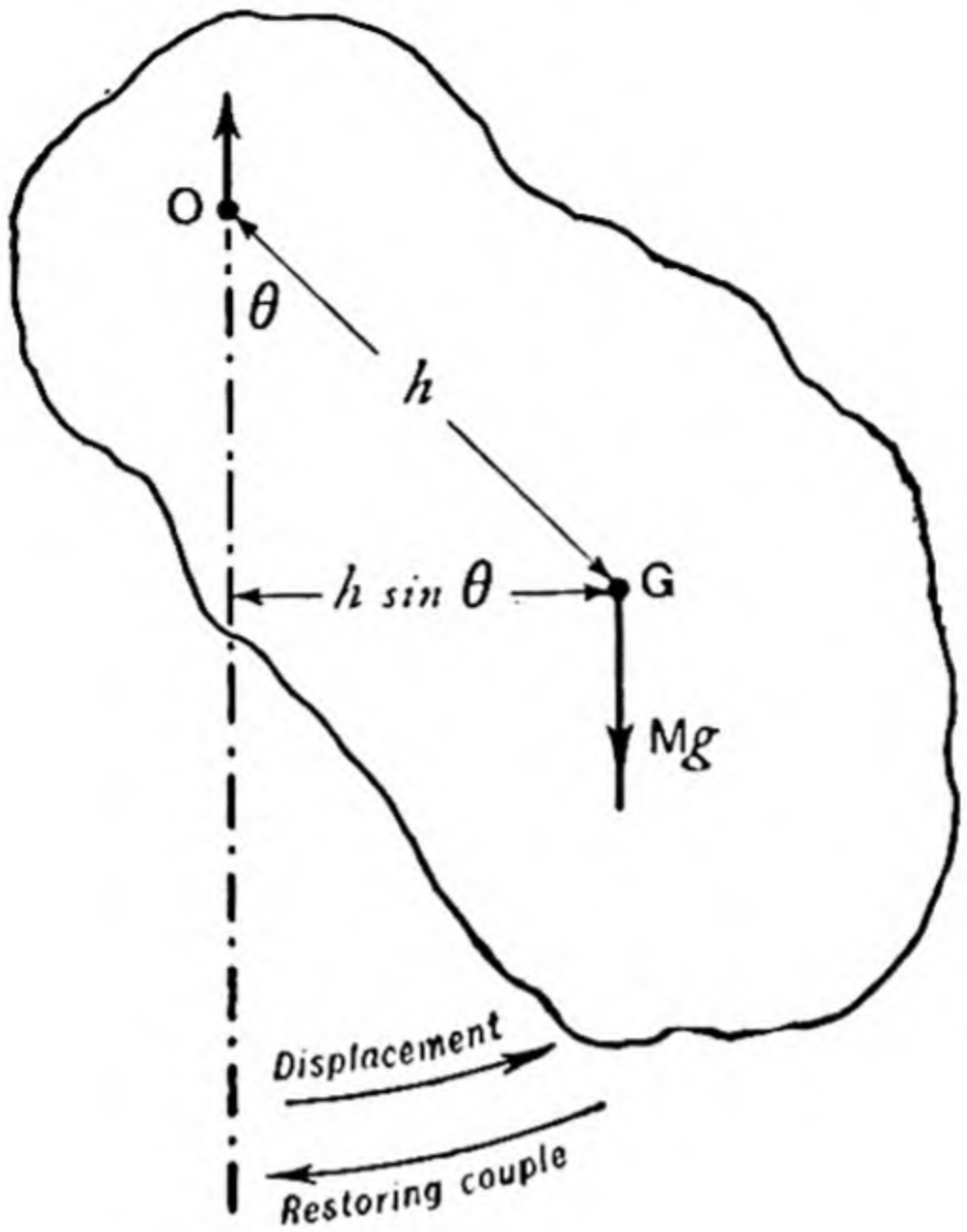
period $T = 2\pi\sqrt{\frac{I}{k}}$ sec.

This is the torsion pendulum, used for the regulation of one type of clock; the arrangement is of considerable importance (see p. 216), and can be used to measure I if k is known, or k if I is known.



Approximate angular S.H.M.; the compound pendulum. A rigid body of mass M gm. is suspended from an axis through some point O, at a distance h cm. from its centre of gravity G (see diagram).

With the symbols of p. 165, the moment of inertia, I gm. cm.^{2},



about O is, by the theorem of parallel axes,

$$I = Mk^2 + Mh^2,$$

where k is the radius of gyration for an axis through G parallel to that used.

Consider a small displacement θ radians ; the restoring couple, C, is $-Mgh \sin\theta$ dyne cm. Using $C = I\alpha$,

$$\alpha = -\frac{Mgh}{I} \sin\theta,$$

or, if θ is so small that θ radians can be written for $\sin\theta$,

$$\alpha = -\frac{Mgh}{I}\theta.$$

This is an angular S.H.M., of period

$$T = 2\pi\sqrt{\frac{I}{Mgh}} \text{ sec.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

The validity of this formula, and the approximation of the motion to S.H.M., depend of course on the angular amplitude being *small*.

Substituting $I = Mk^2 + Mh^2$ in (i),

$$T = 2\pi\sqrt{\frac{k^2+h^2}{hg}} \text{ sec.} \quad \text{.....(ii)}$$

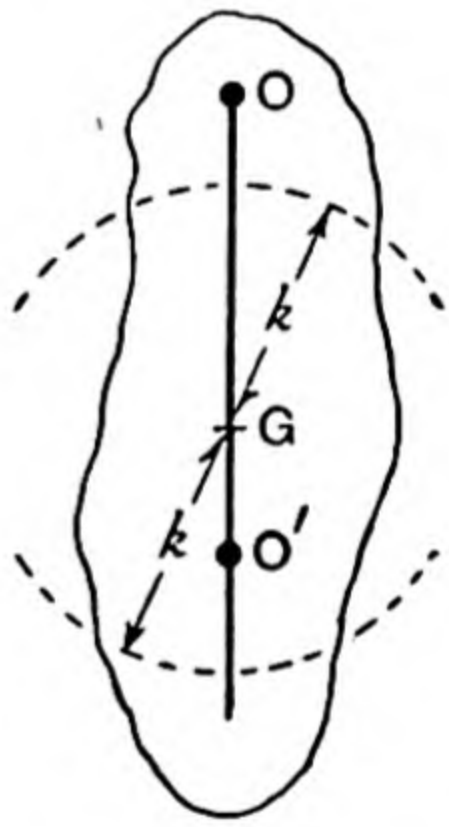
Comparing (ii) with the formula for a simple pendulum of length l cm., which is $T = 2\pi\sqrt{l/g}$ (p. 297), it is seen that

$$l \equiv \frac{k^2+h^2}{h} \text{ cm.}$$

This is the length of the simple pendulum which would have the same period as the compound pendulum; such a simple pendulum is known as the **simple equivalent pendulum.**

Some properties of the compound pendulum. Produce OG to O', such that $OO' = l$ and let $GO' = h'$, then

$$l = (h + h') \text{ (see diagram).}$$



But $l = (k^2/h) + h$, so

$$h' = k^2/h, \quad \text{or} \quad hh' = k^2, \quad \text{or} \quad h = k^2/h'.$$

Substitute for h in $l = (h + h')$,

$$l = \frac{k^2}{h'} + h',$$

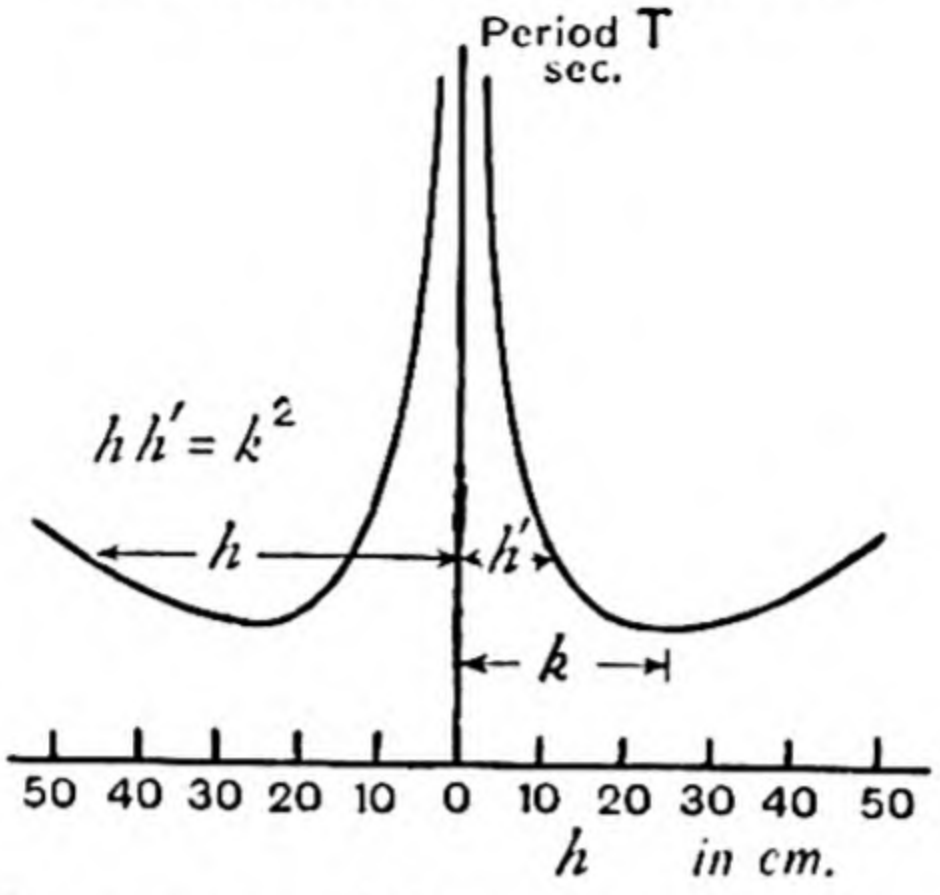
or,

$$l = \frac{k^2 + h'^2}{h'}.$$

Thus, if a fresh pivot were placed at O', the pendulum would have the same length, l, and thus the same period, T.

Hence, two points can be found, one on either side of the centre of gravity, such that the period of oscillation is the same whichever the body is pivoted at. They are called **centre of suspension** and **centre of oscillation** respectively; they are completely interchangeable, and their distance apart is equal to the length of the simple equivalent pendulum.

Since $hh' = k^2$, it follows that if $OG > k$, then $OG' < k$; if, however, we start with $OG < k$, then $OG' > k$. Thus there are in general *two* such pairs of points, though it may happen that all of the four do not lie on the body itself.



Suppose now the position of O is varied ; as

$$T = 2\pi\sqrt{\frac{k^2+h^2}{hg}}$$

the period is very large both when h is very small and when h is very large ; there must therefore be a minimum period for some intermediate value of h.

Write $\dfrac{k^2+h^2}{h}$ as $\dfrac{k^2+h^2-2kh}{h}+\dfrac{2kh}{h}$,

then $$l = \frac{(k-h)^2}{h}+2k,$$

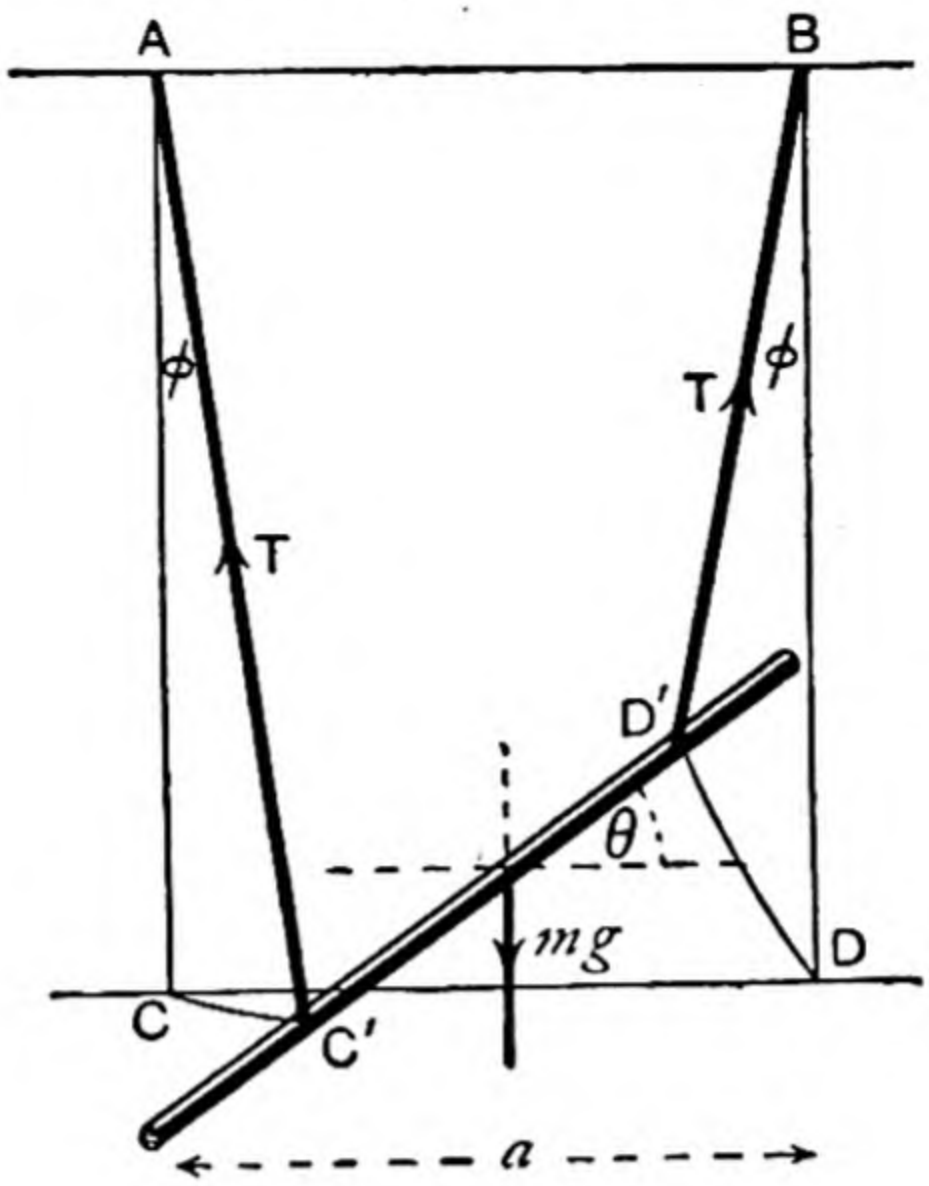
which has its least value when $h=k$, and $l=2k$, giving a minimum period of

$$T = 2\pi\sqrt{2k/g} \text{ sec.}$$

The diagram illustrates the foregoing points for a compound pendulum consisting of an ordinary metre-stick.

Bifilar suspension. A uniform rod CD (p. 302), of mass m gm., and moment of inertia I gm. cm.^{2} about its centre of gravity is suspended by two vertical strings of length l cm. from points A and B which are a distance a cm. apart. The rod is deflected sideways so that it rotates about a vertical axis through a small angle and is then released. Consider the rod at the instant when the angular displacement is θ radians. Call the displacements of the strings from the vertical, CAC' and DBD' in the figure, each ϕ radians ; if T dynes be the tension in each string, resolving these tensions horizontally and vertically,

$$2T\cos\phi = mg,$$



and $$T \sin \phi \,.\, a = G,$$

the restoring couple. If θ is small, ϕ is also small, whence approximately $2T = mg$ and $G = Ta\phi = \frac{1}{2}mga\phi$.

Now $$CC' = \tfrac{1}{2}a\theta = l\phi, \text{ so } \phi = \frac{a\theta}{2l};$$

hence $$G = \tfrac{1}{2}mg \,.\, a \,.\, \frac{a\theta}{2l}$$

$$= \tfrac{1}{4}mg \frac{a^2}{l}\theta$$

towards the zero position. As this is a restoring couple, the angular acceleration α towards the mean position, since $G = I\alpha$, is

$$\alpha = (-)\frac{mga^2}{4lI} \,.\, \theta;$$

so the rod executes angular simple harmonic motion of period

$$2\pi\sqrt{\frac{4lI}{mga^2}} \text{ sec.}$$

Writing $I = mk^2$, where k is the radius of gyration about the centre of gravity, and simplifying, the expression for the period is

$$\frac{4\pi k}{a}\sqrt{\frac{l}{g}}.$$

Composition of simple harmonic motions. A body executing two simultaneous simple harmonic motions in the same straight line has for its displacement at time t the sum of the individual displacements for the two separate motions ; that is, if at time t

$$x_1 = a_1 \sin (2\pi n_1 + \epsilon_1), \quad \text{and} \quad x_2 = a_2 \sin (2\pi n_2 t + \epsilon_2),$$

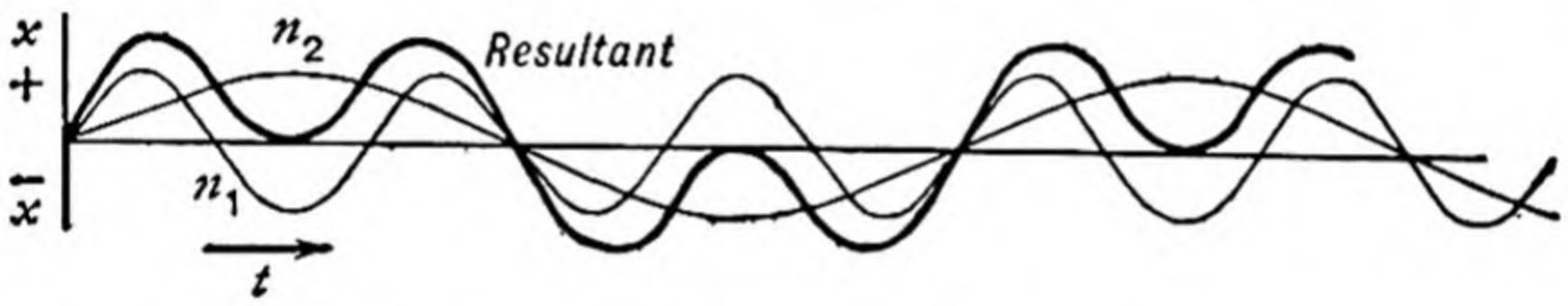
then

$$x = x_1 + x_2 = [a_1 \sin (2\pi n_1 t + \epsilon_1) + a_2 \sin (2\pi n_2 t + \epsilon_2)].$$

In general, the simplest way of compounding two such motions is to draw the displacement-time curve for each, and to add the ordinates for a sufficient number of instants to plot the resultant displacement-time curve.

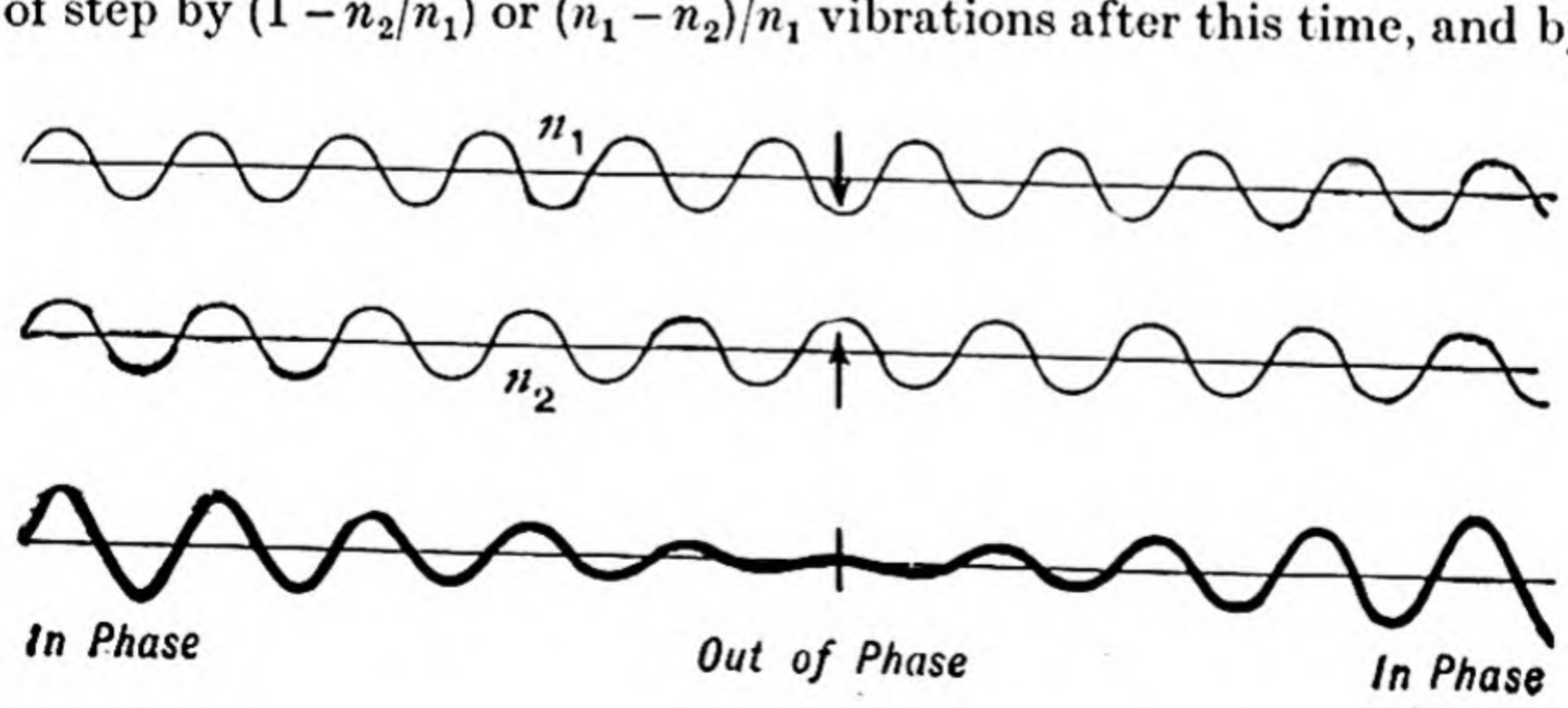
Two particular cases are important :

(*a*) *When n_1 is some multiple of n_2.* The resulting motion then repeats itself with frequency n_2 ; the actual displacement-time relation depends of course on the relative phases of the two at the moment when timing begins, as well as the relative amplitudes.



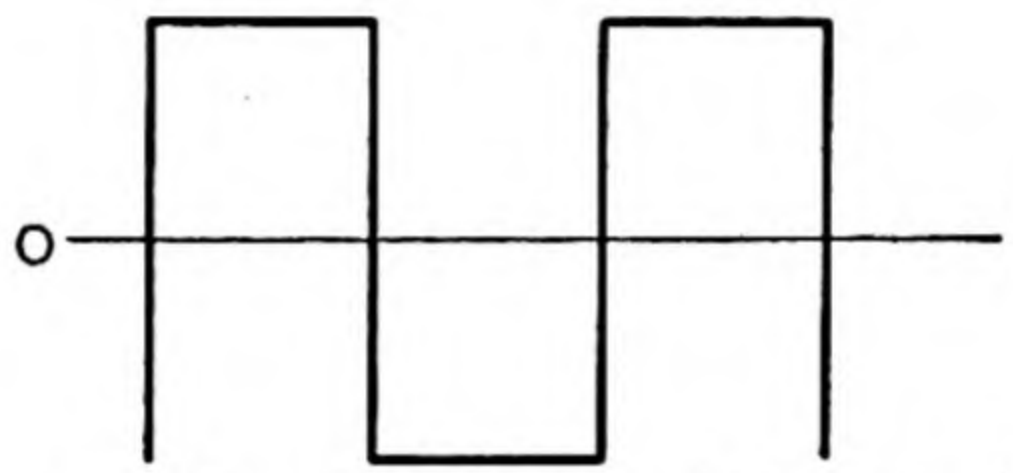
The diagram shows the resultant for $n_1 = 3n_2$, the two starting in phase, and having equal amplitudes.

(*b*) *When n_1 is slightly different from n_2.* If $n_1 > n_2$, in $1/n_1$ sec., the first has made one vibration and the second n_2/n_1 ; so they are out of step by $(1 - n_2/n_1)$ or $(n_1 - n_2)/n_1$ vibrations after this time, and by

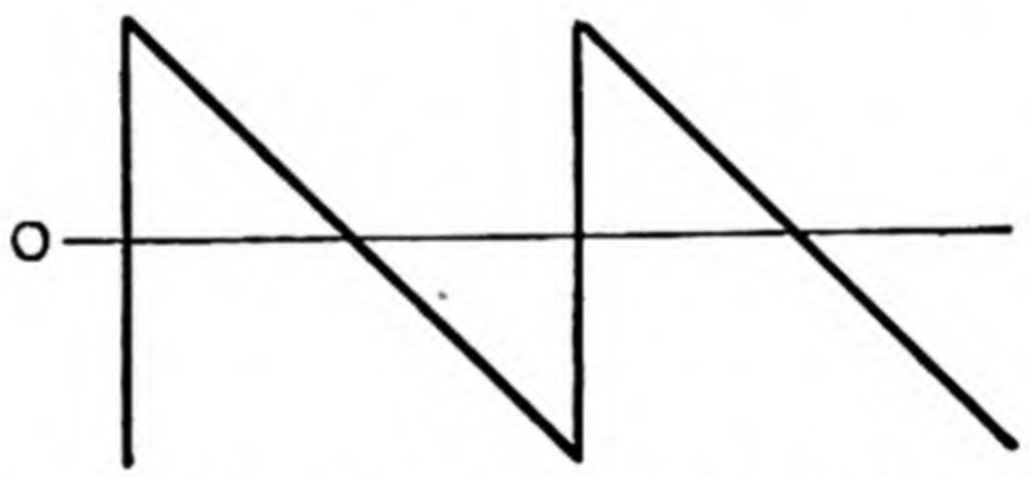


$(n_1 - n_2)$ vibrations after one second. During one second there must then be $(n_1 - n_2)$ occasions when they are exactly in phase, and $(n_1 - n_2)$ occasions when they are exactly out of phase, an effect of alternate reinforcement and destruction known as **beating**. The accompanying diagram illustrates this.

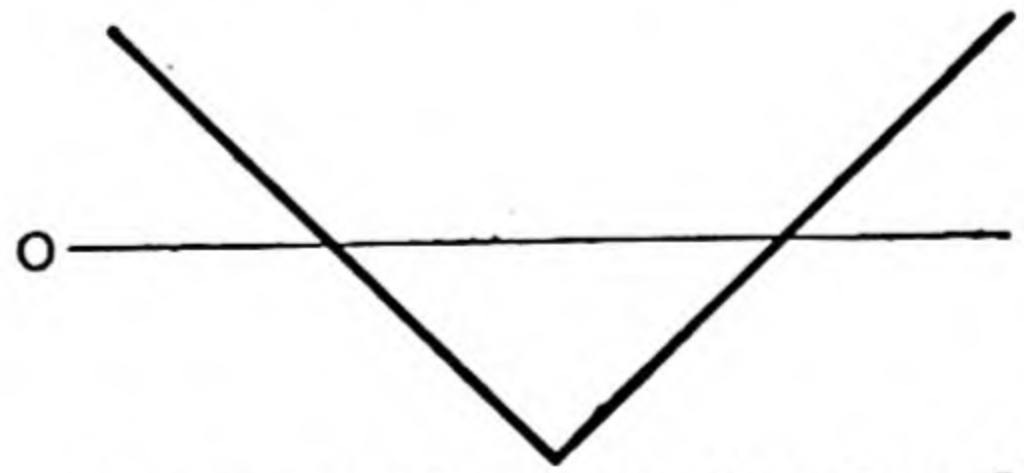
Resolution of periodic motion into simple harmonic motions. Fourier's theorem states that any single-valued finite periodic function can be represented by a series of simple harmonic terms with frequencies that are integral multiples of that of the original function. This means that, although all we may be able to tell by first inspection is that a vibration repeats with frequency n, we can



$$y = a\left[\sin nt + \tfrac{1}{3}\sin 3nt + \tfrac{1}{5}\sin 5nt \ldots\ldots\right]$$



$$y = a\left[\sin nt + \tfrac{1}{2}\sin 2nt + \tfrac{1}{3}\sin 3nt \ldots\ldots\right]$$



$$y = a\left[\cos nt + \tfrac{1}{3^2}\cos 3nt + \tfrac{1}{5^2}\cos 5nt \ldots\right]$$

write it down as the resultant of a set of simple harmonic motions of frequencies 0, n, $2n$, $3n$ ... thus :

$$x = a_0 + a_1 \sin(2\pi nt + \epsilon_1) + a_2 \sin(2\pi \cdot 2nt + \epsilon_2) + a_3 \sin(2\pi \cdot 3nt + \epsilon_3) \ldots,$$

or in the equivalent form

$$x = a_0 + a_1 \sin 2\pi nt + b_1 \cos 2\pi nt + a_2 \sin 2\pi \cdot 2nt + b_2 \cos 2\pi \cdot 2nt + a_3 \sin 2\pi \cdot 3nt + b_3 \sin 2\pi \cdot 3nt \ldots$$

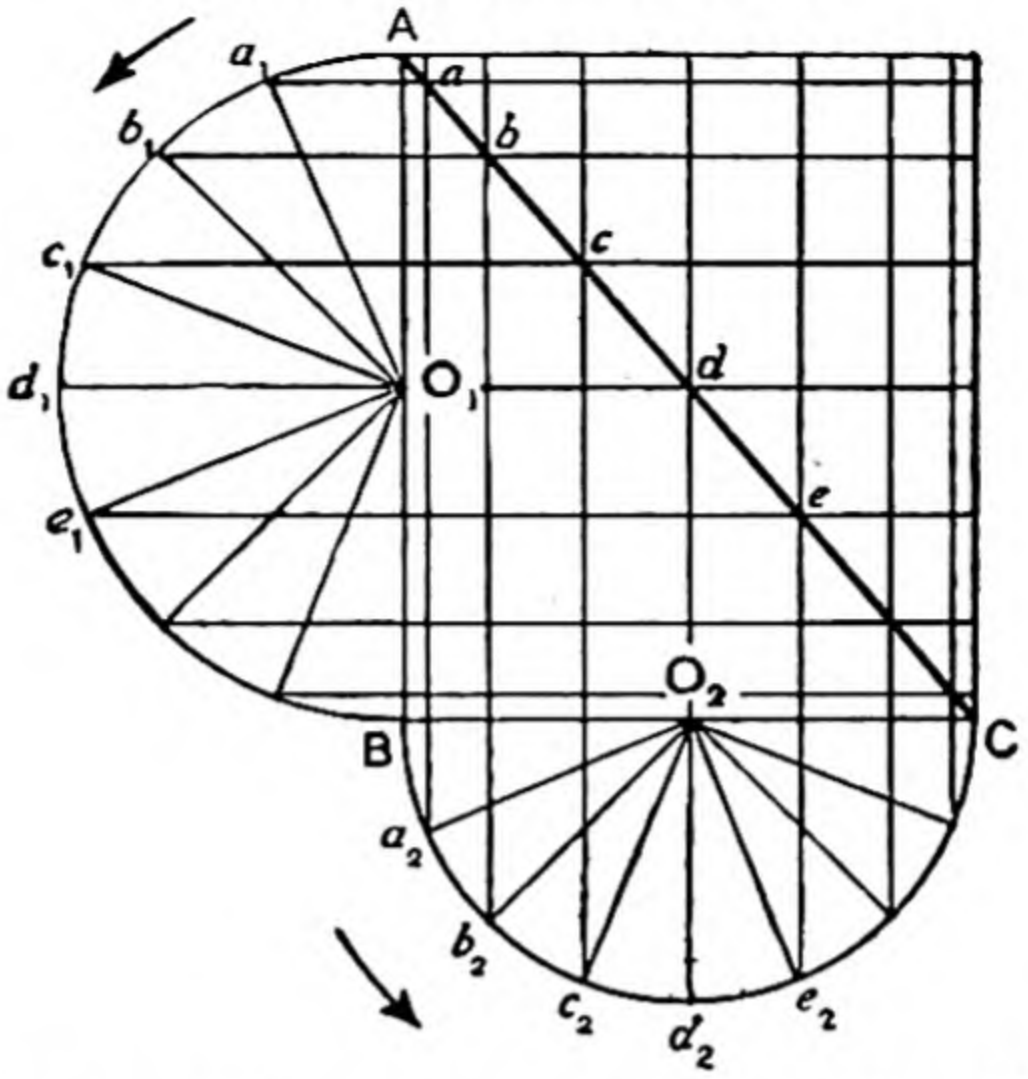
All possible multiples of n are not necessarily represented, and it may often happen that all the a and b coefficients with a few exceptions are zero—thus in the upper diagram of p. 303, a_1 and a_3 are the only coefficients represented.

The three figures reproduced on p. 304 show three important types of oscillation—the square-shaped graph, which represents a periodic "on-off" or reversal, represented by a series of odd values of n, with a proportional to $1/n$; the "saw-tooth" shaped, representing a uniform change with a sudden recovery to the start, represented by a complete series of sine terms with a proportional to $1/n$; and the form which is of importance in the theory of vibrating strings, represented by a series of cosine terms with a proportional to $1/n^2$.

Composition of simple harmonic motions at right angles. As simple harmonic motion is the projection on the diameter of a circle of the motion of a particle describing the circle with uniform speed, we can use the associated circular motion to help construct a time-base when compounding two simple harmonic motions at right angles.

In the diagram (p. 306), two simple harmonic motions of the same period but different amplitudes are compounded. For one motion, the path of the particle is AO_1B, and for the other BO_2C, the mean positions being O_1 and O_2. The two starting-points, at $t=0$, are A and B ; drawing the two circles on AO_1B and BO_2C as diameters, and marking off equal arcs round the circumference for equal time intervals, as $a_1b_1c_1d_1 \ldots$ and $a_2b_2c_2d_2 \ldots$, the corresponding points on the paths of the particle are a, b, c, d, ... , etc., and the motion is a straight line inclined to the paths of the individual vibrations.

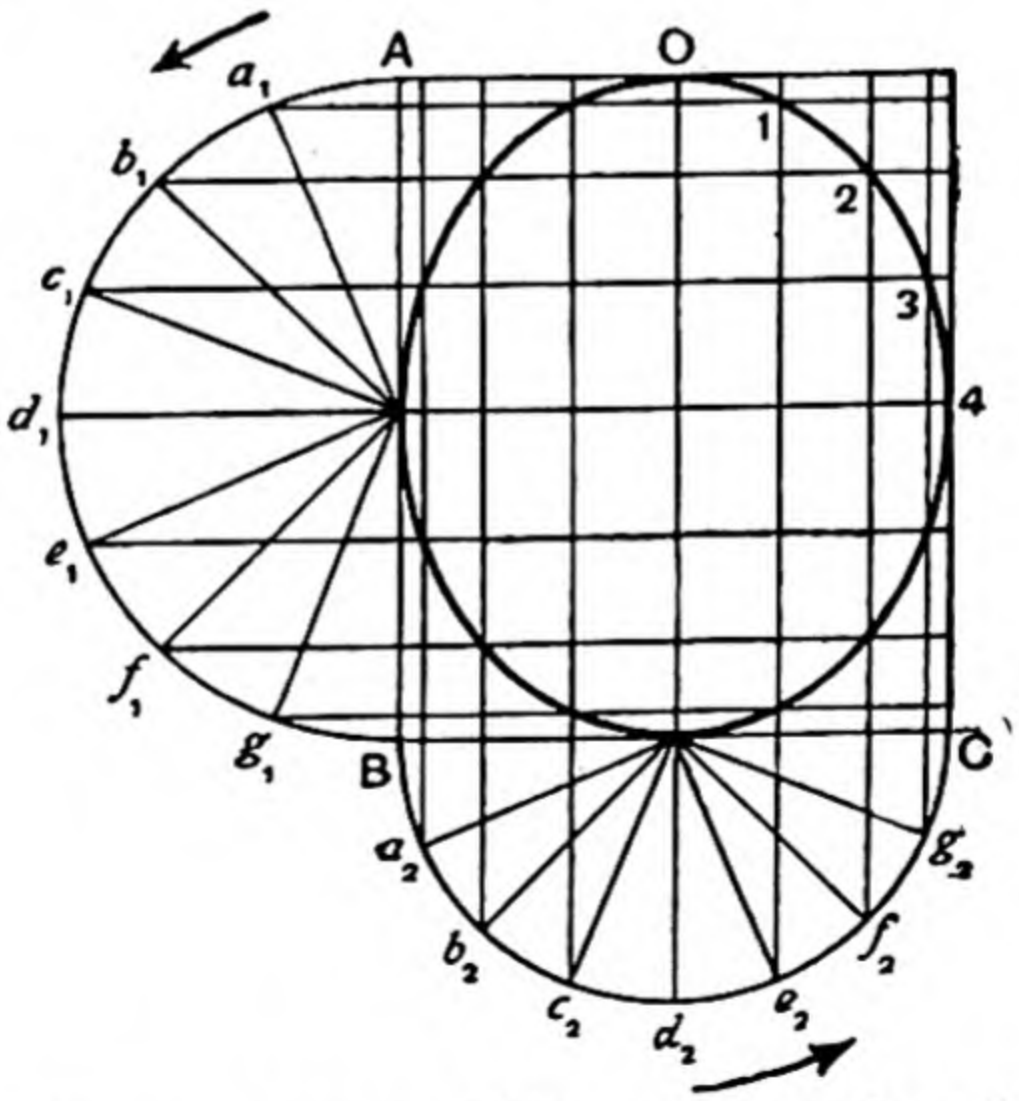
In the diagram below, the periods are the same, but there is a phase difference of $\pi/2$, or a quarter of a period, one starting at A,



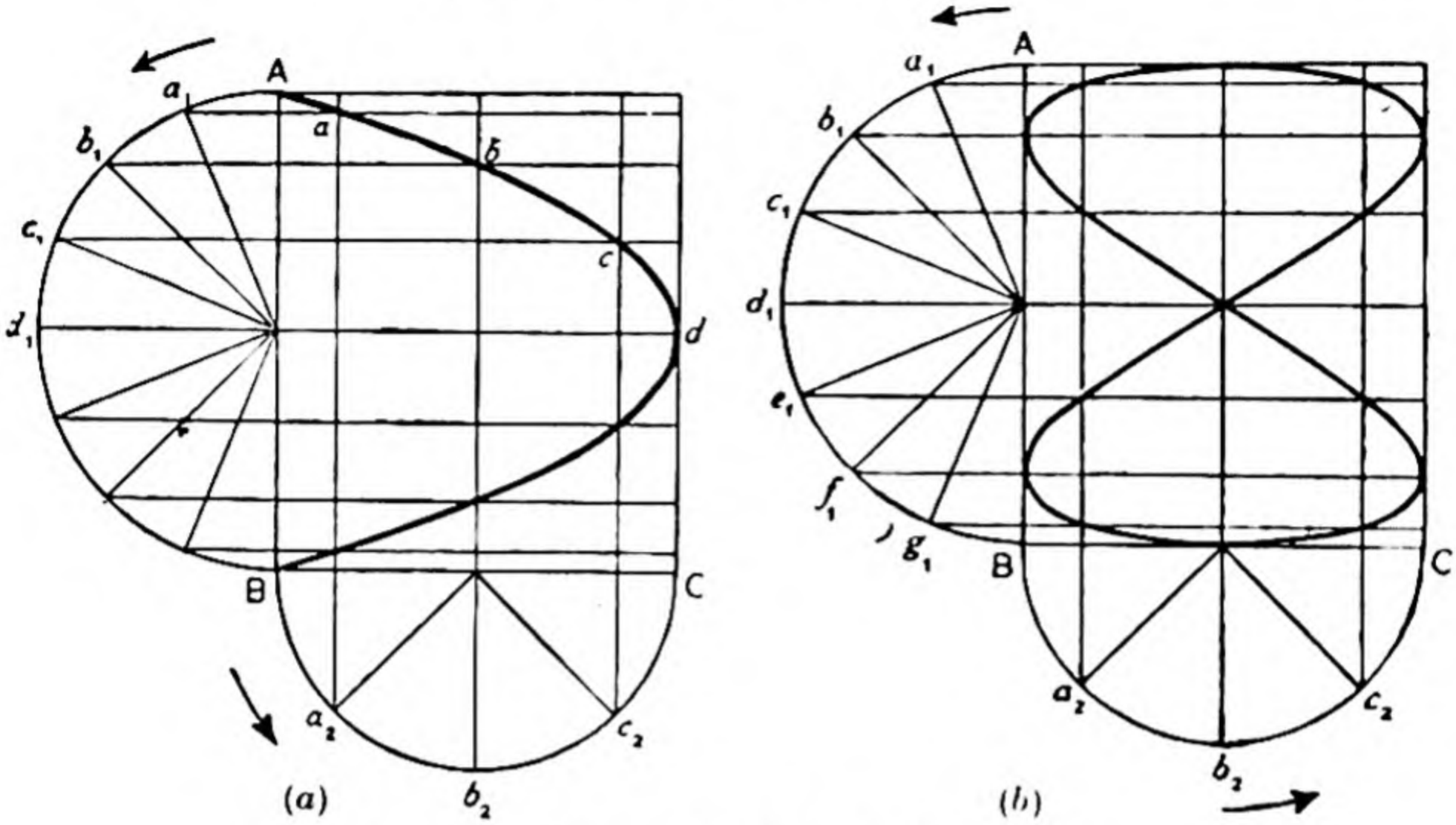
Composition of S.H.M's. at right angles; same period but different amplitudes, phase difference zero

and the other at d_2; working round A, a_1, b_1, ... on the left-hand semicircle, and d_2, e_2, f_2, ... on the lower, the resultant path 0, 1, 2, 3, ... , which is an ellipse, is plotted.

For two simple harmonic motions of different periods, the same method is used, but of course the arcs marked off to denote equal



S.H.M's. at right angles, same period, different amplitudes, phase difference $\pi/2$



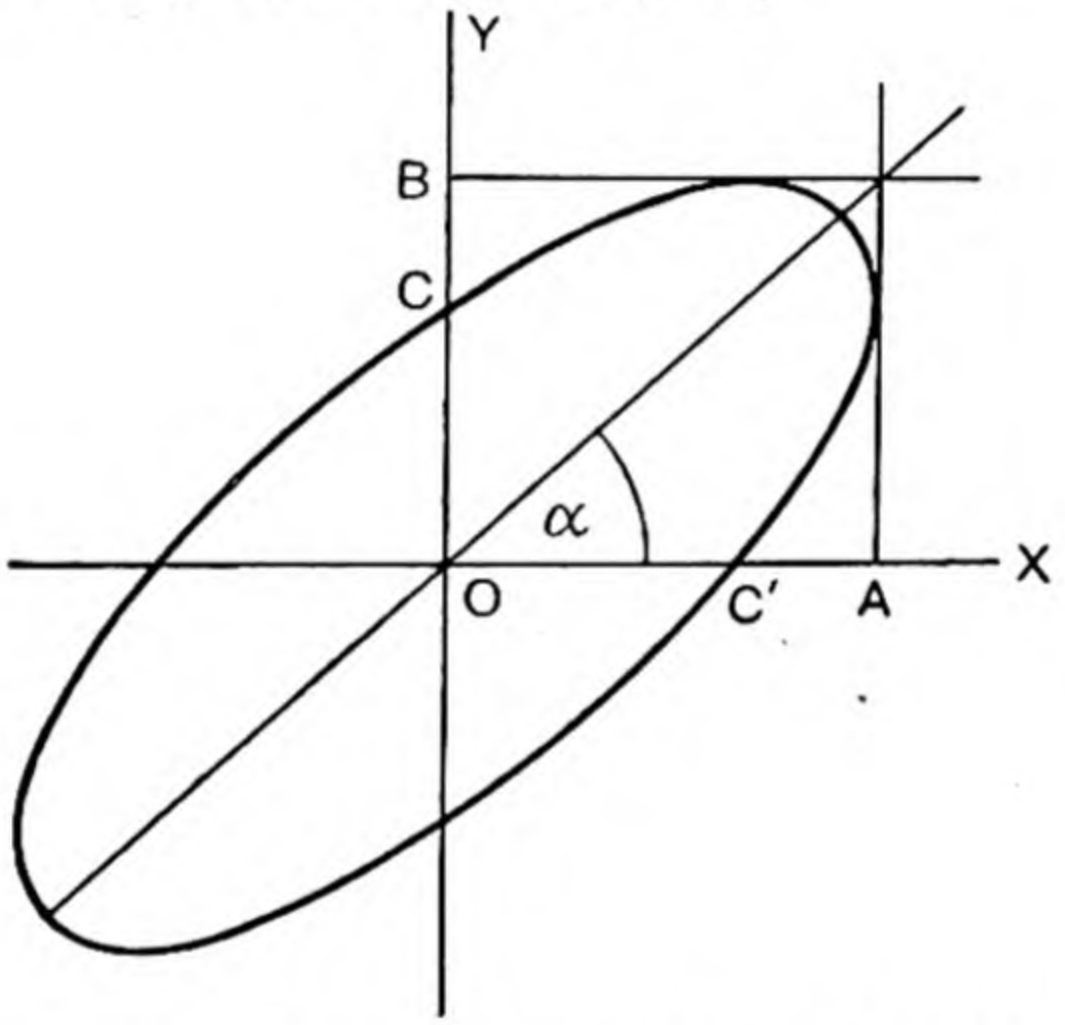
S.H.M's. of different periods at right angles

intervals of time do not now subtend equal angles in the two circles. In Fig. *a*, the ratio of the vertical to the horizontal frequency is 1 : 2, and the two start in phase. Fig. *b* shows the same frequency ratio, with $\pi/2$ phase difference. The patterns thus traced out, which are called **Lissajous' Figures**, are characteristic of the frequency ratio, and enable this to be found at a glance. If this is not a simple whole-number ratio, the patterns alter as the phase difference progressively changes.

Blackburn's pendulum, indicated in the diagram, is a simple means of demonstrating Lissajous' figures. The sand-filled funnel swings as a pendulum of length *BO* in the plane of the figure, and together with the string triangle *ABC* as a pendulum of length *DO* in a plane at right angles to this. Sand issuing from the funnel traces out the motion on paper held below it.

Interpretation of Lissajous' figures. The frequency ratio, if this is expressible in terms of small whole numbers, may be read off at once from a pattern of the type *a*. The ratio *vertical frequency/horizontal frequency* always equals *number of horizontal loops/number of vertical loops*.

In the case of an ellipse, representing two simple harmonic motions of the same period at right angles to one another, as $x = a \sin 2\pi nt$ and $y = b \sin (2\pi nt + \epsilon)$, the ratio of the amplitudes, b/a, is given by $\tan \alpha$, where α is the inclination of the axis of the ellipse shown in the figure to the x-axis. The phase difference ϵ is also determinable, for OB gives the maximum value of y, which is b, and OC, the value of y when $x = 0$, is $b \sin \epsilon$; thus $\sin \epsilon = OC/OB$.



Damped vibrations. A pendulum swinging in air suffers a resistance to its motion, which is at each instant very nearly proportional to the velocity at that instant. As work is done in moving against this resistance, the energy of the pendulum is steadily reduced, and the amplitude of each swing is less than that of the previous swing. This is called **damping**, and it occurs in practice with every kind of vibration.

The nature of the motion depends on the magnitude of the resistance, and three cases can be distinguished.

(*a*) *Slight Damping.* Oscillations are maintained, though gradually dying away ; and the period is slightly greater than it would be

if there were no damping. The extreme displacements a_1, a_2, a_3 ... at the ends of successive swings form a geometrical progression; thus

$$\frac{a_1}{a_2} = \frac{a_2}{a_3} = \frac{a_3}{a_4} \ldots = \delta,$$

where δ is called the **decrement.** The **logarithmic decrement,** $\log_e \delta$, is generally used in correcting for damping.

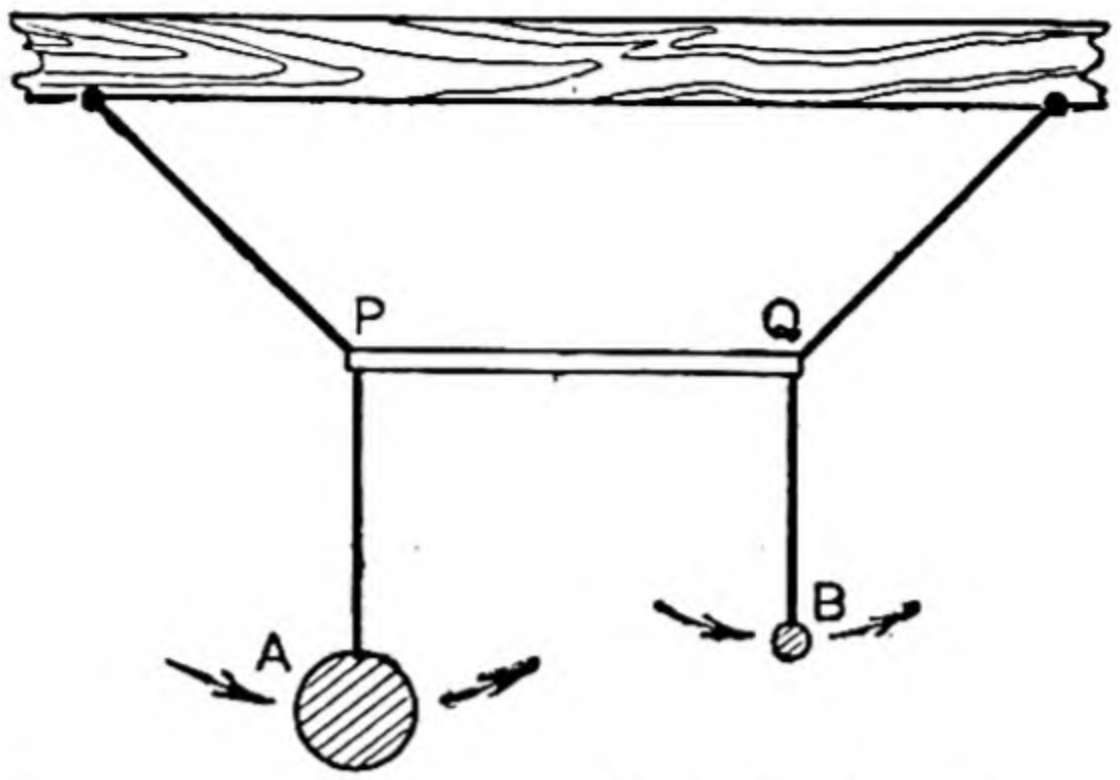
(*b*) *Critical Damping.* In critical damping, the resistance is just sufficient to prevent oscillation, yet not so great that the return to the zero position after displacement is indefinitely delayed. One quarter of a vibration is executed, with a period considerably greater than the undamped period.

(*c*) *Very heavy damping.* In very heavy damping, the resistance is so great that the return to zero after displacement is very slow indeed.

A proper discussion of damped vibrations is beyond the scope of this book, but two further points are worth noting here. First, the abstraction of energy by any means must necessarily result in damping; so that when a vibrating body is used as a source of sound, the very fact that it is communicating energy to the surroundings causes damping. Secondly, vibrations are a nuisance in many instruments with a swinging indicator—such as, for example, an ordinary balance or a moving-coil galvanometer—and it usually happens that sensitiveness is accompanied by a long period of swing; in a well-designed sensitive instrument, matters are arranged so that the movement is just critically damped or **dead-beat.**

Forced vibrations. A system which can execute harmonic vibrations has a **natural free period** and frequency of its own. But if it is subjected to a periodic driving force, it executes vibrations with the same (or very nearly the same) frequency as that of the driving force. These are called **forced vibrations,** or forced oscillations. The driving force itself usually originates from a system executing simple harmonic motion, and is transmitted from driver to driven by some kind of coupling.

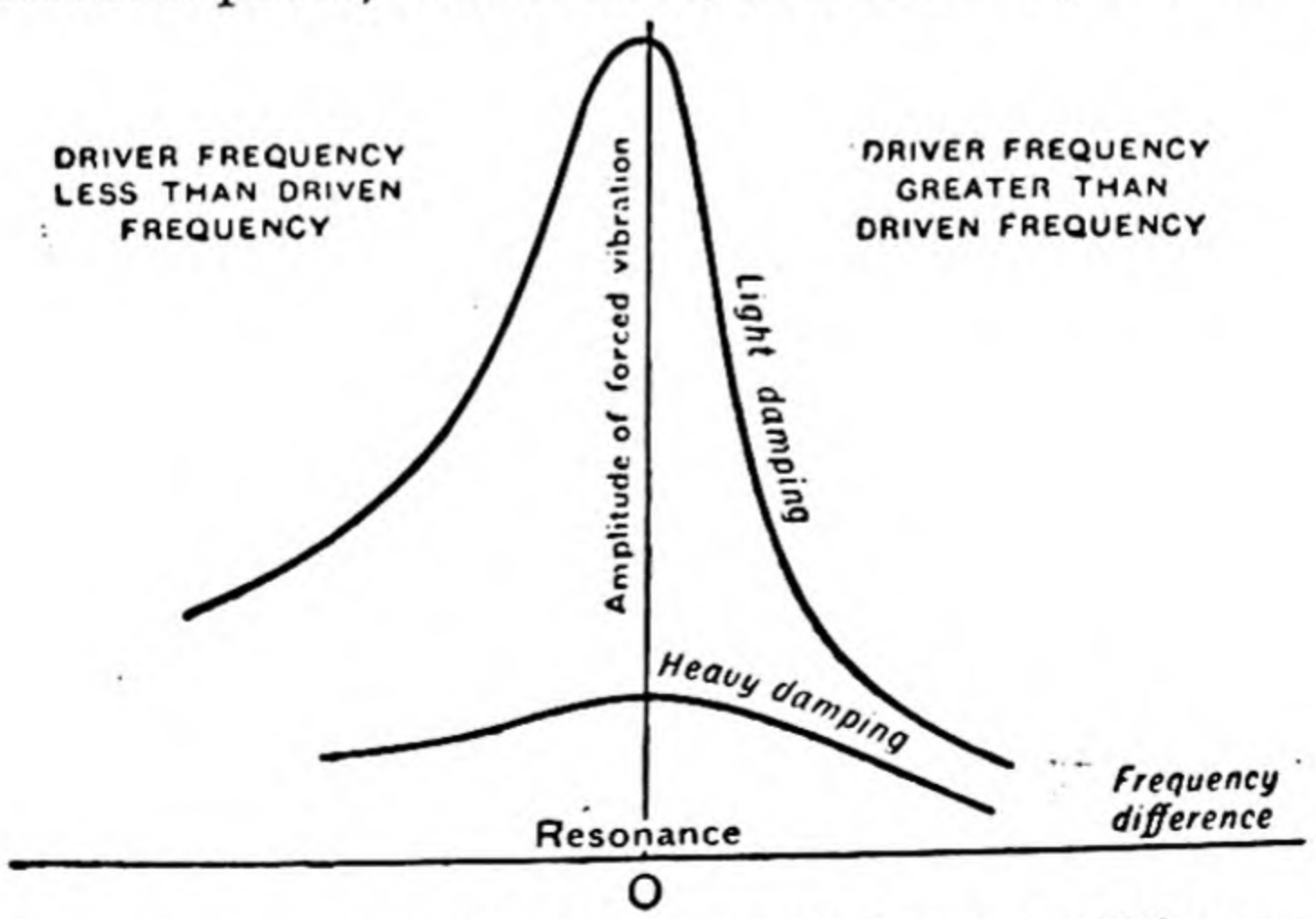
The diagram shows a simple device for illustrating forced oscillations. Two pendulums A and B are suspended from a lath PQ, which is itself suspended by cords. The driver A is set swinging, and B is driven by A via the coupling PQ, and oscillates with the



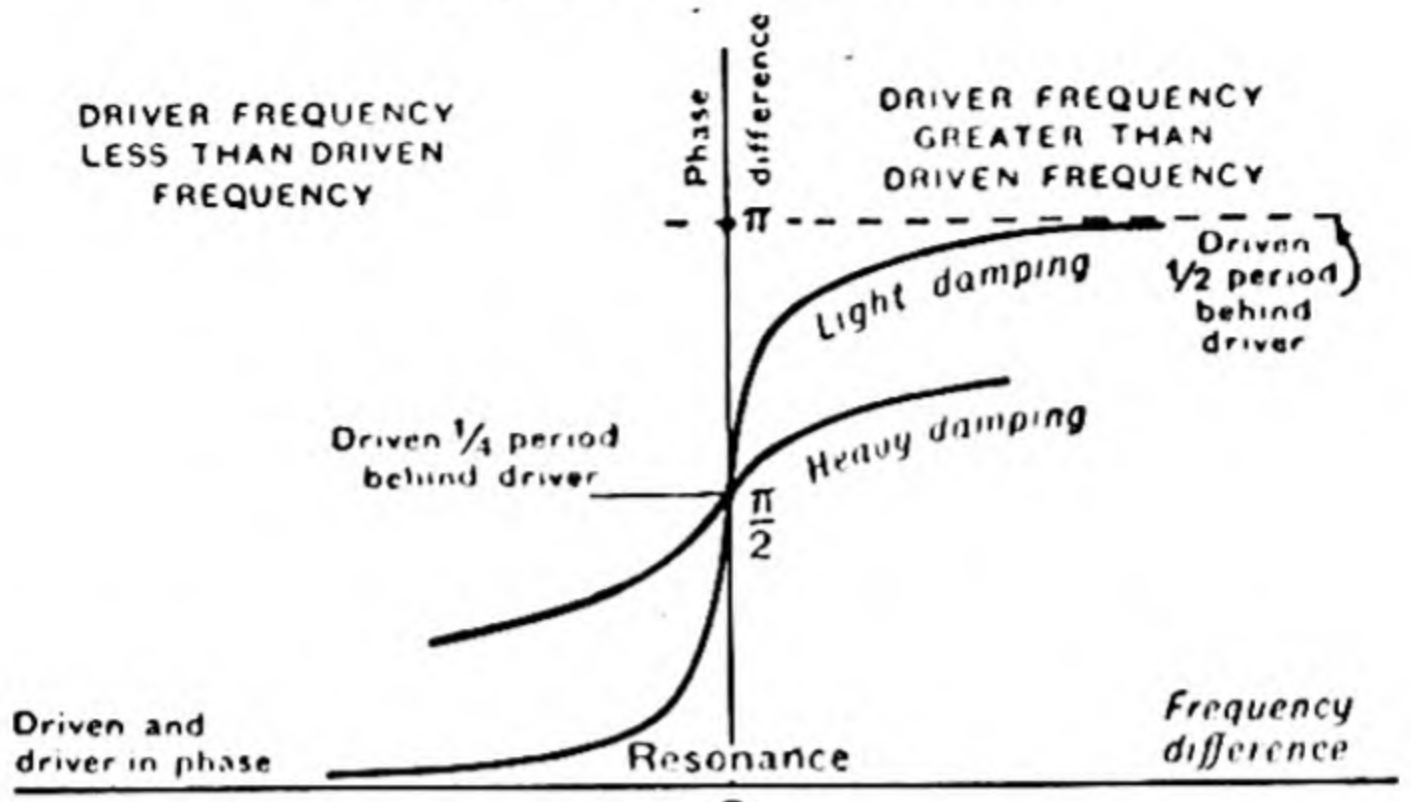
same period after a steady state of affairs has been reached. On starting the systems, the first impulse B receives starts it swinging with its own natural frequency, and some competition between the frequencies of A and B, observable as a **beats** effect, occurs in the early stages of B's motion. This stage is of short duration if the mass of the driver is considerably greater than that of the driven, and forced vibration proper then sets in.

There are three cases to consider :

(1) The frequency of the driver is less than that of the driven (that is, the driver is the larger pendulum, with the longer period). When this happens, the driven vibrates in the same phase (or very nearly the same phase) as the driver, with relatively small amplitude.



Variation of amplitude of driven with frequency difference



Variation of phase difference with frequency difference

(2) The frequencies of driver and driven are the same. The amplitude of the forced vibration is very great, and the condition is called **resonance.** The driven vibrates a quarter of a period (or $\pi/2$ in phase difference) behind the driver.

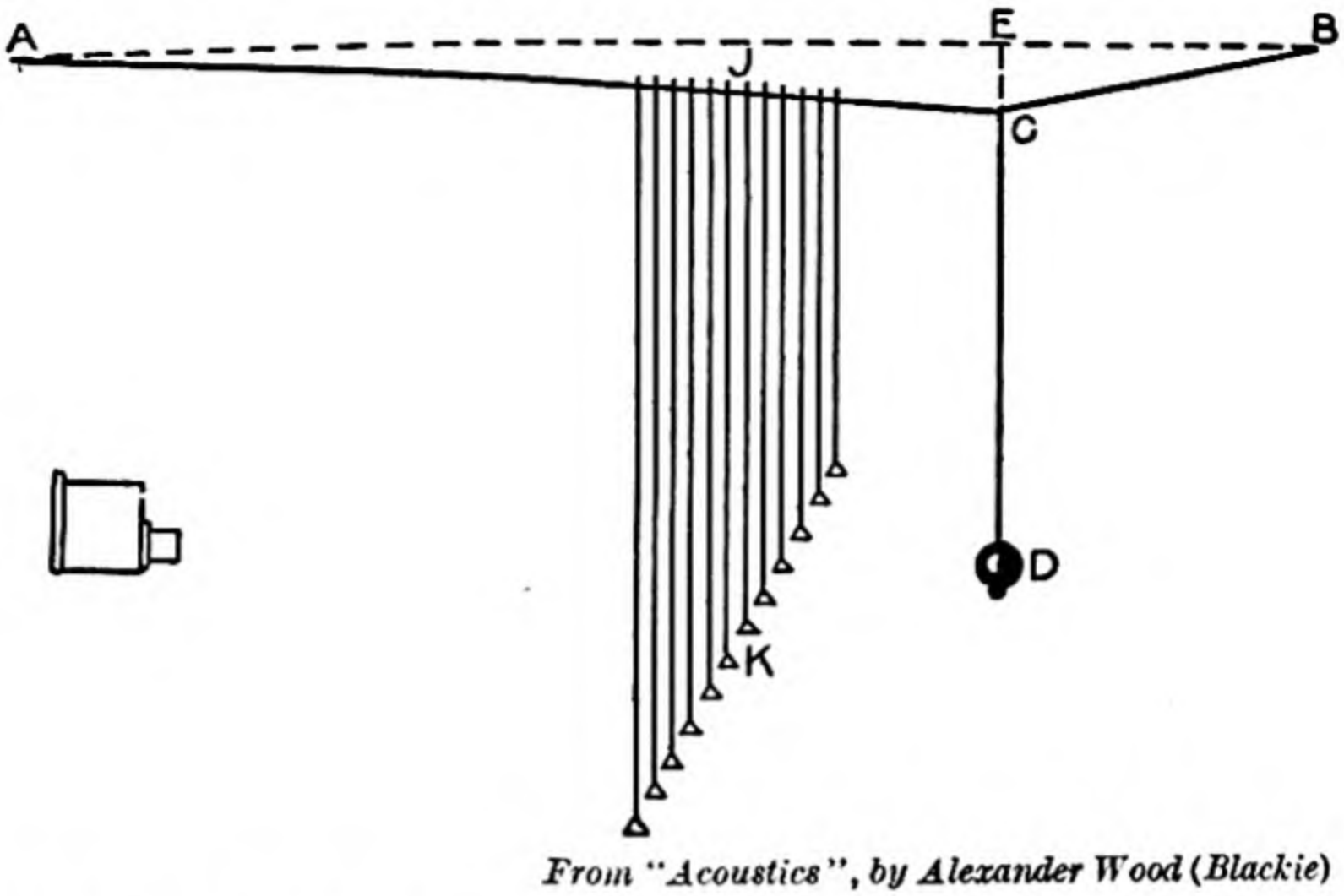
(3) The frequency of the driver is greater than that of the driven (that is, the driver is the shorter pendulum, with the shorter period). In this case, the driven vibrates with relatively small amplitude, and half a period (or π in phase difference) behind the driver.

The phase relations between driver and driven are important, particularly in the case of resonance.

The diagrams illustrate the way in which the amplitude of the driven varies with the frequency of the driver, and also the phase difference between the motions of driver and driven for different driver frequencies.

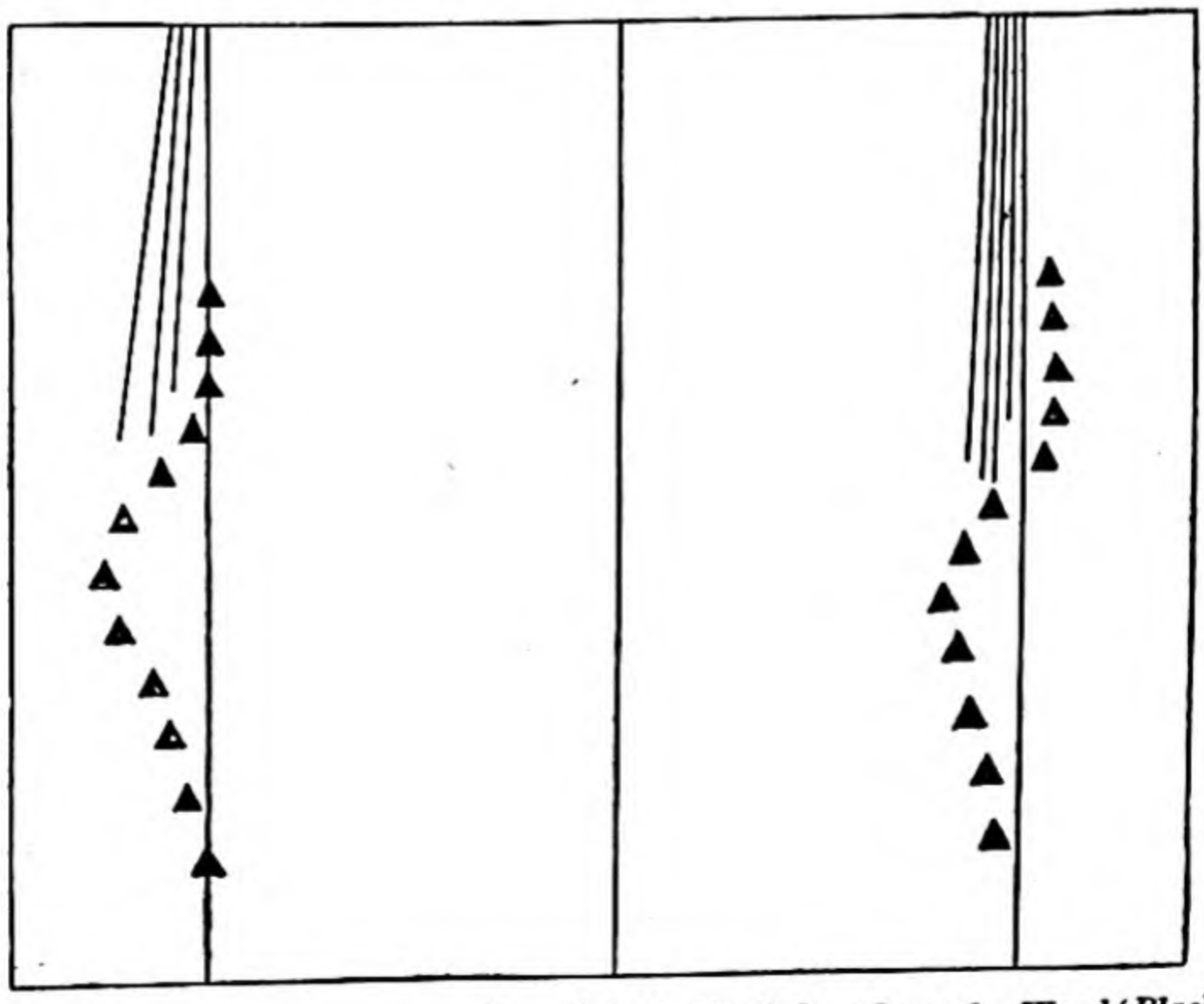
Effect of damping. Damping naturally reduces the amplitude of the forced vibration at all driver frequencies, but here there is no question of decay with time; the amplitude attained is that at which the rates of dissipation of energy by the driven and of supply of energy from the driver are equal. This reduction is made greater at the resonance frequency and frequencies close to it; so that damping reduces the sharpness of resonance—that is, there is a small response by the driven over a wide range of frequencies, instead of a large response over a narrow range.

Barton's pendulums. The diagram shows a set of pendulums of different lengths, attached to a cord ACB by which they are coupled



From "Acoustics", by Alexander Wood (Blackie)

to a heavy driver pendulum of effective length *DE*. The "bobs" are light paper cones which are subject to considerable air-damping. The pendulum labelled *K*, seventh from the top, is of the same length as *DE* and thus has the same period as the driver. The accompanying diagram gives an instantaneous view of the system at the



From "Acoustics", by Alexander Wood (Blackie)

Barton's pendulums; two pictures seperated by a short time interval

moment when D is passing its zero position and moving to the right, and a similar snapshot a fraction of a period later. The resonant pendulum K, seventh from the top, and most displaced, but here unlabelled, is seen to be exactly a quarter of a period behind D, while the shorter pendulums are more or less in phase with D, having moved to the right in the later picture ; the larger ones are more or less out of phase with D, as is shown by the fact that in the later picture they have moved slightly to the left.

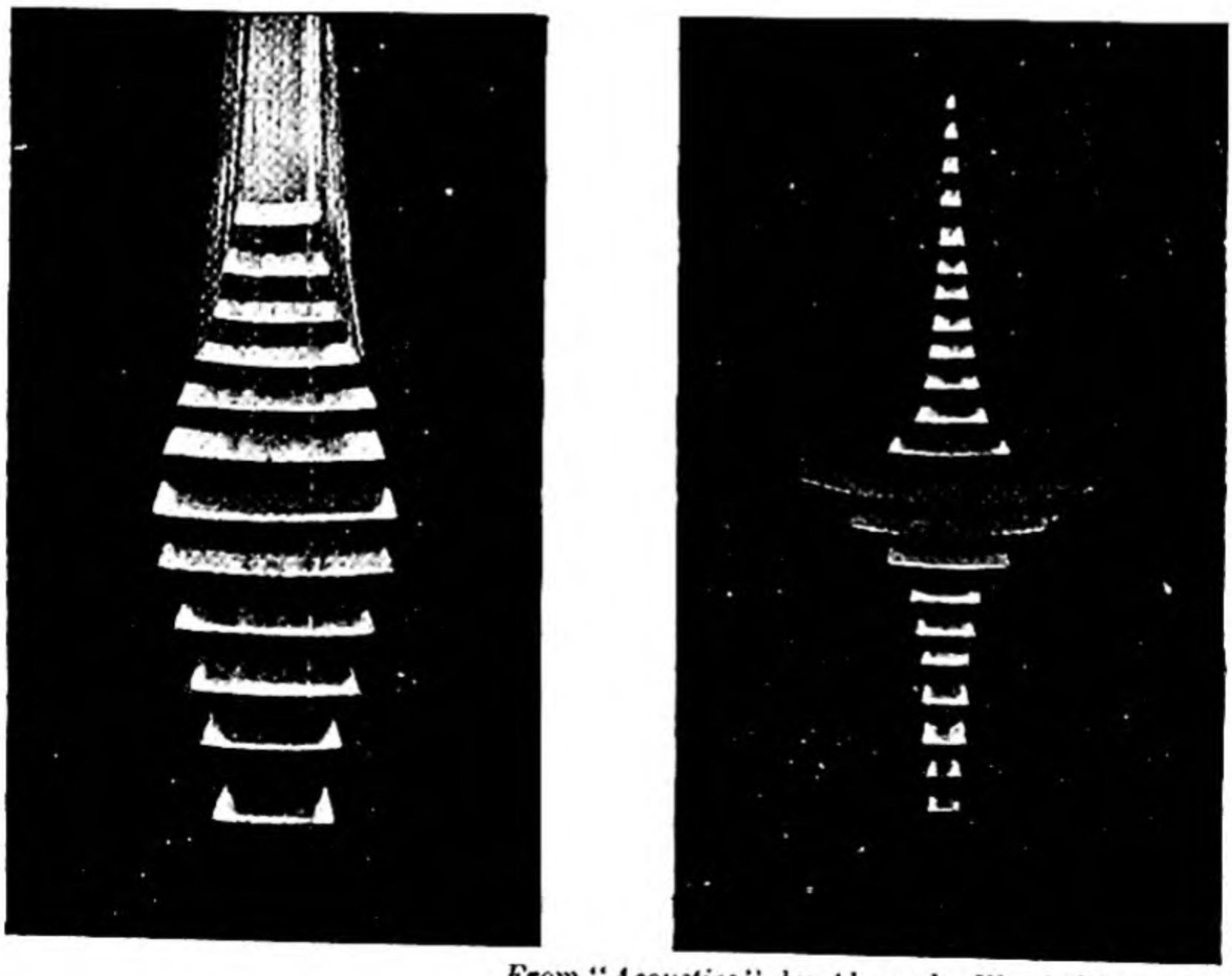
This apparatus illustrates conclusively that :

(a) at resonance, the driven is a quarter period behind the driver ;

(b) if the driven has the shorter period (higher frequency), driver and driven are nearly in phase ;

(c) if the driven has the larger period (lower frequency), the driven is half a period behind the driver.

The result of damping is shown by placing a camera in line with the cones, as indicated and taking time exposures (a) with the paper cones unloaded (left-hand picture), and (b) with heavy curtain rings



From "Acoustics", by Alexander Wood (Blackie)

Barton's pendulums: left, unloaded ; right, loaded

slipped over the cones (right-hand picture), which increases their inertia and so reduces the effect of damping. The absence of any marked selective resonance in the first case, and the great response of one pendulum only in the second case, are clearly shown in the photographs.

Energy exchange at resonance. Suppose driver and driven are pendulums with bobs of comparable mass, how is resonance maintained? It might be expected that, since the coupling between driver and driven acts impartially towards both, once the driver A has set the driven B in motion, B would react via the coupling and return energy to A. But a consideration of the phase relation shows that this cannot happen ; for as B is a quarter of a period behind A, any reaction on A will be a quarter of a period behind B, and thus half a period behind A—exactly timed to reduce A's vibration further.

The usual sequence of events in these circumstances is then as follows :

1. The amplitude of A steadily decreases, while that of B steadily increases, B swinging all the while a quarter of a period behind A.

2. A is brought to rest, all its energy having been communicated to B.

3. B now acts as driver, and A vibrates in resonance, a quarter of a period behind B, and matters continue until B has been arrested and A swings with its maximum amplitude. All B's energy has been returned to A.

The exchange is then repeated again as from the first stage. It will be noted that neither plays the part of both driver and driven at the same time, and that the whole energy of the system is transferred to and fro from one to the other in a series of exchanges.

EXAMPLES ON CHAPTER 10

1. Explain what is meant by simple harmonic motion.
Deduce an expression for the periodic time of the vertical oscillations of a mass suspended by a helical spring, the extension of which is proportional to the load. The mass of the spring may be neglected.
Describe in some detail how such an arrangement might be used to obtain an estimate of the acceleration due to gravity. (N.)

2. The velocity of a particle describing simple harmonic motion is

16 cm. per sec. at a distance of 8 cm. from the centre of the motion and 8 cm. per sec. at 12 cm. from the centre. Starting from the definition of simple harmonic motion, calculate the period and amplitude of the above motion.

A particle is subjected to two simple harmonic motions, of equal amplitudes and at right angles to one another, the frequencies being in the ratio of 1 : 2. Assuming the two motions to be in phase at the start, determine, by a graphical method, the resultant motion of the particle. Describe an experiment to illustrate this motion, and indicate how the accuracy of the frequency ratio may be tested. (N.)

3. Define *simple harmonic motion*.

Show that a heavy body supported by a light spiral spring executes simple harmonic motion when displaced vertically from its equilibrium position by an amount which does not exceed a certain value and then released. How would you determine experimentally the maximum amplitude for simple harmonic motion?

A spiral spring gives a displacement of 5 cm. for a load of 500 gm. Find the maximum displacement produced when a mass of 80 gm. is dropped from a height of 10 cm. on to a light pan attached to the spring. (N.)

4. Explain the meaning of the term *moment of inertia*.

Describe in detail how you would find experimentally the moment of inertia of a bicycle wheel about the central line of its hub.

A uniform cylinder 20 cm. long, suspended by a steel wire attached to its mid-point so that its long axis is horizontal, is found to oscillate with a period of 2 seconds when the wire is twisted and released. When a small thin disc, of mass 10 gm., is attached to each end the period is found to be 2·3 seconds. Calculate the moment of inertia of the cylinder about the axis of oscillation. (N.)

5. A steel ball-bearing of mass m and radius r is allowed to roll freely, without slipping, in a shallow saucer which forms part of a concave spherical surface of large radius R. Explain what forces are acting on the ball. Show that the motion of its centre of gravity will be approximately simple harmonic and obtain an expression for the period of oscillation. (Moment of inertia of a solid sphere about an axis through its centre of gravity $=\frac{2}{5}mr^2$.)

Find the frictional force acting on the ball when it is at a point where the surface of the saucer makes an angle θ with the horizontal, and calculate the least coefficient of friction required to prevent slipping at this point. (N.)

6. Define simple harmonic motion and discuss its importance.

A horizontal diaphragm with sand sprinkled on it is oscillating vertically with simple harmonic motion. The amplitude of the vibration is gradually increased. If the frequency of the vibration is 100 cycles per sec., at what value of the amplitude will the sand just be thrown clear of the diaphragm? (C. S.)

7. A circular hoop is suspended with its plane horizontal by equally-spaced vertical threads. Find the time of a small rotational oscillation. (C. S.)

8. Obtain a relation between the period of a simple harmonic motion and the acceleration at unit displacement.

A metal cylinder of height 15 cm. floats upright in mercury of density 13·6 gm./c.c. It is set into vertical oscillation and the period is found to be 0·62 sec. Find the density of the metal, assuming friction to be negligible. (C. S.)

9. Discuss the behaviour of solid bodies subjected to deforming forces, and describe a method of determining Young's modulus for material in the form of a wire.

Assuming that the extension of a spring is proportional to the tension, find the period of oscillation of a light spring with a weight of 100 gm. attached to it, if a weight of 1 gm. extends the spring by 1 mm. (C. S.)

10. In the HCl molecule the force required to alter the distance between the atoms from its equilibrium value is $5{\cdot}4 \times 10^5$ dynes per cm. What is the fundamental frequency of vibration of the molecule, assuming the vibration to be simple harmonic, and the mass of the Cl atom to be infinite compared to that of the H atom, which is $1{\cdot}66 \times 10^{-24}$ gm.? (C. S.)

11. Define *simple harmonic motion*, and show how to calculate the period of a system executing such a motion.

A mass M is suspended at the lower end of a light helical spring, and is found to produce a statical extension L. The mass is displaced downwards through a distance d below its equilibrium position, and then released. Assuming that the spring obeys Hooke's law, show that the motion is simple harmonic, and find an expression for the period. Calculate, also, the maximum kinetic energy of the mass. (O. & C.)

12. Deduce an expression for the time of vibration of a particle executing a simple harmonic motion in a straight line.

When a mass of 100 gm. is suspended from the end of a vertical spring, the latter is stretched 2 cm. Show that if the mass be depressed slightly further and then released, the motion will be simple harmonic, and find the time of vibration. The extension of the spring may be assumed to be proportional to the load, and the mass of the spring may be neglected. (O. & C.)

13. Define "simple harmonic motion", and find the period of a particle executing such motion.

A cylindrical air-tight piston, of mass M and cross-sectional area A, slides without friction in the vertical neck of a bottle of volume V filled with air originally at atmospheric pressure P. Show that if the piston is displaced from its position of equilibrium, under isothermal conditions, it will oscillate with simple harmonic motion, and find an expression for the period. (O. & C.)

14. Under what conditions will a vibrating body execute simple harmonic motion? Deduce an expression for the period of oscillation of a body moving with S.H.M.

A particle vibrates with S.H.M. along a straight line. Its greatest acceleration is $5\pi^2$ cm. per sec. per sec., and when its distance from the equilibrium position is 4 cm. the velocity of the particle is 3π cm. per sec. Find the amplitude and the period of oscillation of the particle. (O. & C.)

15. Explain what is meant by simple harmonic motion. Obtain an expression for the relation between the velocity and the displacement of such motion.

A vertical spiral spring of negligible mass supports a heavy particle which stretches it through a length L cm. Show that the period of the vertical oscillations of the system is the same as the period of a simple pendulum of length L cm. (O. & C.)

16. A test-tube of mass 6 gm. and of external diameter 2 cm. is floated vertically in water by placing 10 gm. of mercury at the bottom of the tube. The tube is depressed a small amount and then released. Find the time of oscillation. (O. & C.)

17. Define simple harmonic motion, and obtain an expression for the velocity of a body moving with simple harmonic motion in terms of its period, amplitude, and distance from its mean position.

A body moving with simple harmonic motion has a velocity of 4 ft. per sec. at a distance of 3 ft. from the mean position, and 3 ft. per sec. at a distance of 4 ft. from the mean position. What is its velocity at the mean position? (O. & C.)

18. Calculate the period of oscillation of a uniform rigid rod swinging about a fixed horizontal axis under gravity. Show that there are three other positions for the axis of oscillation which will give the same period, and show the connection between them. (O. & C.)

19. A body oscillates under gravity about an axis at a distance h from its centre of mass. Find the value of h for which the time of oscillation is a minimum, and the value of this minimum time, in terms of the mass of the body and of its moment of inertia about a parallel axis through the centre of mass. (O. & C.)

20. Obtain a formula for the period of a simple pendulum executing small oscillations.

The period of a simple pendulum is 2·0 sec., and the mass of the bob is 50 gm. The bob is pulled aside through a horizontal distance of 12 cm., and then released. Find the displacement and the kinetic energy of the bob 0·7 sec. after the instant of release.

21. Give an account of the chief phenomena of forced vibrations and resonance.

22. A suspended galvanometer coil swings with damped harmonic motion. The first deflection from rest is 18·8 divisions, and the deflection observed on the same side of the scale after ten complete swings is 6·5 divisions. Calculate the logarithmic decrement, and find what the value of the first deflection would have been in the absence of damping.

23. By drawing the displacement-time curves and adding ordinates, compound the following pairs of S.H.M's. in the same straight line :

(i) Frequencies n and $2n$, equal amplitudes, in phase.
(ii) Frequencies n and $3n$, amplitudes $2a$ and a, in phase.
(iii) Frequencies n and $1{\cdot}5n$, amplitudes a and $2a$, phase difference $\pi/2$.

24. Compound the following pairs of S.H.M's. at right angles :

(i) $x = 3 \sin 2\pi nt$, $y = 5 \sin 2\pi nt$.
(ii) $x = 3 \sin 2\pi nt$, $y = 5 \sin (2\pi nt + \pi/2)$.
(iii) $x = 3 \sin 2\pi nt$, $y = 5 \sin (2\pi nt + \pi/6)$.

25. Compound graphically the following pairs of S.H.M's. at right angles :

(i) $x = a \sin 2\pi nt$, $y = a \sin 4\pi nt$.
(ii) $x = a \sin 2\pi nt$, $y = 2\ a \sin (6\pi nt + \pi/2)$.

CHAPTER 11

WAVE MOTION

A **wave** is defined as a disturbance, usually periodic, which travels with finite velocity through a medium, and remains unchanged in type as it travels.

The chief characteristics common to all kinds of waves are as follow :

1. The particles of the medium execute relatively small vibrations about their mean positions, but do not themselves suffer any permanent displacement in the direction of propagation of the wave.

For example, a swimmer in deep sea water is carried vertically up and down (actually following a more or less circular path in a vertical plane), but is not transported bodily towards the shore, showing that the water in his immediate neighbourhood is behaving similarly. Again, if one end of a taut rubber cord be held in the hand, and plucked sideways with the other hand, a wave is seen to travel down the cord. The wave moves along the cord, but each portion of the cord as the wave passes it moves to and fro at right angles to the direction of travel of the wave.

2. The wave is a transfer of energy, not of portions of the medium.

3. Each successive particle of the medium performs a motion similar to that of its predecessors along the line of travel of the wave, but later in time ; this is rather like a rank of soldiers " dressing " on parade. This handing on of similar motion from one particle to the next constitutes the advance of the wave itself.

A single pulse may fulfil all these conditions. But most of the important examples of wave motion in Physics arise from a continuous vibration of the particles of the medium, and are regularly periodic waves. The simplest kind of vibration that an individual particle can execute is a simple harmonic motion ; in this case, we speak of the wave as a **simple harmonic wave.** If the individual particles execute a more complicated type of motion, attributable to

the superposition of several simple harmonic motions of different periods and amplitudes, then the wave itself can be regarded as several independent simple harmonic waves superposed on one another.

In many important practical examples, a wave radiates in all directions from a relatively small source, and the amplitude of vibration of the individual particles falls off as the distance from the source increases. But we shall only consider the propagation of waves in which there is no such change in amplitude as the wave progresses, and each particle duplicates exactly the motion of its predecessors. This means that we are thinking of waves travelling out along a single straight line, such as waves propagated along cords or in cylindrical pipes. Most of the chief properties of waves can be demonstrated in this way, and shown experimentally with the help of a long piece of ordinary bunsen tubing (about 20 ft. long) firmly attached to a support at one end, and manipulated by hand at the other end; or with simple acoustical apparatus described later.

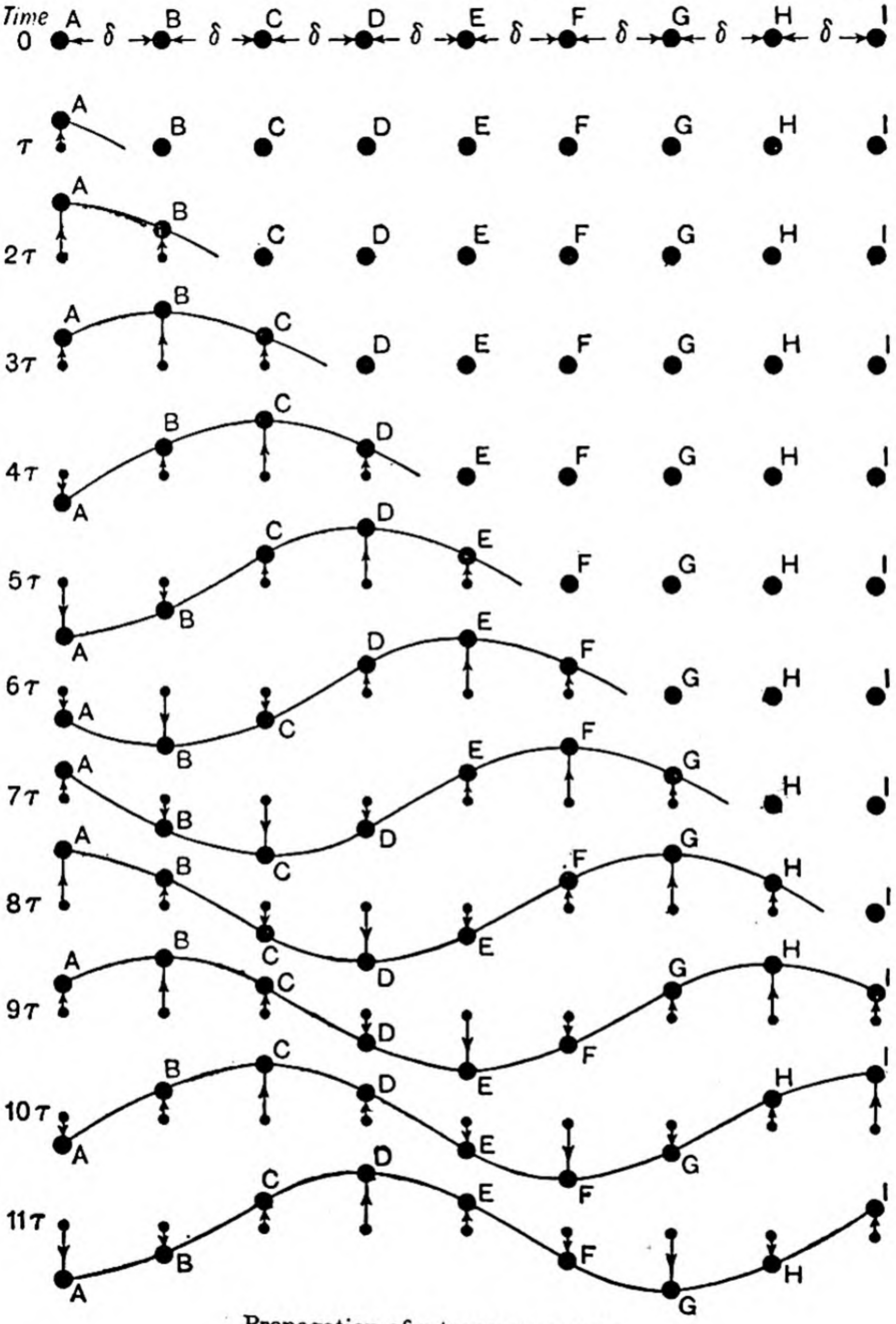
Transverse waves. A transverse wave is one in which the particles of the medium vibrate in a direction at right angles to the line of propagation of the wave. Such a wave may be produced on a long rubber cord by waving the free end up and down, or from side to side. Transverse waves occupy a definite plane, and are said to be **plane polarised.** A narrow slit, set across the cord, will allow the wave motion to pass beyond it only if the slit itself lies in this plane. If the slit is inclined to this plane, there is partial transmission, of that component of the vibration of the particles which lies in the plane containing the slit and the cord; while if the slit is at right angles to the original plane of polarisation of the wave, there is no transmission at all.

To understand the propagation of such a wave, consider a set of particles $A, B, C, \ldots$ set out at equal intervals of δ cm. along a line. Suppose A starts to move at right angles to this line with S.H.M., its motion being taken up and copied by $B, C, D, \ldots$ in turn, each starting τ sec. after the other; that is,

B repeats A's motion but τ sec. later,

C repeats A's motion but 2τ sec. later,

D repeats A's motion but 3τ sec. later.



Propagation of a transverse wave.

This successive repetition of the simple harmonic motion of A constitutes the propagation of a simple harmonic wave moving to the right in the figure. The velocity of propagation of the wave is

$$v = \delta/\tau \text{ cm. per sec.}$$

The full line in the diagram indicates for each instant the position of the particles and thus shows for each instant the way in which the displacement y of the particles at right angles to the direction of propagation varies with the distance x from the chosen origin (particle A). Each picture is thus a picture of the wave itself at the instant concerned.

The maximum displacement a cm. of the individual particles is called the **amplitude of the wave.**

Considering the lowest of the drawings (that is, the latest, for time 11τ), the particles are most displaced upwards at D and downwards at A, and G; at these points the particles are at rest momentarily in their own simple harmonic motions, or the particle velocity is here zero.

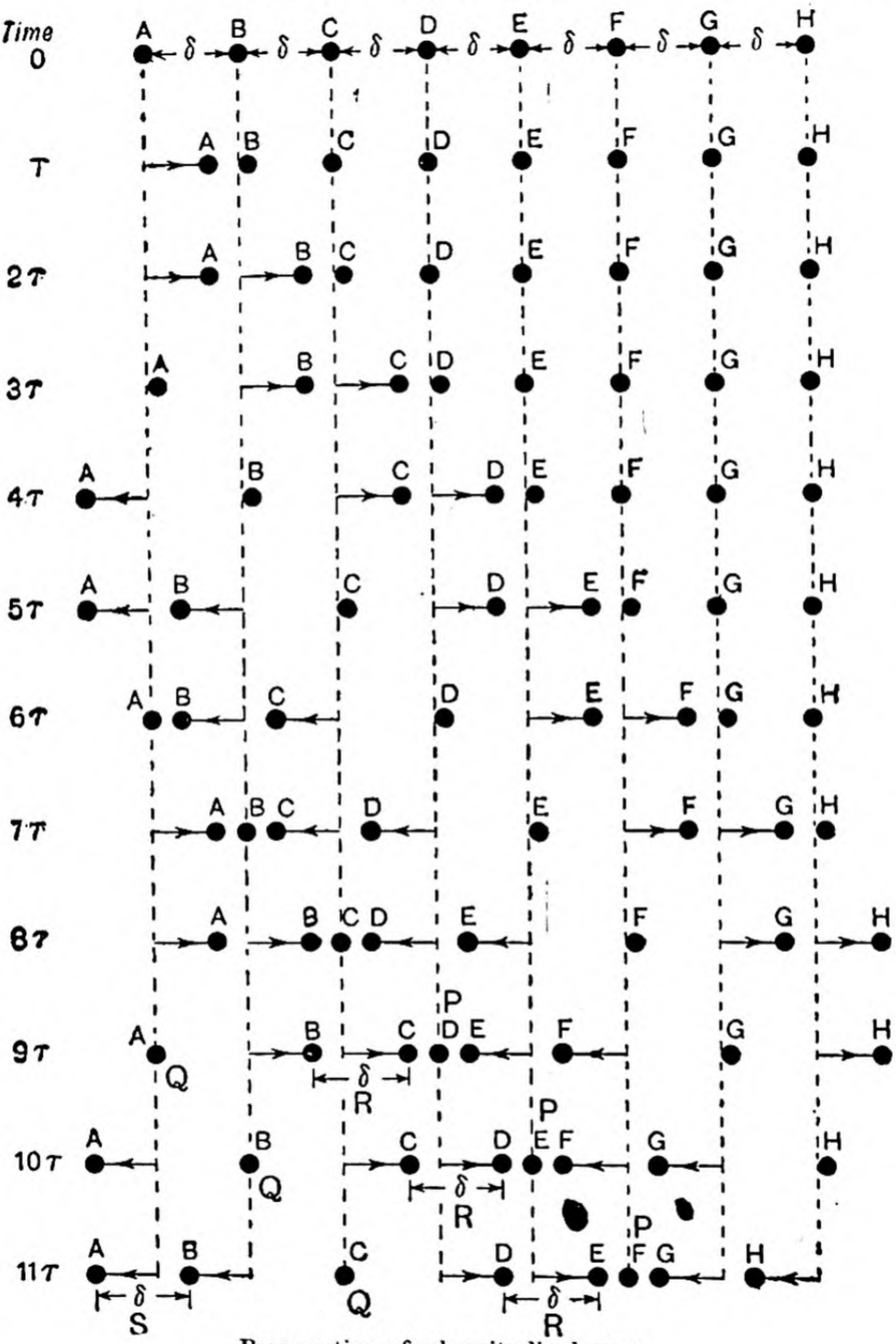
The particles are moving most rapidly where they are passing through the mean positions of their simple harmonic motions. No particle doing this is shown exactly in the drawing, but B and H are approaching it moving upwards, C and I have passed it moving upwards, and E and F are near it moving downwards.

The particle velocity is *greatest* where the displacement is least; the particle velocity is *least* where the displacement is greatest.

Successive pairs of particles in the same phase of their own simple harmonic motions, such as A and G, B and H, C and I, can be picked out. The distance between such successive pairs of particles in the same phase (or, simply, between successive pairs of particles in the same stage of their simple harmonic motions) is called the **wave-length** of the wave (written λ).

Longitudinal waves. In longitudinal waves, the excursions of the individual particles are made in the direction of propagation of the wave itself.

If a stretched rubber cord is fixed at one end and held taut by one hand at the other, the free hand can be used to pull a part of the cord a few inches towards one, the pull being along the cord; on releasing this part, a longitudinal wave travels along the cord.



Propagation of a longitudinal wave.

Each particle of the cord vibrates in its turn along a line in the direction of the original plucking, that is, along the string itself. This may be shown by making suitably placed chalk marks on the cord, and observing their displacement carefully; a close watch is needed, as on a cord such as this the longitudinal waves travel much faster than transverse waves, and it is difficult to avoid exciting a transverse wave of small amplitude as well.

Models to illustrate the propagation of a longitudinal wave are described in many of the standard works on Sound. But probably the best way of understanding the propagation of such a wave is to draw out for oneself a series of diagrams showing the successive stages in the motions of the individual particles.

The diagram on p. 323, shows successive stages of the S.H.M. in the direction of the cord propagated along from particle A to B, from B to C, and so on; each instantaneous picture shows the state of the wave itself at that moment.

As with a transverse wave, if the individual particles vibrate with amplitude a cm., this is called the **amplitude** of the wave.

Instead of there being an obvious "sine curve" of displacements, with "crest" followed by "trough", we see that the disturbance is passing along as a succession of *compressions* and *rarefactions*.

Consider the three pictures, for the instants, 9τ, 10τ, and 11τ. The particles are crowded together most densely at the points marked P, where they are moving most rapidly in their own simple harmonic motions in the direction of the wave. **In a longitudinal wave the maximum compression accompanies zero particle displacement and maximum particle velocity in the direction of travel of the wave.**

The particles are farthest apart at the points marked Q, where they are moving most rapidly in their own simple harmonic motions in the opposite direction to the direction of travel of the wave. Thus, **maximum rarefaction accompanies zero particle displacement and maximum velocity in the opposite direction to the direction of travel of the wave.**

At the points marked R, the particles are most displaced forwards; at S, which can be shown only in the lowest figure, they are most displaced in the opposite direction. In these cases the spacing of the particles is the same as that (δ) obtaining before the wave arrived. **So the compression or rarefaction is zero when the particle displacement is a maximum.**

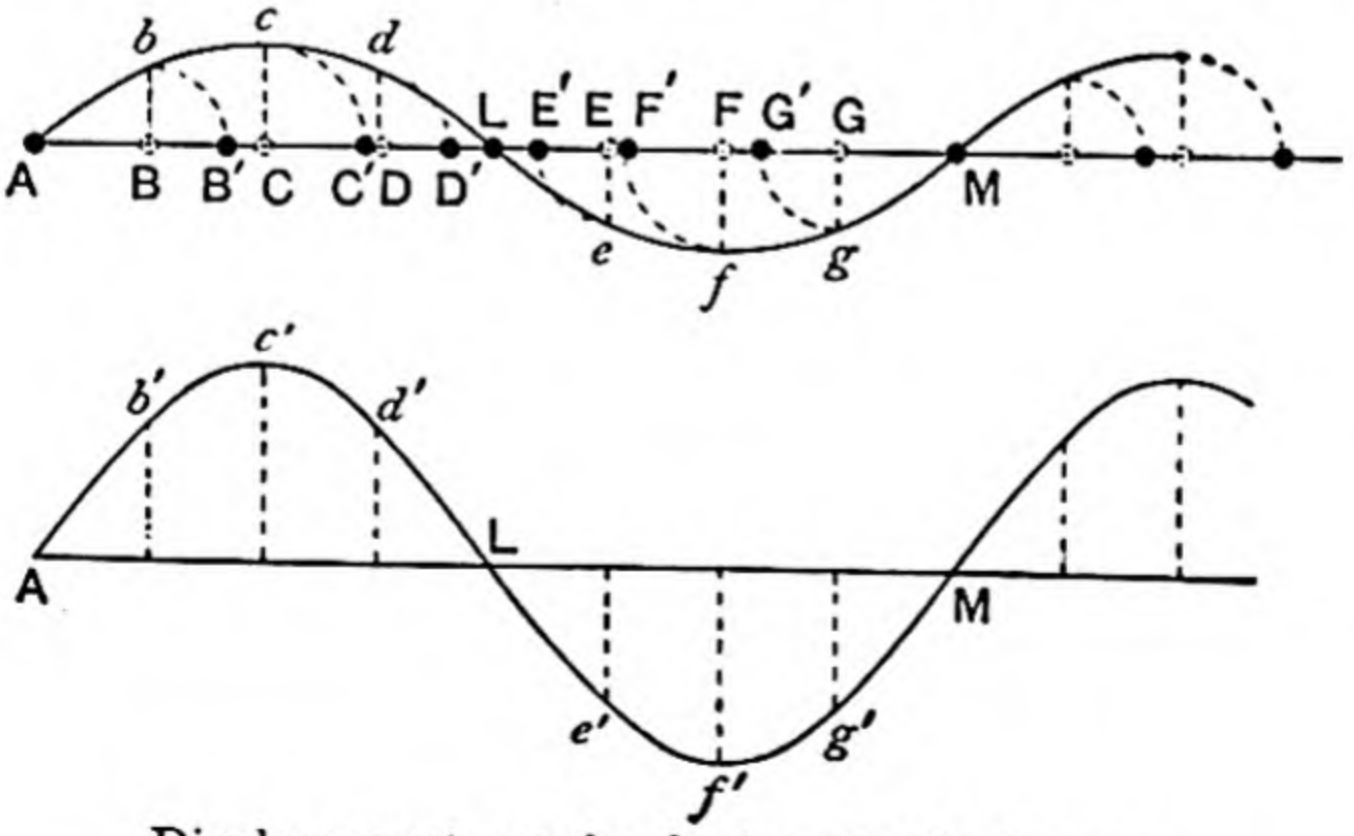
As before, the distance between successive pairs of particles in the same phase of their own simple harmonic motions is called the wave-length of the wave and indicated by λ.

If, instead of thinking primarily of the displacement of the individual particles, we fix attention on the density change that is really the observed result of this, we see that one whole wave-length contains one compression and one rarefaction; and further, the compression wave is a quarter of a wave-length out of step with respect to the displacement wave, since zero displacement accompanies maximum density change, and maximum displacement accompanies zero density change.

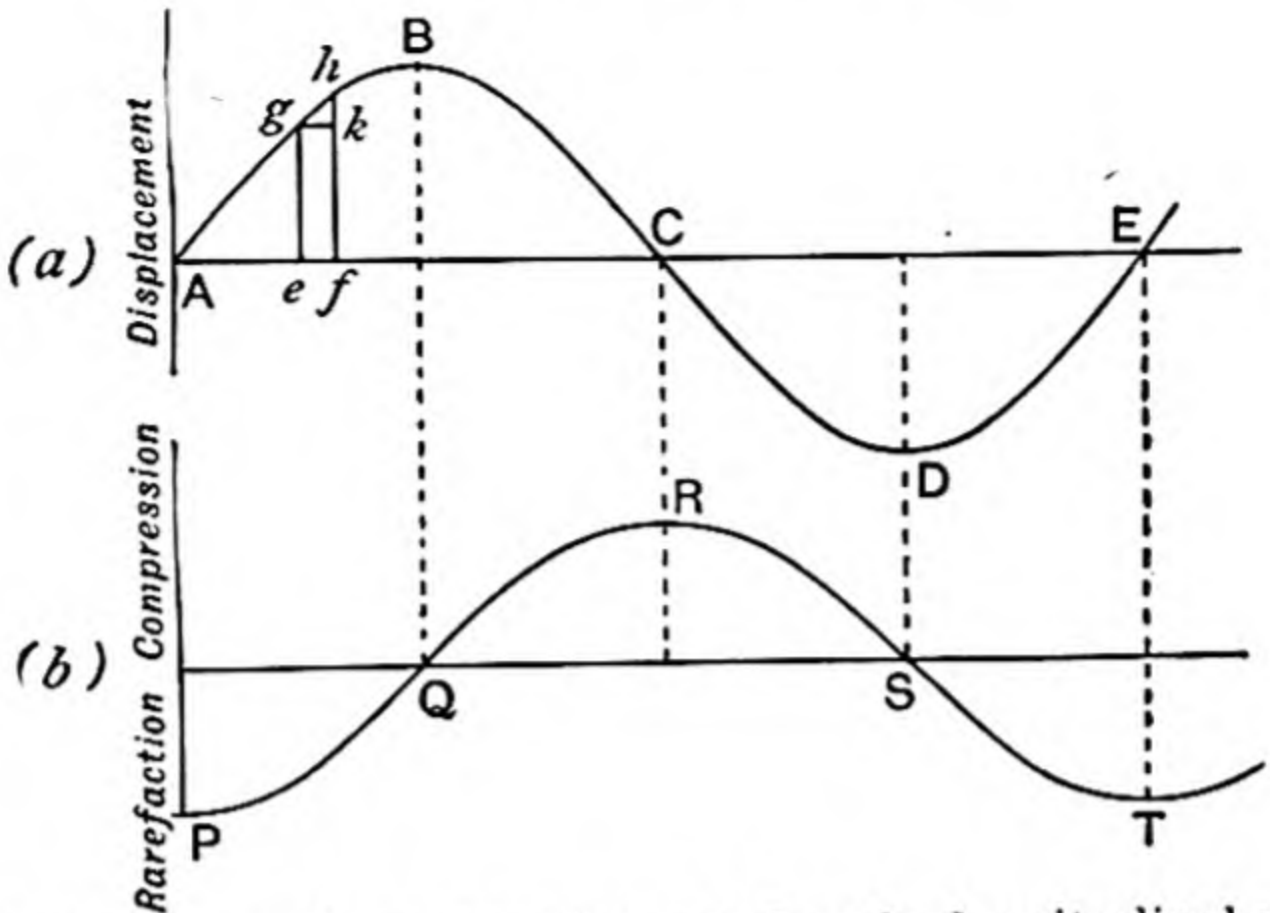
Graphical representation of longitudinal wave. Instead of drawing an actual picture of the displaced particles, these displacements can be represented along the x-axis by lines drawn up or down in the y-direction; thus a figure can be drawn which shows, according to the usual conventions of graphs, the way in which particle displacement varies with distance along the wave, at a chosen instant.

The diagram shows the method of constructing such a graph. Displacements such as BB' to the right, which is the direction of travel of the wave, are represented by lines such as Bb above the x-axis (with y positive).

Displacements, such as EE', to the left, or in the opposite direction to the direction of travel of the wave are drawn, as is Ee, below the x-axis (with y negative). The lower diagram shows the same graph drawn with a larger scale for displacements, which are usually so



Displacement graph of a longitudinal wave.



Displacement and compression graphs of a longitudinal wave.

small with a longitudinal wave that a faithful copy of the displacements would be of little use.

A compression graph can also be constructed ; this can be regarded as representing the wave in its capacity as a travelling wave of compression and rarefaction, rather than a travelling wave of particle vibration.

Consider any two particles, say at e and f (in the displacement-distance graph of diagram (a). The displacement of the particle at e is eg, that of the particle at f is fh, and the difference in their displacements is hk. This difference hk divided by their original separation ef, that is, the ratio hk/ef, measures the extent to which their distribution has been altered and shows the degree of compression or rarefaction. But $ef = gk$, so $hk/ef = hk/gk$, which is the gradient of the displacement graph. If the displacement curve is sloping backwards, as from B to D, then particles farthest from A are less displaced than those nearer, so the particles are crowded together and there is a compression ; if it is sloping forwards, as from A to B, the particles are more separated than normal, and there is a rarefaction. The steeper the slope of the displacement graph, the greater the compression or rarefaction.

The ordinate of the compression curve is obtained from the slope of the displacement graph at the point concerned, and in this way the compression-distance graph may be constructed, as diagram (b).

As the terms " compression " and " rarefaction " refer to changes in the density of distribution of the particles, they must also refer

to the changes in the density of the medium which are caused as the wave passes along.

If the displacement-distance graph is a "sine curve", the compression-distance graph is a "cosine curve", starting with a rarefaction.

Velocity, frequency, and wave-length. If the individual particles make n vibrations per second, as this is characteristic of all particles involved, n can be termed the **frequency of the wave.**

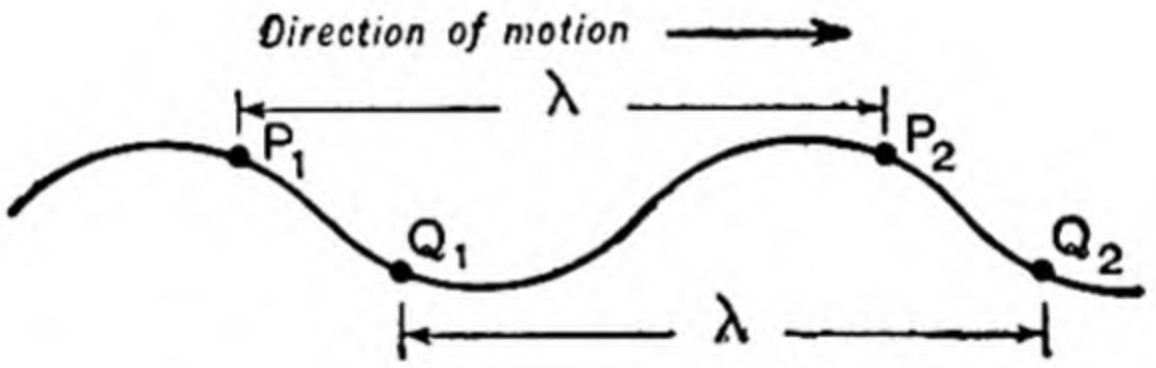
Pairs of particles such as P_1, P_2 or Q_1, Q_2 in the displacement-distance diagram, are in the same phase of their own simple harmonic motions, and are therefore a distance λ cm. apart. But the disturbance that travels from P_1 (or Q_1) to P_2 (or Q_2) must take exactly $1/n$ sec., since P_2 (or Q_2) is always one vibration ahead of P_1 (or Q_1) in space.

Thus the velocity of propagation of the wave, which covers a distance of λ cm. in $1/n$ second, is

$$v = \lambda \div 1/n,$$

or

$$v = n\lambda \text{ cm. per sec.}$$



Various types of waves: expressions for velocities. It has been seen how, with a mass on a spiral spring, the period of oscillation depended on the mass M and the stiffness factor k of the spring (p. 295). If M is large and k small, the vibration is sluggish; if M is small and k large, vibration is rapid. The frequency of vibration of the spring is proportional to $\sqrt{k/M}$, or

$$\sqrt{\frac{\textit{stiffness factor}}{\textit{inertia factor}}}.$$

It is difficult in any elementary treatment to explain the handing on of the wave disturbance from one particle to another in proper mathematical terms. But we can imagine some kind of connection like a small spring between adjacent particles, and can see that,

whatever the details of the actual transference of the wave from particle to particle, it is reasonable to suppose that the speed of such transference, and hence the velocity of the wave along its line of propagation, is governed by the ratio

$$\frac{\textit{stiffness factor}}{\textit{inertia factor}}$$

for the particular type of displacement in the particular type of medium.

This is indeed the case, and some important formulae, which should be known (but which it is not proposed to prove here), are given below.

(1) *Transverse wave along a cord.* The " stiffness factor " is the tension in the cord, and the " inertia factor " the linear density or mass per unit length.

If T dynes be the tension, and m gm. per cm. the mass per unit length, then the velocity of propagation of a transverse wave is

$$v = \sqrt{T/m} \text{ cm. per sec.}$$

Alternatively, if A sq. cm. be the area of cross-section, the longitudinal stress in the cord is T/A dynes per sq. cm., while the density ρ in gm. per c.c. is $\rho = m/A$, we can write

$$v = \sqrt{\textit{stress}/\rho} \text{ cm. per sec.}$$

(2) *Longitudinal wave along a cord or thin rod.* Here the small excursions of the particles constitute small longitudinal elastic displacements, and we should expect Young's modulus to be involved. This is, in fact, the " stiffness factor " required. The " inertia factor " is the density of the material. Thus the velocity of a longitudinal wave is given by

$$v = \sqrt{Y/\rho} \text{ cm. per sec.,}$$

where Y is Young's modulus in dynes per sq. cm., and ρ is the density in gm. per c.c.

(3) *Compression wave (sound wave) in a gas.* The " stiffness factor " is the adiabatic modulus of elasticity, which is γp, where p is the pressure in dynes per sq. cm., and γ is the ratio between the specific heat at constant pressure (C_p) and the specific heat at

constant volume (C_v), the value of which is about 1·41 for air under ordinary conditions. The density is the " inertia factor ".

Thus $v = \sqrt{\gamma p/\rho}$ cm. per sec., where p is the pressure in dynes per sq. cm., ρ the density in gm. per c.c., and γ the ratio C_p/C_v for the gas.

(4) *Electromagnetic waves in space.* Light, X-rays, γ-rays, ultra-violet and infra-red rays, and radio waves, are all believed to be propagated as electromagnetic waves through space, and are stated to be transverse waves, since they can all be polarised. In a plane polarised beam of light, it is supposed that there is an alternating electric intensity in one plane, with an alternating magnetic intensity in a plane at right angles to this. It is difficult here to isolate the " stiffness " and inertia factors; for, if we consider the electric intensity, the " stiffness " depends on the electrical properties of space, represented by its dielectric constant k_0, while the " inertia " depends on the magnetic properties, represented by the permeability of space m_0. These roles are interchanged if we consider the magnetic intensity.

It can be shown that the velocity, c, of any electromagnetic wave in free space is given by

$$c = 1/\sqrt{k_0 m_0},$$

where k_0 and m_0 are the dielectric constant and permeability, both measured on the same system of electrical units (that is, both on the absolute electrostatic, or both on the absolute electromagnetic system).

The velocity v in a medium of refractive index μ, is $1/\mu$ of the velocity in free space,

or $$v = c/\mu \text{ cm. per sec.}$$

(5) *Waves on the surface of a liquid.* The formula for the velocity of waves of wave-length λ cm. on the surface of a liquid of density ρ gm. per c.c. and surface tension T dynes per cm. is

$$v = \sqrt{\frac{\lambda g}{2\pi} + \frac{2\pi T}{\rho\lambda}} \text{ cm. per sec.}$$

For very large values of λ the first term only is important, and v is very nearly given by

$$v = \sqrt{\lambda g/2\pi} \text{ cm. per sec.,}$$

the " deep-water waves " velocity.

Similarly, for small values of λ (" surface-tension ripples "), the expression becomes

$$v = \sqrt{2\pi T/\rho\lambda} \text{ cm. per sec.},$$

where T is surface tension.

Equation for a progressive wave. Using the symbol y for the displacement of a particle from its mean position, the displacement-time equation for simple harmonic motion, and taking $y=0$ when $t=0$, is:

$$y = a \sin \sqrt{\mu}\, t.$$

Since the period of oscillation is $T=2\pi/\sqrt{\mu}$ and the frequency n is $1/T=\sqrt{\mu}/2\pi$, we can write $\sqrt{\mu}=2\pi n$; so that this equation becomes

$$y = a \sin 2\pi n t.$$

This will apply to each of the individual particles if for each of them t is taken to be the " local time " t_L, measured from the instant when each particle begins its motion.

The wave travels outwards with a velocity v cm. per sec. If we start timing when the wave reaches a certain particle, as A in the diagram of p. 321, and also take A as origin for measuring distances along the path of the wave, a particle at distance x cm. from A will receive the disturbance x/v sec. later than A, and so at any later stage will have been oscillating for x/v sec. less time. That is, the local time, t_L, is $(t - x/v)$.

Hence the equation

$$y = a \sin 2\pi n (t - x/v)$$

represents the displacement at time t *after the disturbance arrives at the reference point* A of a particle at distance x cm. *measured along the path of the wave from the reference point* A, and thus describes the motion executed by every particle involved in the transmission of the wave.

As $n = v/\lambda$, $y = a \sin 2\pi n (t - x/v)$ can be rewritten

$$y = a \sin \frac{2\pi v}{\lambda} (t - x/v),$$

or

$$y = a \sin \frac{2\pi}{\lambda} (vt - x).$$

Again, since $n = 1/T$, and $v = n\lambda$, $y = a \sin 2\pi n (t - x/v)$ can be written

$$y = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right).$$

The three forms in which the wave equation may appear, namely,

$$\mathbf{y = a \sin 2\pi n (t - x/v)},$$

$$\mathbf{y = a \sin \frac{2\pi}{\lambda} (vt - x)},$$

and

$$\mathbf{y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)},$$

should be committed to memory.

The foregoing equations apply to the particle displacement. The density change, pressure change, and condensation in a longitudinal wave are all a quarter-period out of phase with the displacement. Thus the density change has its maximum value when $t = 0$ and $x = 0$, etc., and the graph representing its variation with distance is a cosine graph. If Y represent the instantaneous density change or pressure change or condensation, and A the maximum value or amplitude of such change, then

$$Y = A \cos 2\pi n (t - x/v),$$

$$Y = A \cos \frac{2\pi}{\lambda} (vt - x),$$

or

$$Y = A \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right),$$

are the equations which represent the propagation of this change and correspond to the sine equations for displacement.

On p. 326 it was shown that the compression or density-change at any place in the path of a longitudinal wave is given by the corresponding gradient of the displacement-distance graph, negative values of the gradient indicating compressions, and positive rarefactions. Also, it was seen on p. 324 that the particle velocity is in the direction of the wave, or positive, during compressions, and negative during rarefactions. Differentiation of $y = a \sin 2\pi n (t - x/v)$ with respect to x gives the gradient of the y/x graph, $-2\pi na/v \times \cos 2\pi n (t - x/v)$; differentiation with respect to t gives the particle velocity, $2\pi na \cos 2\pi n (t - x/v)$. When the cosine term is positive,

the gradient is negative and the velocity positive, so that this condition represents a compression. We can thus correctly represent a compression wave of amplitude A as $Y = A \cos 2\pi n(t - x/v)$, positive values of Y indicating compressions and negative values of Y rarefactions.

Whatever the actual form in which the wave equation is written, and whether it refers to particle displacement or to density change, the terms x and vt (or x/v and t) appear together in the phase part. Any equation that does not give (or cannot be arranged to give) these two terms together cannot represent a progressive wave.

The wave equation has been derived by considering the motions of the individual particles. It may also be obtained from the definition of wave motion as given on p. 319. Any of the lower lines of the diagram of p. 321, giving an instantaneous picture of the particles, shows what at first sight appears to be a succession of "sine curves", each λ cm. long. Thus, a distance λ cm. along the abscissa represents a phase change of 2π radians in the angle of which the sine is plotted, and a distance along this axis of d cm. (in either direction) a phase change of $2\pi d/\lambda$.

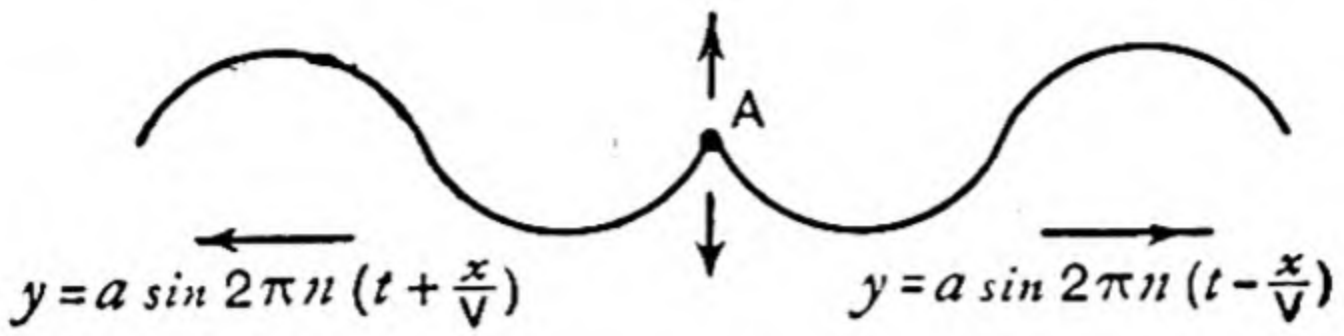
But, on closer examination it is seen that the sine curve is inverted, or reversed in direction. It is the graph $y = a \sin 2\pi X/\lambda$, where X is the distance measured, not from A, but *backwards from the leading edge of the wave*. This, at time t sec. after it has passed A, has reached a distance vt cm. beyond A; and at this instant the value of X for a particle x cm. to the right of A is $(vt - x)$.

Substituting $(vt - x)$ for X, we have

$$y = a \sin \frac{2\pi}{\lambda}(vt - x).$$

Twin waves arising from a single disturbance. So far we have considered the motion of a wave in one direction from the moment it passed a chosen particle A, without considering how such a wave is usually started. It must surely begin by the communication to the medium at some point or other of an external force which, if the motion of the affected point is simple harmonic, we can call a simple harmonic force.

Still thinking of transverse waves along a cord, suppose that A (see diagram) is a point on the cord to which a simple harmonic force of frequency n is applied. The motion of A and all particles to



the right of it is exactly as previously described, and the simple harmonic wave

$$y = a \sin 2\pi n\,(t - x/v)$$

travels to the right.

A corresponding wave, with v reversed, moves out to the left of A. As A is taken as origin, distances measured to the left are accounted negative, from the point of view of A. The velocity of the wave is also reversed, but this is only involved in the calculation of "local time" for particles on its path. A particle x cm. to the left of A, at distance $-x$ reckoned algebraically, has a "local time" $t_L = \left\{t - \left(-\frac{x}{v}\right)\right\} = t + x/v$; thus the equation for the wave travelling to the left is

$$y = a \sin 2\pi n\,(t + x/v).$$

The action of a simple harmonic force at any point A thus originates *two* waves as a rule, represented by

$y = a \sin 2\pi n\,(t - x/v)$ travelling to the right,

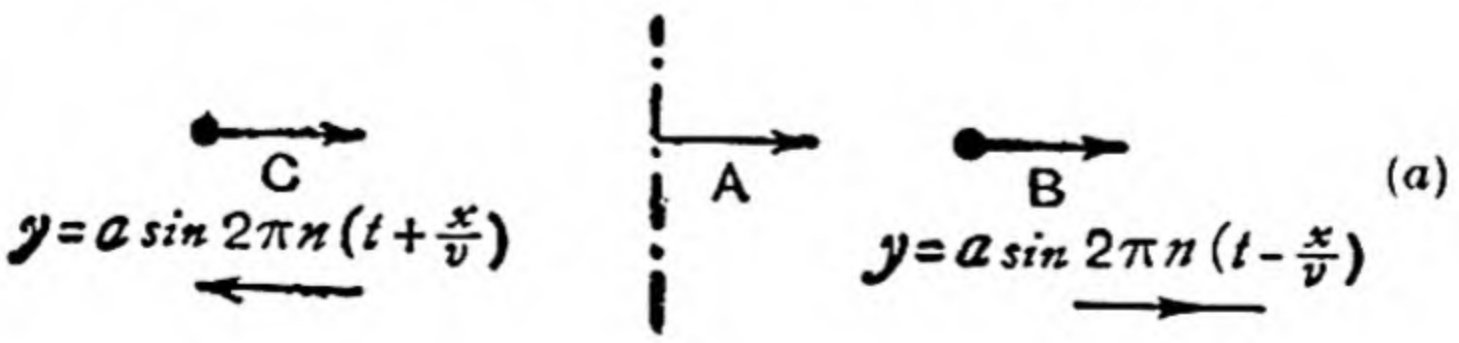
and

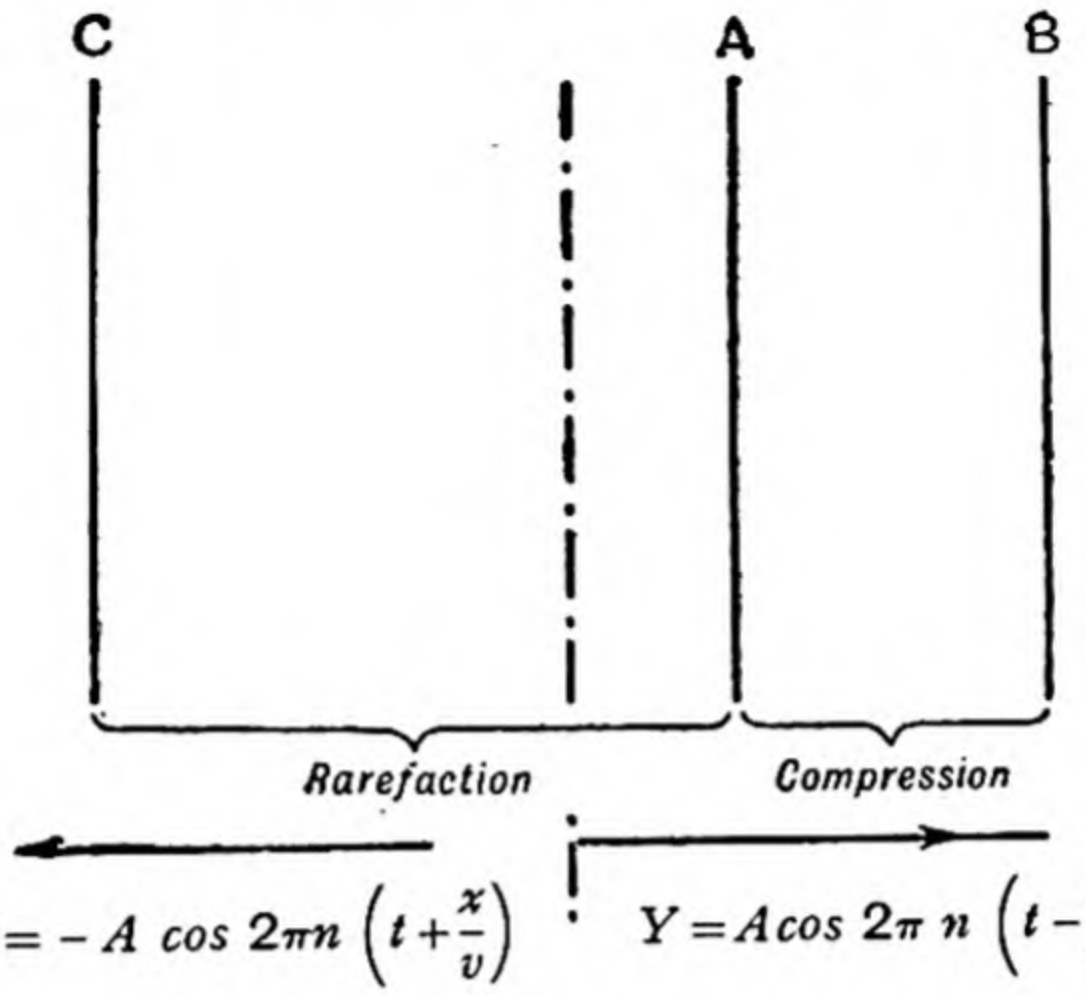
$y = a \sin 2\pi n\,(t + x/v)$ travelling to the left.

With longitudinal waves the result is a little different. If the motion of A under an applied simple harmonic force is to the right at any instant, twin waves travel in both directions from A, impelling both B (on the right) and C (on the left) towards the right very shortly afterwards. The displacement waves are thus in phase and oppositely directed, and can be represented by

$y = a \sin 2\pi n\,(t - x/v)$ to the right,

and $y = a \sin 2\pi n\,(t + x/v)$ to the left (diagram a).





(*b*)

But particle *B* is travelling in the same direction as the direction of propagation of the wave, and thus belongs to a compression; particle *C* is travelling in the opposite direction to that of the leftwards wave, and thus belongs to a rarefaction. The twin compression waves are thus exactly out of phase, a compression travelling to the right, and a rarefaction to the left, and can be written

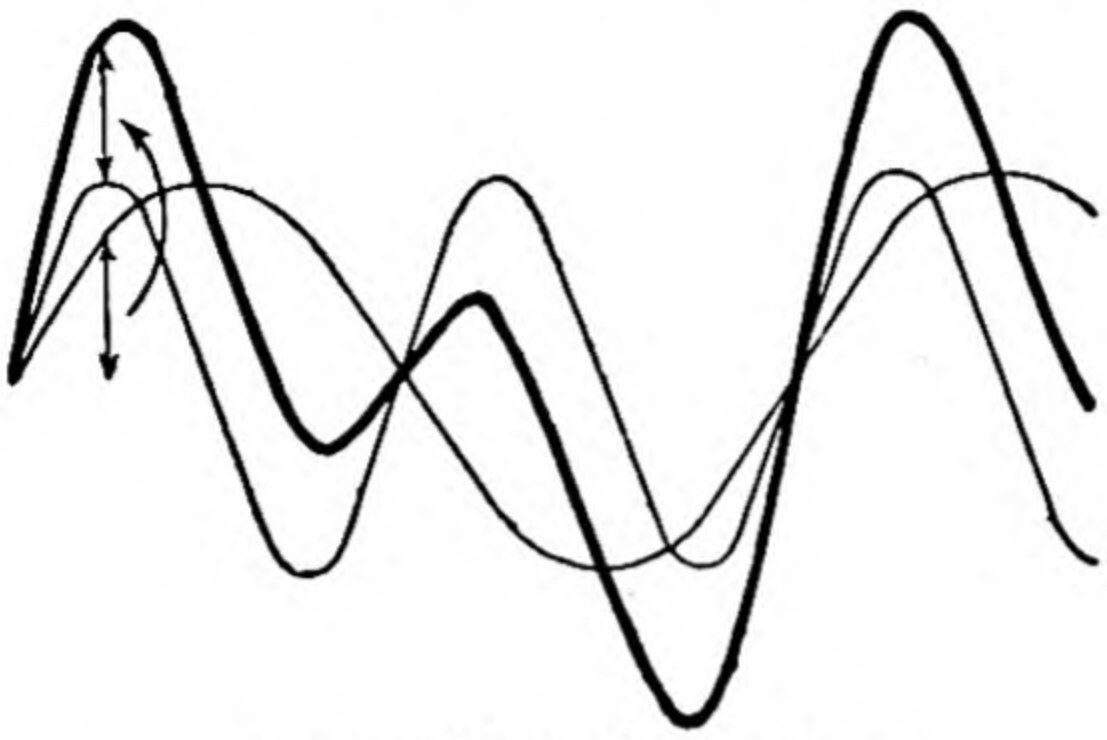
$$Y = A \cos 2\pi n (t - x/v) \text{ to the right,}$$

and $$Y = -A \cos 2\pi n (t + x/v) \text{ to the left (diagram } b).$$

Superposition of two waves. A particle executing two simultaneous simple harmonic motions follows a motion compounded of the two independently, and its displacement at any instant is the algebraic sum of the two displacements, if both are in the same line.

Thus, when two simple harmonic waves of the same type are travelling through the same body of medium, or along the same cord, the particle displacement at any instant is the algebraic sum of the two individual displacements. This resultant displacement at any instant for all the individual particles is often most conveniently found by drawing the two displacement-distance graphs (or instantaneous pictures of the waves) for the same instant, and adding the ordinates, as in the diagram, where the thick line is the resultant displacement-distance graph obtained by adding the ordinates of the two separate thin-line curves, and represents the resultant *wave*.

But in some cases it is very instructive to use the wave equations



Superposition of two waves.

instead. These are usually simple cases in which the waves have the same amplitude, frequency, and velocity and differ only in phase or direction. The phase difference corresponding to half a period is most simply shown as a difference in the *sign* of the amplitude.

Case 1. Two displacement waves, a positive for both, travelling either in the same or opposite directions.

Representing the separate waves by

$$y_1 = a \sin 2\pi n (t - x/v),$$

and

$$y_2 = a \sin 2\pi n (t \pm x/v),$$

the resultant wave is represented by $y = y_1 + y_2$, so

$$y = a \{\sin 2\pi n (t - x/v) + \sin 2\pi n (t \pm x/v)\}.$$

The trigonometrical formula required is that for the sum of two sines, which in terms of any two angles B and C is

$$\sin B + \sin C = 2 \sin \tfrac{1}{2}(B + C) \cos \tfrac{1}{2}(B - C)$$

(" twice the sine of half their sum, times the cosine of half their difference ").

Case 2. Two displacement waves, a positive for one, negative for the other.

The two waves can be represented by

$$y_1 = a \sin 2\pi n (t - x/v),$$

and

$$y_2 = -a \sin 2\pi n (t \pm x/v).$$

The resultant, $y = y_1 + y_2$, is

$$y = a \{\sin 2\pi n (t - x/v) - \sin 2\pi n (t \pm x/v)\},$$

and the formula needed is that for the difference of two sines,

$$\sin B - \sin C = 2 \cos \tfrac{1}{2}(B+C) \sin \tfrac{1}{2}(B-C)$$

(" twice the cosine of half their sum, times the sine of half their difference ").

Case 3. Two compression waves, A the same sign for both.

If the waves are

$$Y_1 = A \cos 2\pi n (t - x/v),$$

and

$$Y_2 = A \cos 2\pi n (t \pm x/v),$$

then

$$Y = Y_1 + Y_2.$$

Hence

$$Y = A \left\{ \cos 2\pi n \left(t - \frac{x}{v} \right) + \cos 2\pi n \left(t \pm \frac{x}{v} \right) \right\},$$

and the appropriate formula for the sum of two cosines is

$$\cos B + \cos C = 2 \cos \tfrac{1}{2}(B+C) \cos \tfrac{1}{2}(B-C)$$

(" twice the cosine of half their sum, times the cosine of half their difference ").

Case 4. Two compression waves, A positive for one, negative for the other.

If the waves are represented by

$$Y_1 = A \cos 2\pi n (t - x/v),$$

and

$$Y_2 = -A \cos 2\pi n (t \pm x/v),$$

then

$$Y = Y_1 + Y_2.$$

Hence

$$Y = A \left\{ \cos 2\pi n \left(t - \frac{x}{v} \right) - \cos 2\pi n \left(t \pm \frac{x}{v} \right) \right\},$$

and the formula is that for the difference of two cosines,

$$\cos B - \cos C = -2 \sin \tfrac{1}{2}(B+C) \sin \tfrac{1}{2}(B-C)$$

(" minus twice the sine of half their sum, times the sine of half their difference ").

Reflection of a transverse cord wave at a rigid boundary. Consider, again, a stretched rubber cord, fixed at one end to a rigid boundary, with the hand at the other end initiating a simple harmonic wave that travels along the cord to the support. The cord will exert a periodically varying force on the support, and the support will at

each instant exert an equal and opposite reaction on the end of the cord, which is thus subjected to a periodically varying external force.

Such a force applied at any other point on the cord would inevitably originate *twin* waves, travelling outwards in both directions. But it is known that (*a*) the wave originally sent down the cord is not continued into the rigid boundary ; and also (*b*) only one of the two twin waves that ought to be caused by the support can actually be propagated, and that is the reflected wave travelling backwards along the cord.

These observations can be reconciled with the expected generation of twin waves by the reaction of the support if the " twin " that ought to have been propagated into the support is such as to neutralise completely the continuation of the original wave into the support.

Hence, to find the reflected wave by graphical construction, we must (diagram (*a*)) :

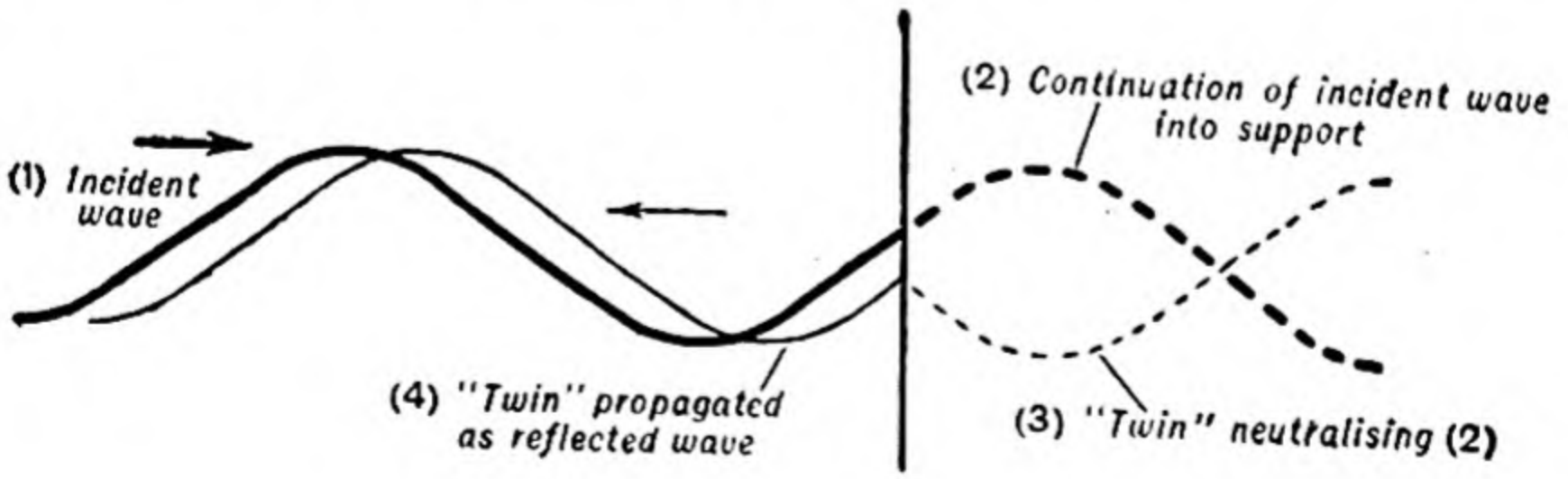
(1) draw the incident wave up to the support ;

(2) continue the original wave into the support ;

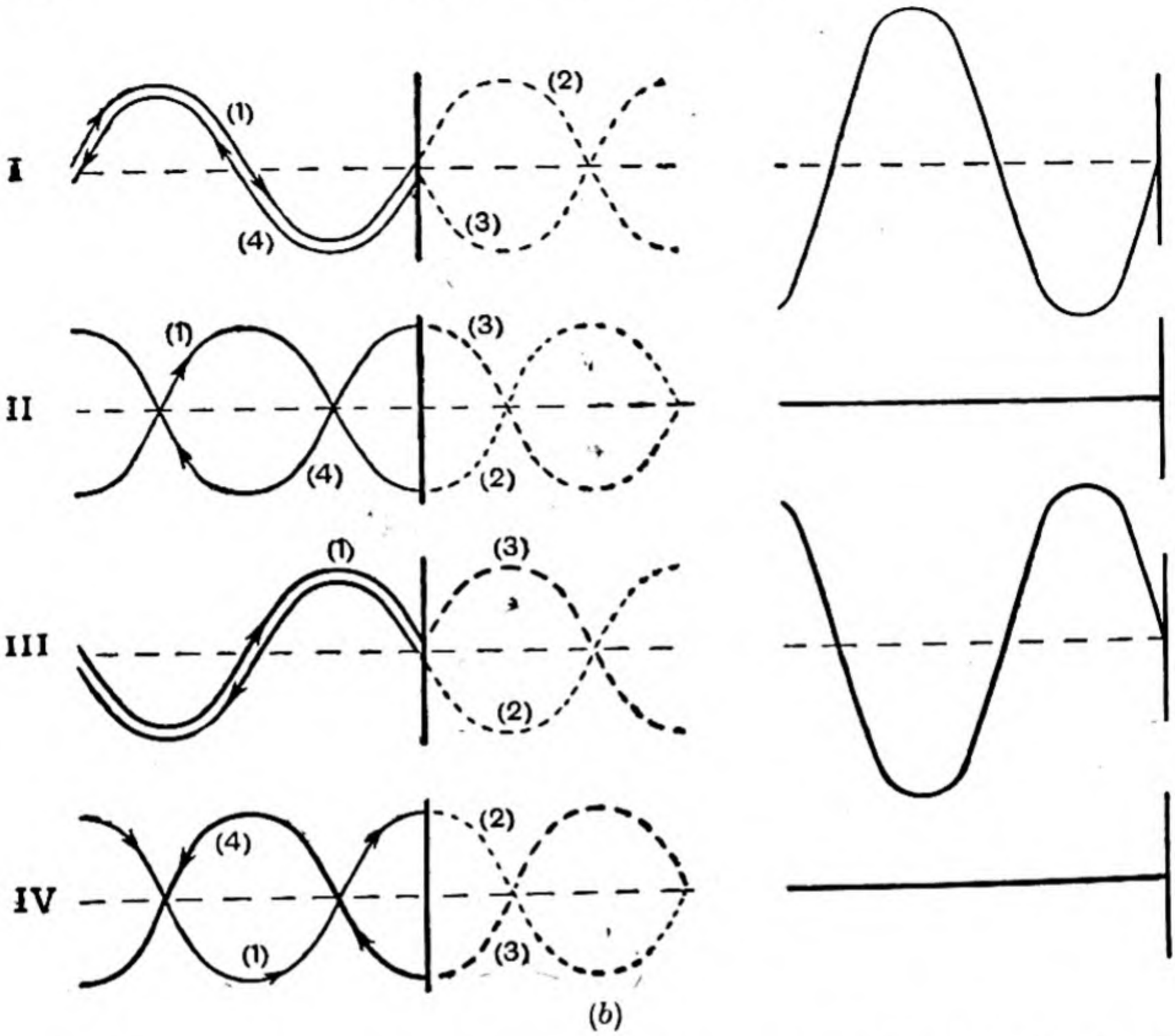
(3) construct in the support a wave which, superposed on this, destroys it everywhere ;

(4) construct the twin of this, propagated backwards along the cord ; this is the required reflected wave.

Finally, to describe what is actually observed, we must remember that after part of the original wave has been reflected, the remainder is still approaching the support. The observed displacement of the cord in the neighbourhood of the support at any instant is obtained by superposing the approaching portion of the incident wave and the returning portion of the reflected wave.



(*a*)



(*b*)

Diagram (*b*) shows the successive stages of the reflection of a single pulse at a rigid support. The left-hand side of each drawing shows the four construction steps, with the reflected wave drawn a little below the incident wave for clearness; the right-hand side shows the observed state of the cord.

The same calculation can be done using the wave equation, with the following usual conventions: if x/v is positive for a wave travelling to the right, x/v is negative for a wave travelling to the left; and if the two waves within the support are to neutralise one another, their displacements at every instant must be equal and opposite, which is best shown by writing their amplitudes as equal and opposite. Time t and distance x are still measured from some chosen instant, and from some fixed reference point, which need not be in the neighbourhood of the reflector.

In this case: (1) consider $y_1 = a \sin 2\pi n (t - x/v)$;

(2) write down the equation for the original wave, continued into the support,

$$y_1 = a \sin 2\pi n (t - x/v) ;$$

(3) neutralise this within the support by a wave of reversed amplitude :

$$y_2' = -a \sin 2\pi n (t - x/v) ;$$

(4) write down the twin of wave (2), with $(t + x/v)$ for $(t - x/v)$ since x/v has been reversed in sign :

$y_2 = -a \sin 2\pi n (t + x/v)$. This represents the reflected wave.

The observed state of the cord at time t can be deduced by adding together the displacements of the incident and reflected waves, and is thus represented by

$$y = y_1 + y_2 ;$$

this is studied more fully on pp. 342–344.

Some care is required in describing the observations. It is seen (*a*) that *the reflected wave has been reversed in direction*, and (*b*) that *it is reversed in amplitude*, that is, a crest is reflected as a trough. These are indicated by the substitution of $+x/v$ for $-x/v$, and $-a$ for a. But from the point of view of the wave itself, there has been no change of phase, since a trough has succeeded a crest at exactly the same time interval, just as if there had been no boundary and no reflection.

Reflection of a longitudinal wave at a rigid boundary. A longitudinal wave can be considered from two points of view : as a displacement wave, and also as a compression wave.

For the longitudinal displacement wave, the same arguments as for a transverse displacement wave hold.

The continuation of the *displacement wave* into the rigid boundary is

$$y_1 = a \sin 2\pi n (t - x/v).$$

This is neutralised within the rigid boundary by

$$y_2' = -a \sin 2\pi n (t - x/v) ;$$

the twin of this, propagated as the reflected displacement wave, is

$$y_2 = -a \sin 2\pi n (t + x/v).$$

But for the *compression wave*, it must be remembered that any small reaction of the boundary that would send a compression into

the body of the boundary would send as its twin a rarefaction to be reflected back ; any reaction which would propagate a rarefaction into the boundary would send as its twin a compression to be reflected back.

The compression wave striking the boundary is represented by

$$Y_1 = A \cos 2\pi n (t - x/v).$$

Its imaginary continuation within the boundary is given by

$$Y_1 = A \cos 2\pi n (t - x/v).$$

The wave to neutralise this within the boundary is given by

$$Y_2' = -A \cos 2\pi n (t - x/v).$$

The twin of this has x/v reversed because it is travelling in the *opposite* direction, and A *positive* because twin compression waves are generated with amplitudes of opposite sign, so the reflected compression wave is

$$Y_2 = +A \cos 2\pi n (t + x/v).$$

Thus, *a compression is reflected as a compression* ; and *a rarefaction as a rarefaction.* This does not conflict at all with the displacement reversal that occurs, but is to be expected, since displacement in the direction of travel of the wave causes compression, and if both displacement and direction are reversed, the compression state should be unaltered.

Reflection of a longitudinal wave at an open boundary. The most important example of this is the reflection of a sound wave propagated down a tube at the open end of the tube. To begin with, the displacement wave is of minor interest here, and we shall consider only the compression wave. Next, it is obvious that only part of the energy is thus reflected, as a considerable portion is propagated out through the open boundary ; and we shall consider the compression wave travelling down the tube as two superposed waves : one of amplitude A that is entirely reflected, and another of amplitude A' that travels on unhindered, the actual amplitude of the compression wave within the tube being $A + A'$.

As the boundary is free from any kind of constraint, there is no solid wall to exert any pressure change on the air, and so the change in pressure at the mouth of the tube must always be practically zero.

This means that, when a compression reaches the mouth, the surrounding air rushes away causing a corresponding rarefaction, which initiates a pair of exactly similar twin waves, one neutralising the compression in the outer air, the other travelling back down the tube. *A compression is reflected as a rarefaction, and a rarefaction as a compression.*

Using the equations for a compression wave, if the incident wave is, for the purpose of reflection, given by

$$Y_1 = A \cos 2\pi n (t - x/v),$$

the wave neutralising it outside the tube by pressure equalisation of the surrounding air is given by

$$Y_2' = -A \cos 2\pi n (t - x/v) ;$$

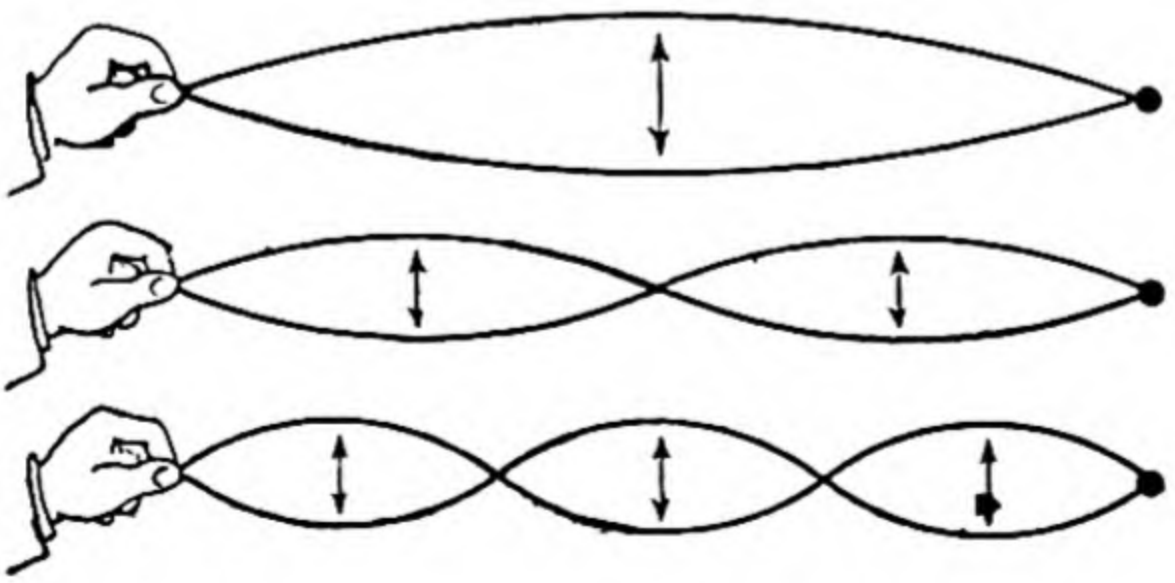
the twin of this, travelling back down the tube as the reflected wave, is given by

$$Y_2 = -A \cos 2\pi n (t + x/v).$$

For the displacement wave, the amplitude a is not reversed in sign on reflection at an open boundary. The particles in a compression moving to the right are moving to the right ; those in a rarefaction moving to the left are moving to the right. Thus the displacement wave returns with a " crest " reflected as a " crest " ; or, in terms of the wave equations, if the incident wave is given by $y_1 = a \sin 2\pi n (t - x/v)$, the reflected wave is given by

$$y_2 = a \sin 2\pi n (t + x/v).$$

Stationary waves. When a succession of transverse waves is sent by hand down a long cord, the waves reflected at the fixed far end travel back quite independently of their successors on the outward journey, and the incident and returning waves are superposed. If



the hand is agitated at random, a confused motion of the cord results, in which progressive waves in both directions can be more or less identified from time to time. But if the motion of the hand is suitably timed, the cord appears to vibrate transversely as a whole, with no sign of any progressive wave. It is then said to be exhibiting a **stationary wave, or standing wave system.**

This can be done in several ways, as indicated in the diagram. If the cord is reasonably long and not too tightly stretched, as many as six or more independent segments may be obtained. The frequency of vibration of the hand must be twice as great for two segments as for one, three times as great for three segments as for one, and so on.

It is observed that :

(1) there is no sign of any progressive wave in either direction ;

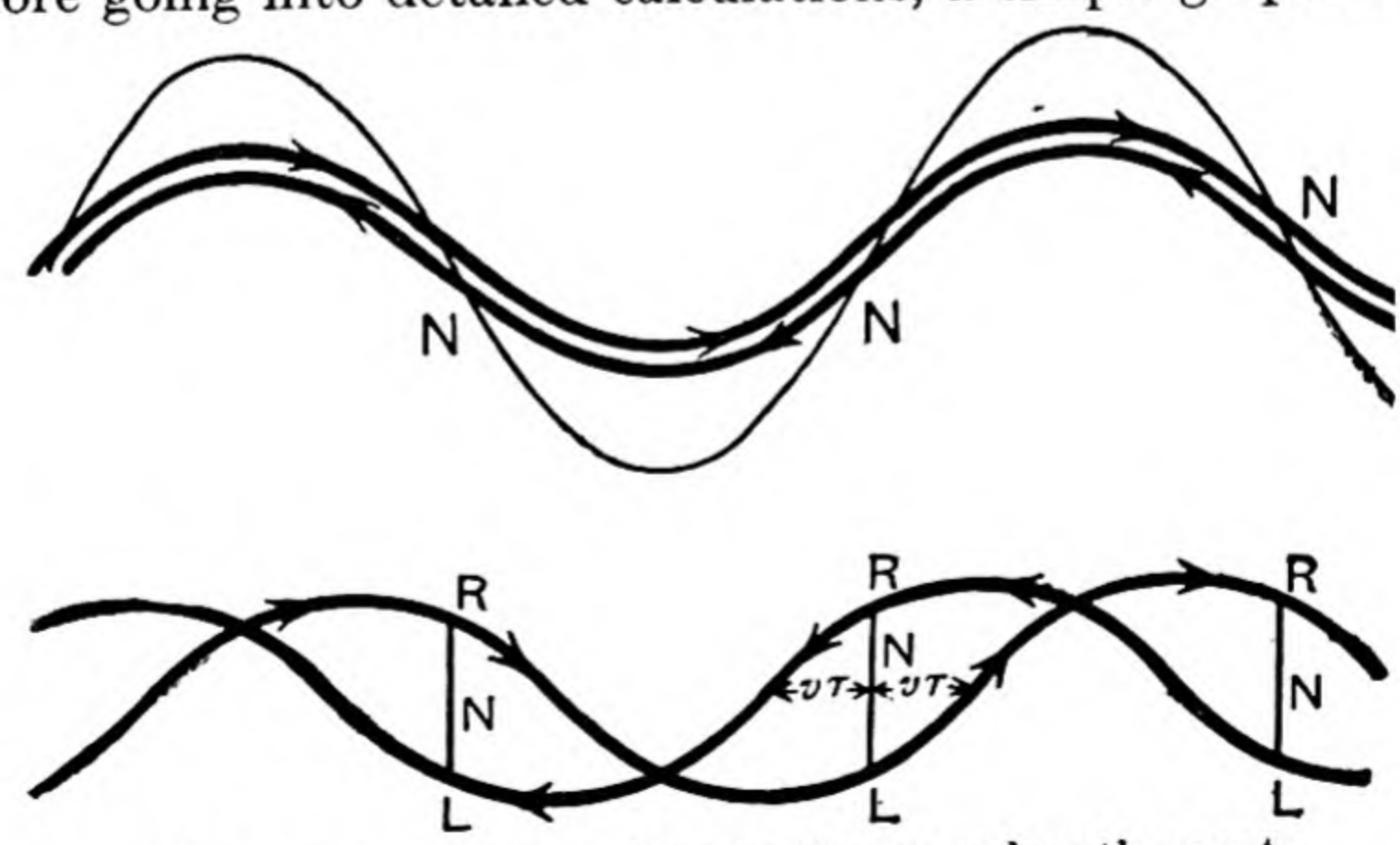
(2) the individual particles of the cord are executing simple harmonic motions at right angles to the cord ;

(3) the amplitude of this simple harmonic motion is greatest at equally spaced points which are called **antinodes,** and zero at a set of equally spaced points which are called **nodes** ; the ends are always nodes ;

(4) the pattern of nodes and antinodes divides the cord up into a number of equal segments ;

(5) all particles in the *same segments* are vibrating exactly in phase, and *adjacent segments* are exactly out of phase with one another.

Before going into detailed calculations, a simple graphical super-



Nodes must exist, and be half a wave-length apart.

position will show that, if we can maintain two trains of similar waves in opposite directions on the cord, there should at least be nodes somewhere on it.

For consider two waves of equal amplitude and wave-length travelling in opposite directions (see diagram). At the instant shown in the upper figure, the waves shown by the thick lines give a resultant displacement shown by the thin line, but points $N, N, N \ldots$, half a wave-length apart, are points of zero displacement. A very short time, say τ sec. later, as shown in the lower figure, one wave has moved a distance $v\tau$ to the right, giving a displacement at the N points of NR upwards ; the other has travelled a distance $v\tau$ to the left, giving each N point a displacement NL downwards. As NR must equal NL, the resultant displacement of each N point is zero, and this must always happen whatever the relative positions of the two waves. Thus, *nodes must exist, and they must be half a wave-length apart.*

Next, what is the condition that the stationary wave system shall be maintained? It must be remembered that a wave starting at the hand and reflected at the fixed end returns again to the hand, where it is reflected again, and so on. Maintenance of the system is only possible if the waves reflected for the second time at the hand exactly reinforce in phase those initiated at that moment at the hand. The time taken for a wave to travel to and from the hand is $2l/v$ sec., where l cm. is the length of the cord and v cm. per sec. the velocity of the wave ; and this must be a whole number of vibration-periods.

If n is the frequency of the waves and λ their wave-length, then this condition is satisfied if

$$2l/v = 1/n,\ 2/n,\ 3/n,\ 4/n \ldots \text{etc.},$$

or,

$$2l = v/n,\ 2v/n,\ 3v/n,\ 4v/n \ldots \text{etc.},$$

or, since $v = n\lambda$,

$$2l = \lambda,\ 2\lambda,\ 3\lambda,\ 4\lambda \ldots \text{etc.},$$

whence

$$l = \lambda/2,\ 2\lambda/2,\ 3\lambda/2,\ 4\lambda/2 \ldots \text{etc.},$$

and **the length of the cord must be a whole number of half wave-lengths.** Only waves the frequency of which is such as to give wave-lengths of

$$2l,\ 2l/2,\ 2l/3,\ 2l/4,\ \text{etc.},$$

can satisfy this condition for maintenance.

Analytical treatment. If the equation of the forward wave from the band is $y_1 = a \sin 2\pi n(t - x/v)$, that of the reflected wave is (p. 339) $y_2 = -a \sin 2\pi n(t - x/v)$.

The resultant displacement at distance x is $y = y_1 + y_2$, or

$$y = a \sin 2\pi n\,(t - x/v) - a \sin 2\pi n\,(t + x/v),$$

which, using the formula for the *difference* of two sines (p. 336) gives

$$y = 2a \cos 2\pi nt \sin 2\pi nx/v,$$

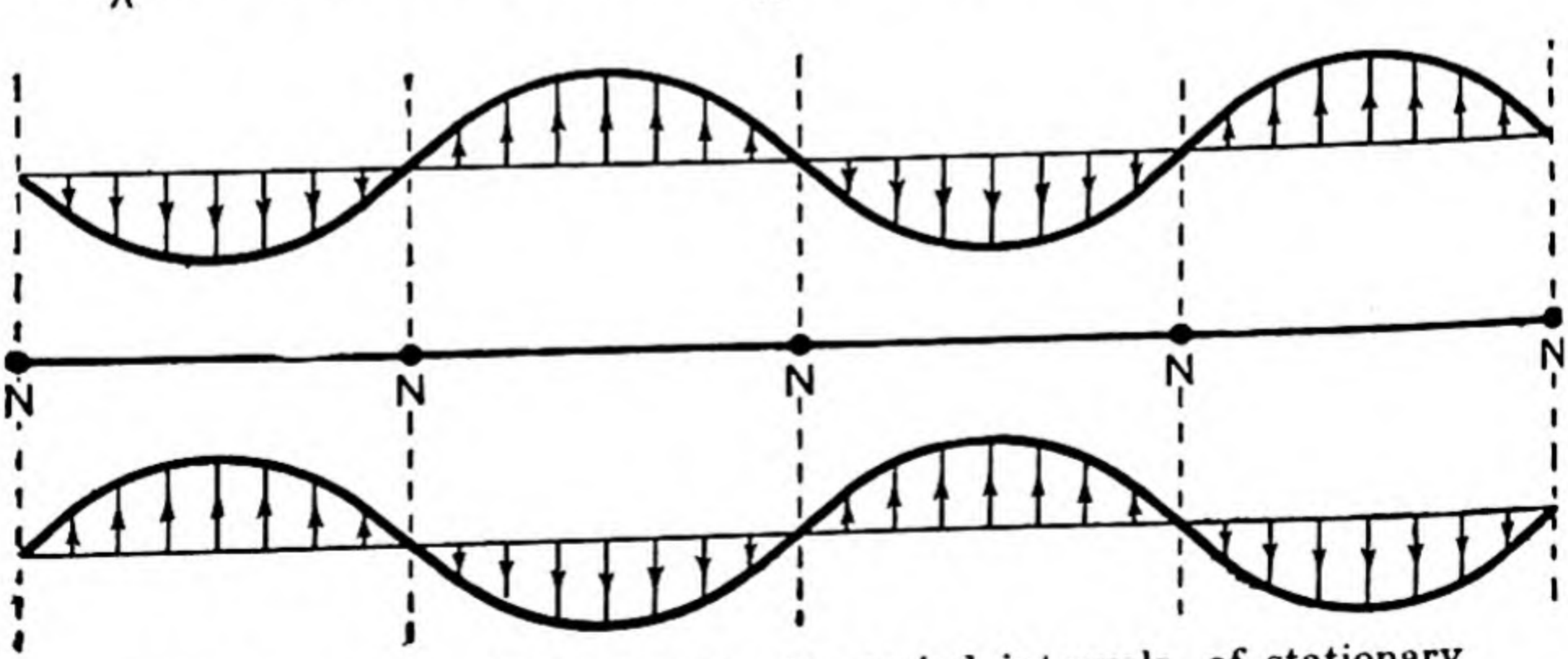
or rearranging and remembering that $v = n\lambda$,

$$y = 2a \sin \frac{2\pi}{\lambda} x \cos 2\pi nt.$$

This cannot represent a progressive wave, since the x/v and t terms no longer appear together, as they must in any progressive wave equation (p. 331). It means that a particle at distance x from the hand executes a simple harmonic motion of frequency n and amplitude $2a \sin \frac{2\pi}{\lambda} x$.

This amplitude is zero when $\sin \frac{2\pi}{\lambda} x = 0$, that is, when $\frac{2\pi}{\lambda} x = 0$, π, 2π, . . . etc., or when $x = 0$, $\lambda/2$, $2\lambda/2$, $3\lambda/2$, . . . etc., up to l. So we see again that the length l must be a whole number of half wave-lengths. These points of zero amplitude are the nodes.

The amplitude is $\pm 2a$, the maximum value possible, when $\sin \frac{2\pi}{\lambda} x = \pm 1$, which is when $\frac{2\pi}{\lambda} x = \pi/2$, $3\pi/2$, $5\pi/2$, . . . etc., or

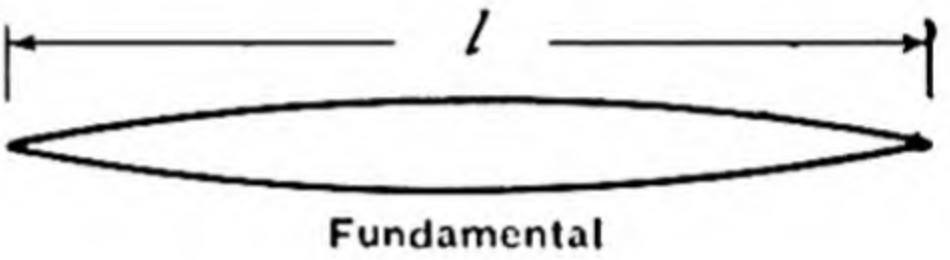


Three successive stages, at quarter-period intervals, of stationary vibration of a string; N indicates a node.

when $x = \lambda/4, 3\lambda/4, 5\lambda/4, \ldots$ etc. These points of maximum amplitude are the antinodes.

Demonstration of stationary waves on strings. It is not proposed here to deal at all fully with the musical aspect of the sound waves propagated into the air by a suitably mounted vibrating string. But it is convenient to be able to use one's ear as a means of recognising the related frequencies of the various modes in which a string can vibrate.

The sound waves sent out by the mount, which is set into forced vibration by the string, must have the same frequency (or frequencies) as those of the string itself. It should, however, be quite clear that in the waves travelling through the air we have *longitudinal* sound waves; while those travelling along the string and giving the stationary wave pattern which is being studied are *transverse* elastic waves.



Fundamental

Second harmonic

Third harmonic

Fourth harmonic

Fifth harmonic

A musical note is recognised by its pitch, which depends on the frequency; we can thus recognise the frequencies of the vibrations of the string if we can identify the pitches of the notes emitted when the string sets the air into vibration and propagates sound waves.

The common chord *doh mi soh doh'* has three easily recognisable intervals:

the major third, *doh—mi*;

the fifth, *doh—soh*;

the octave, *doh—doh'*.

In the scale of *C* major, these notes are:

c′, e′, g′, c″;

and their frequencies on the scale used in Physics are :

c′, 256 ; e′, 320 ; g′, 384 ; c″, 512 per sec.

The frequency ratios corresponding to the intervals mentioned are:

the major third, $320/256 = 5/4$,
the fifth, $384/256 = 3/2$,
the octave, $512/256 = 2/1$.

These simple frequency ratios are characteristic of the intervals, whatever the keynote doh *may be.*

The velocity, u, of string waves is given by

$$u = \sqrt{T/m} \text{ cm./sec.,} \quad \text{(p. 328)}$$

where T is the tension in dynes, and m is the mass per unit length in grams.

Possible wave-lengths for stationary waves on a string of length l cm. are :

(1) $\lambda_1 = 2l$, known as the fundamental.
(2) $\lambda_2 = 2l/2$, the second harmonic.
(3) $\lambda_3 = 2l/3$, the third harmonic.
(4) $\lambda_4 = 2l/4$, the fourth harmonic.
(5) $\lambda_5 = 2l/5$, the fifth harmonic.
(x) $\lambda_x = 2l/x$, the xth harmonic.

As $u = n\lambda$, the corresponding frequencies are :

(1) $n_1 = \frac{1}{2l}\sqrt{\frac{T}{m}}$, fundamental. (2) $n_2 = \frac{2}{2l}\sqrt{\frac{T}{m}}$, second harmonic.

(3) $n_3 = \frac{3}{2l}\sqrt{\frac{T}{m}}$, third harmonic. (4) $n_4 = \frac{4}{2l}\sqrt{\frac{T}{m}}$, fourth harmonic.

(5) $n_5 = \frac{5}{2l}\sqrt{\frac{T}{m}}$, fifth harmonic, and in general

$$n_x = \frac{x}{2l}\sqrt{\frac{T}{m}}, \; x\text{th harmonic.}$$

The frequencies are thus in the ratio $1 : 2 : 3 : 4 : 5 : \ldots : x$.

The frequency ratio n_2/n_1 is 2 : 1. Thus the second harmonic is an octave above the fundamental ;

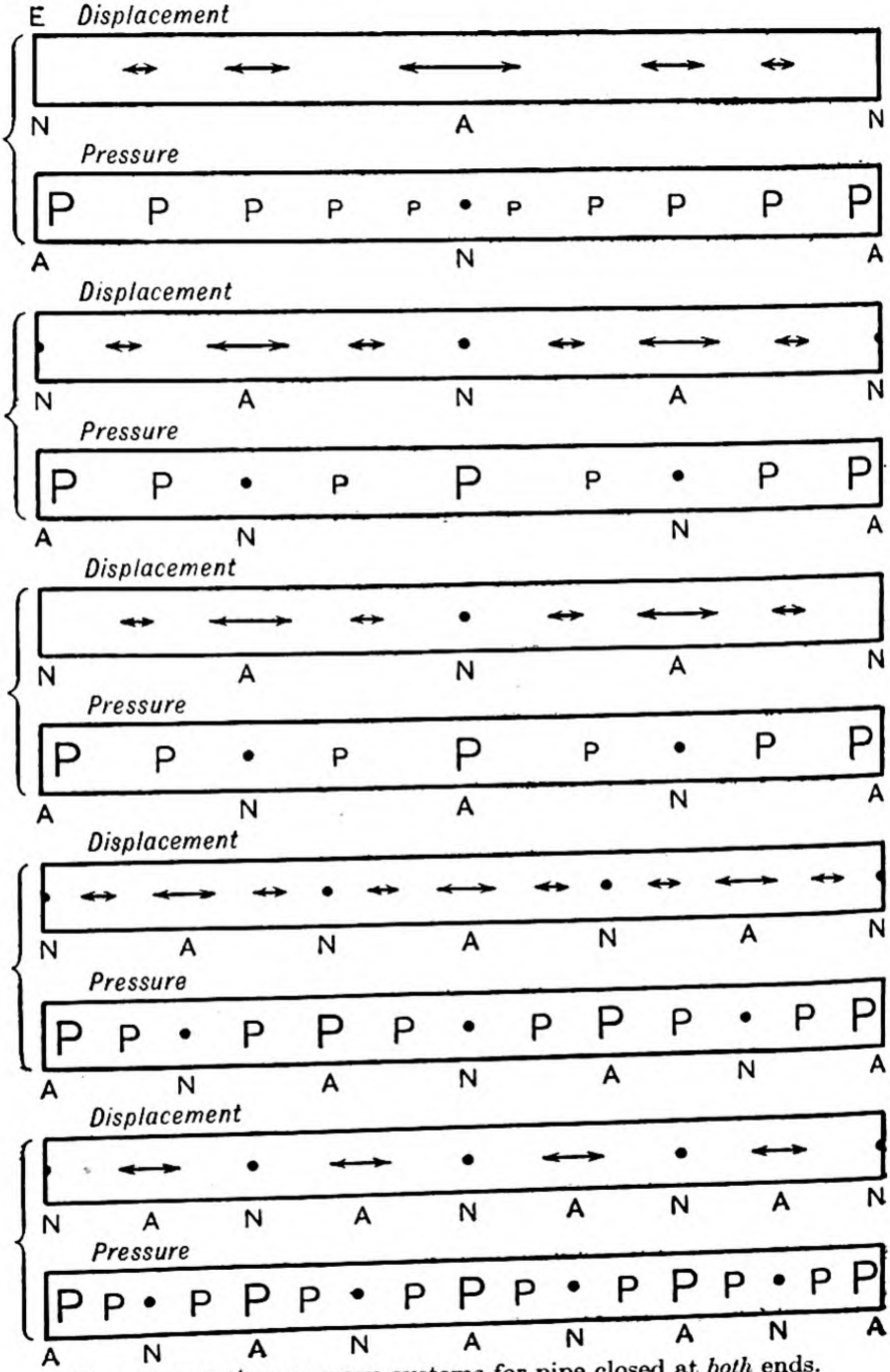
n_3/n_2 is 3/2, so the third harmonic is a fifth above the second ;
n_4/n_n is 4/2, so the fourth harmonic is an octave above the second ;
n_5/n_4 is 5/4, and the fifth harmonic is a major third above the fourth harmonic,

speaking now, of course, of the sound waves to which the vibrations of the string give rise.

Many harmonics are present together when the string is excited in the ordinary way, the main note heard being that corresponding to the fundamental as a rule. If the string, while so vibrating, is touched lightly with the corner of a handkerchief or the finger tip at a point corresponding to a node for any particular mode of vibration, then all other modes are practically extinguished, and that one persists alone, to be recognised by the note it propagates.

Thus, bowing or plucking the string anywhere at random, the second harmonic (accompanied by the fourth, sixth, eighth, which are usually feeble) is obtained by touching lightly at the mid-point ; the note is recognised as an octave above the fundamental. The third harmonic is isolated by touching one third the length along the string, and recognised as a fifth above the previous note, and so on. After proceeding like this for a little time, one can usually hear the notes corresponding to the higher harmonics superposed on the fundamental when the string is excited and not touched at all. A keen ear will detect considerable differences in the quality of the note as such when different means of excitation (bowing, stroking, plucking) and different points of attack on the string are employed, and also a change in quality as the note fades. The **quality of a musical note** is determined by the number and relative intensities of the higher notes or overtones superposed on the fundamental, but that is a matter best left for the present. Also, the higher members of the sequence produced by an actual instrument may not exactly coincide in frequency with the expected harmonic.

Longitudinal stationary waves. Here there are really two independent wave patterns to examine : that for the *displacement wave*, and that for the *compression wave*. Suppose that the medium under discussion is a column of air or other gas enclosed in a tube of uniform bore, closed at each end by a solid reflecting boundary, with waves being propagated by means which we can imagine to excite small longitudinal vibrations close to one of these ends, say E (see diagram).


E Displacement
N A N
Pressure
P P P P P • P P P P P
A N A
Displacement
N A N A N
Pressure
P P • P P P • P P
A N N A
Displacement
N A N A N
Pressure
P P • P P P • P P
A N A N
Displacement
N A N A N A N
Pressure
P P • P P P • P P P • P P
A N A N A N A
Displacement
N A N A N A N A N
Pressure
P P • P P P • P P P • P P P • P P
A N A N A N A N A

First five stationary-wave systems for pipe closed at *both* ends.

At each end the *displacement wave* reaching it is reflected with amplitude reversal, as p. 339 ; and each end must be a displacement node. The case is exactly the same as that for transverse displacement waves on a cord, and the argument of p. 343 can be applied to show that a stationary wave pattern can be maintained only if the frequency has one of the values which give a wave-length such that the length of the column l is a whole number of *half* wave-lengths.

If y here denote the longitudinal displacement of a particle at distance x from the reference point E at time t, then the forward wave is given by the equation

$$y_1 = a \sin 2\pi n (t - x/v),$$

and the wave reflected from the other boundary (p. 339) by

$$y_2 = -a \sin 2\pi n (t + x/v).$$

Superposing, the resultant displacement $y = y_1 + y_2$ is (exactly as on p. 344 by adding sines, and using $v = n\lambda$)

$$y = 2a \sin \frac{2\pi x}{\lambda} \cos 2\pi n t.$$

This is the same stationary wave pattern with nodes at

$$x = 0,\ x = \lambda/2,\ x = 2\lambda/2,\ x = 3\lambda/2,\ \text{etc.},$$

up to $x = l$, which must be a whole number of half wave-lengths ; and antinodes at $x = \lambda/4,\ x = 3\lambda/4,\ x = 5\lambda/4,$ up to $x = (l - \lambda/4)$.

The closed ends of the tube must always be displacement nodes.

The diagram indicates various possible stationary displacement wave systems.

For the *pressure wave*, however, matters are somewhat different. First, the maxima of the compression wave are $\lambda/4$ out of step with the displacement-wave maxima, so that the closed end of the pipe must necessarily be a pressure antinode. Secondly, a compression is reflected back as a compression, without reversal of amplitude.

Thus, if the forward compression wave is represented by

$$Y_1 = A \cos 2\pi n (t - x/v),$$

the returning wave is given by

$$Y_2 = A \cos 2\pi n (t + x/v),$$

and, superposing, $Y = Y_1 + Y_2$,

$$= A \cos 2\pi n(t - x/v) + A \cos 2\pi n(t + x/v),$$

$$= 2A \cos 2\pi nx \,.\, \cos 2\pi nt,$$

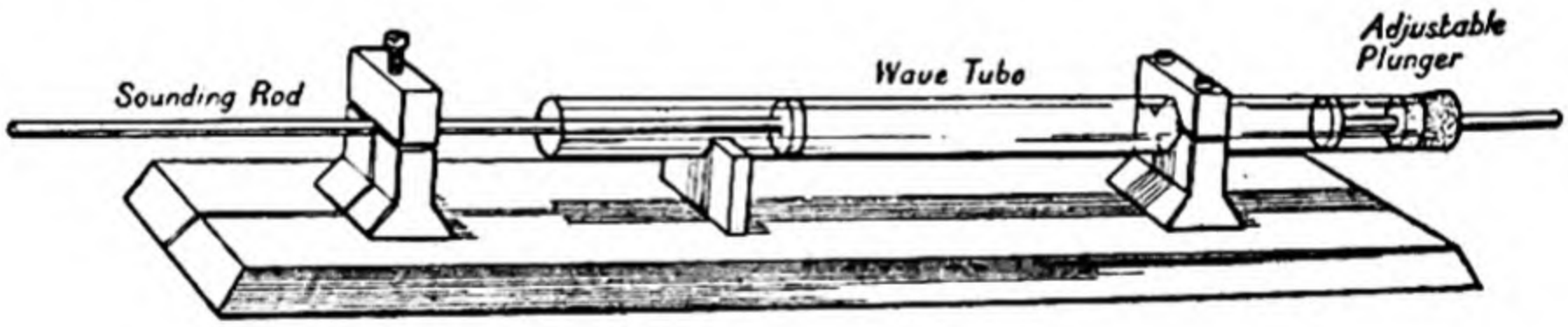
$$= 2A \cos \frac{2\pi x}{\lambda} \,.\, \cos 2\pi nt.$$

Thus, the pressure change varies harmonically with time, the frequency being n, and the amplitude

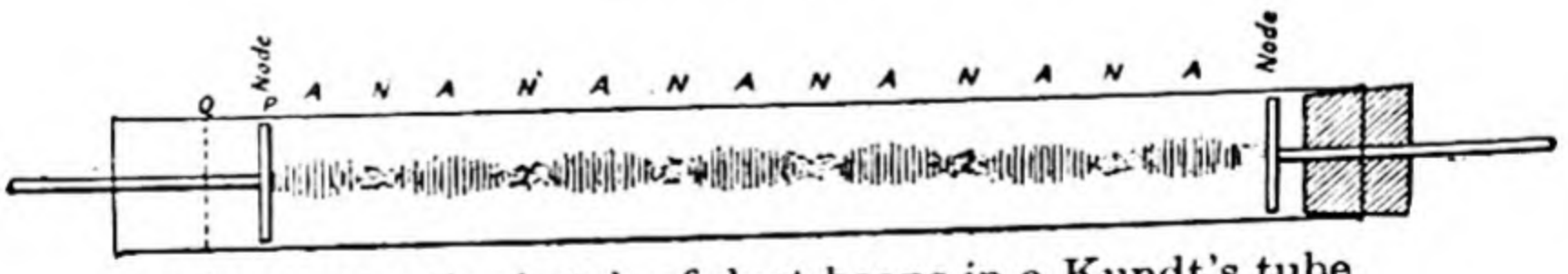
$$2A \cos \frac{2\pi x}{\lambda},$$

which is a maximum ($\pm 2A$) when $x = 0, \lambda/2, 2\lambda/2, 3\lambda/2, \ldots$ up to l, giving the positions of the pressure antinodes; and zero when $x = \lambda/4, 3\lambda/4, 5\lambda/4, \ldots$, giving the positions of the pressure nodes.

The stationary wave pattern may be shown in two ways. In Kundt's tube (see diagram) sound waves of frequency about 4000/sec. are propagated down the tube from the left end where a disc attached to the end of a glass rod excited into longitudinal vibration by stroking with a resined cloth acts as source. Adjustment of the plunger at the other end to make the length of the air column very nearly a whole number of half wave-lengths for sound of this



Kundt's tube



Diagrammatic sketch of dust heaps in a Kundt's tube

By courtesy of Prof. E. N. da C. Andrade

Photograph of dust striations

frequency causes the stationary wave system to be set up. The displacement antinodes are made visible by sprinkling a line of dry lycopodium powder along the tube. This is thrown into violent agitation while the rod is sounding, and sets into a pattern of striations centred around each antinode when the sound ceases. Continued operation may cause small heaps of powder to collect at the nodes.

The plunger P, closing the left-hand side of the tube, cannot both set the air in its neighbourhood in vibration and also keep it stationary at a node ; somewhere between the inscription " Node " at P and the dotted line at Q the actual node must lie, and the position of P must be such that the amplitude of its oscillation is the same as that of the air particles surrounding it.

Rubens' tube, filled with coal gas burning at a series of small jets at the top, demonstrates the pressure-change distribution ; the jets are longest above the pressure antinodes.

Tube open at both ends. For the compression wave, there is an amplitude reversal on reflection at an open end. If

$$Y_1 = A \cos 2\pi n (t - x/v) \text{ represents the forward wave,}$$

then $Y_2 = -A \cos 2\pi n (t + x/v)$, represents the reflected wave, the resulting stationary wave pattern, *if formed*, being given by

$$Y = Y_1 + Y_2 = -2A \sin \frac{2\pi x}{\lambda} . \sin 2\pi n t,$$

with pressure nodes (and therefore displacement antinodes) at both the open ends, and spacing depending on λ.

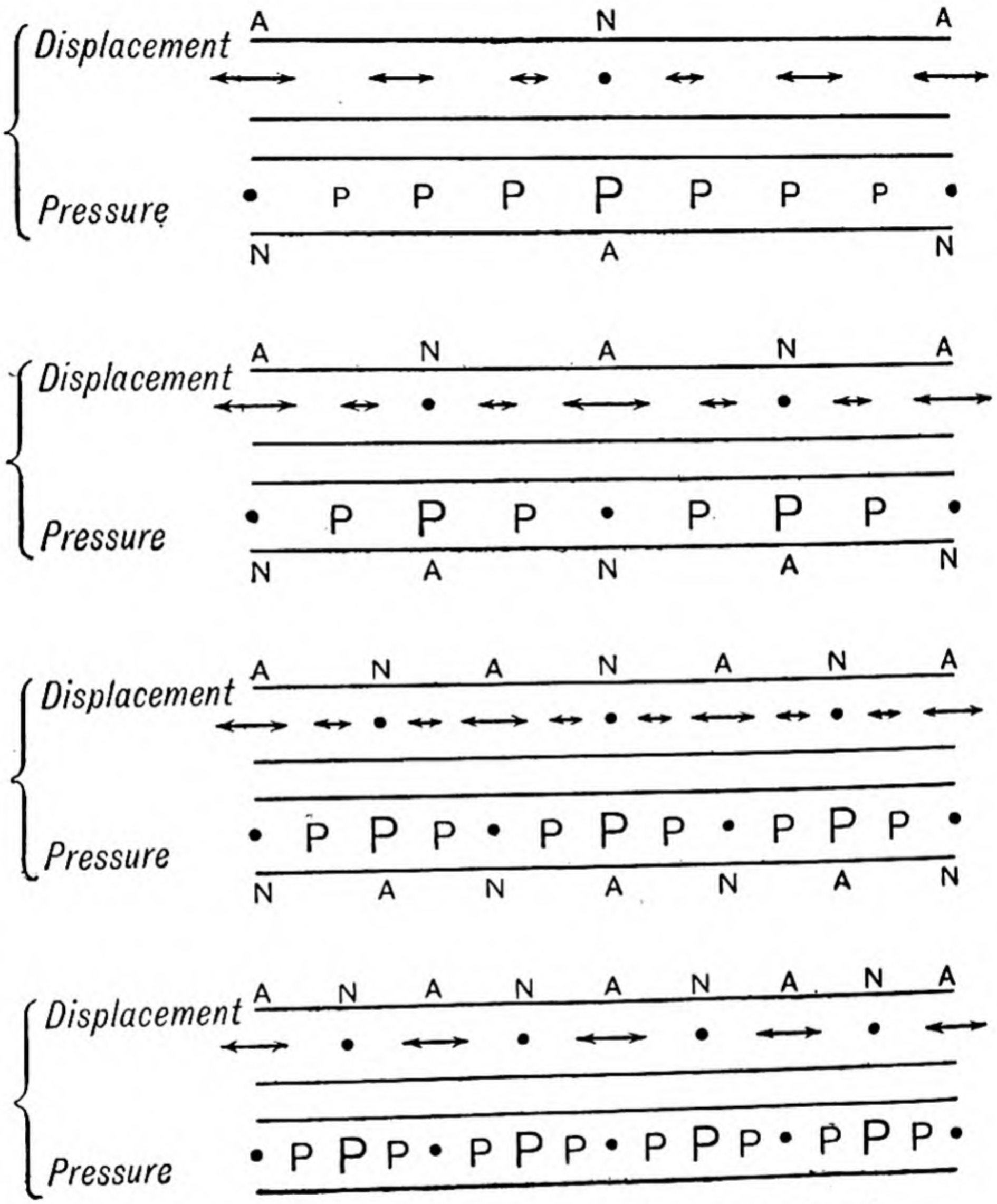
If we consider the progress of a single pulse travelling several times up and down the tube, arriving at one end as a compression, travelling back as a rarefaction, reflected again as a compression in phase with a succeeding pulse, it can be seen that the condition for maintenance is that twice the length of the tube is a whole number of wave-lengths. The length of the tube must be a whole number of half wave-lengths, and possible wave-lengths to give the pattern are

$$\lambda = 2l, \quad \lambda = 2l/2, \quad \lambda = 2l/3, \quad \lambda = 2l/4, \ldots \text{ etc.,}$$

with frequencies in the ratio $1 : 2 : 3 : 4 \ldots$ the whole of the possible members of the harmonic series.

The diagram of p. 252 shows the pressure and displacement nodes and antinodes for the first four possible modes.

Open organ pipes are of this type.


Displacement
A N A
Pressure
P P P P P P P P
N A N
Displacement
A N A N A
Pressure
P P P P P P P P
N A N A N
Displacement
A N A N A N A
Pressure
P P P P P P P P
N A N A N A N
Displacement
A N A N A N A N A
Pressure
P P P P P P P P
N A N A N A N A

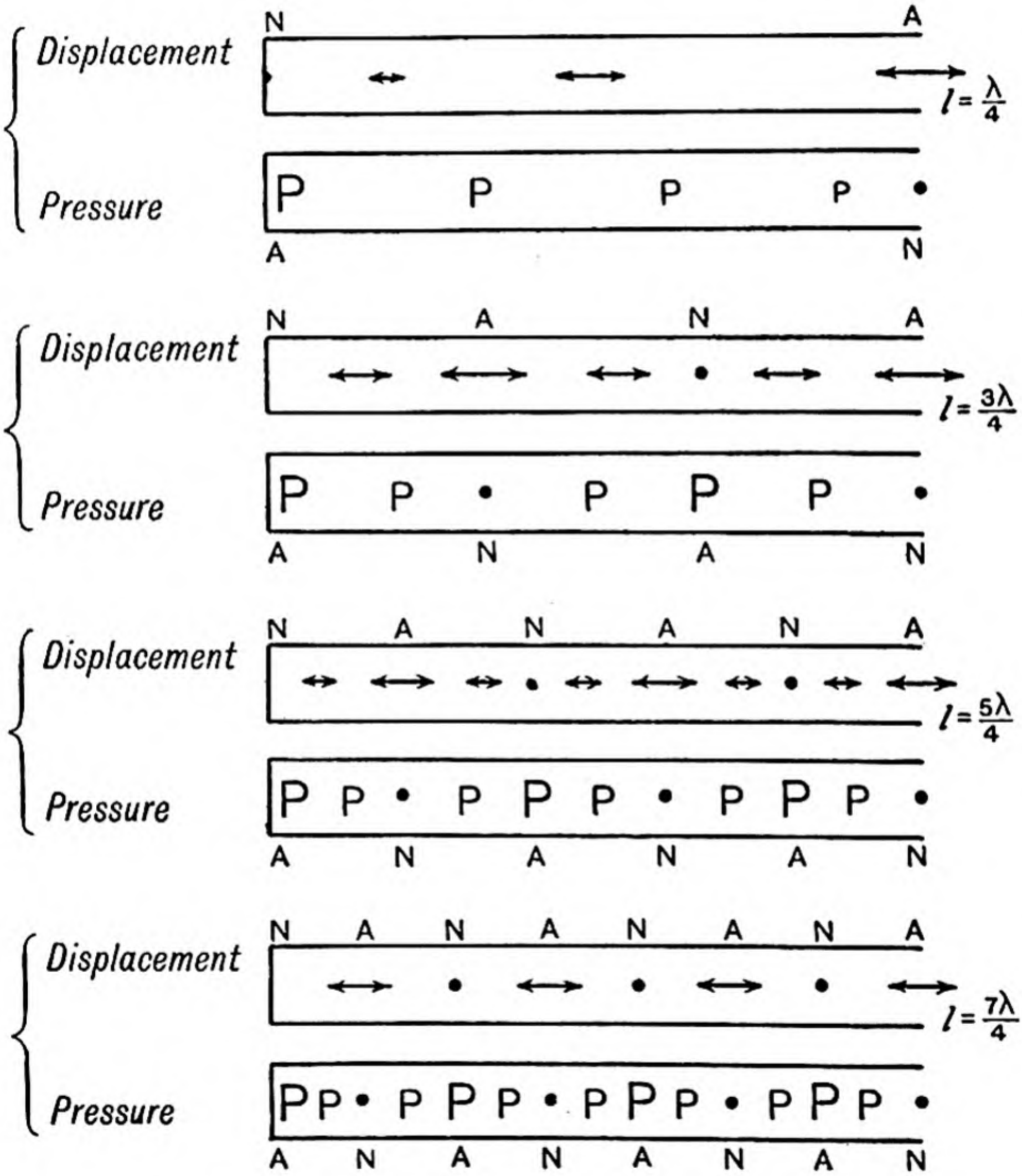
First four stationary wave-systems of tube *open at both ends.*

Tube open at one end, closed at the other. The only additional point arising in this case is the condition for maintenance of a stationary wave system to be possible.

Let a single pulse start at either end, say the open end, and follow its progress.

It is reflected as a compression at the closed end, and returns to be reflected as a rarefaction at the open end. It will be in phase with a succeeding rarefaction if the length of the tube, l, is such that $2l = \lambda/2$ (since compressions and rarefactions are $\lambda/2$ apart), or any odd number of half wave-lengths.

Thus, possible tube lengths for a given λ are $l = \lambda/4,\ 3\lambda/4,\ 5\lambda/4 \ldots,$



Stationary waves in tube closed at one end.

or, possible wave-lengths for a given l are $\lambda = 4l$, $\lambda = \frac{4}{3}l$, $\lambda = \frac{4}{5}l$ . . . etc., with frequencies in the ratio 1 : 3 : 5 . . . etc., giving only the *odd* members of the harmonic series.

The distribution of pressure and displacement nodes and antinodes is shown in the diagram.

The wave-length of the longest wave which will give a stationary wave pattern with a closed tube is thus *twice* that for an open tube of the same length. From the acoustical point of view, this means that the frequency is half, and the pitch of the note an octave below, the corresponding open pipe note. Stopped organ pipes furnish an example of this type of stationary wave pattern.

Beats. Consider two waves of frequencies n_1 and n_2, of comparable amplitude, travelling with the same velocity over the same expanse of medium.

Let them start, at the time we start timing, in step. Suppose n_1 is the greater.

At time $1/n_2$ sec. from the start, wave 2 has executed one vibration, and wave 1, n_1/n_2 vibrations.

So wave 1 is $\left(\frac{n_1}{n_2} - 1\right)$, or $\frac{n_1 - n_2}{n_2}$ vibrations ahead of wave 2.

Hence, after $\frac{n_2}{n_1 - n_2}$ such intervals, wave 1 will be $\frac{n_1 - n_2}{n_2} \times \frac{n_2}{n_1 - n_2}$, or one whole vibration ahead of wave 2, and the two will be in phase (or " in step ") once more, reinforcing one another.

The time interval between these reinforcements is

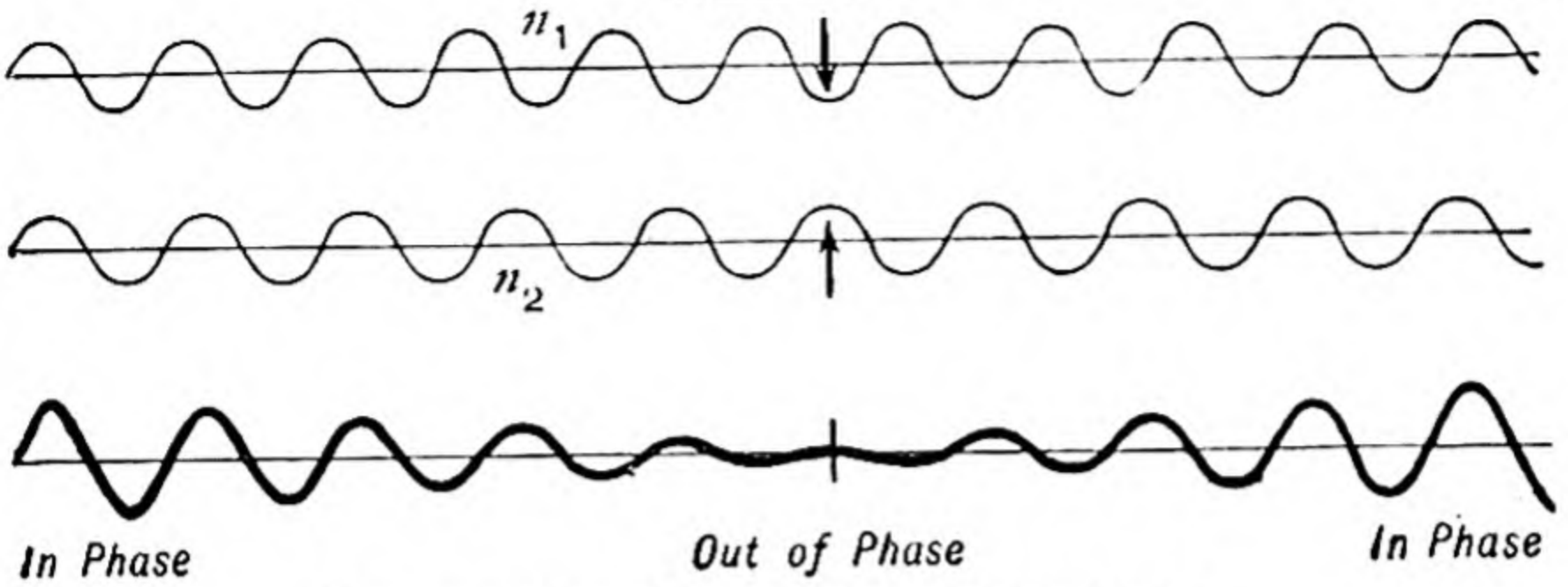
$$\frac{n_2}{n_1 - n_2} \times \frac{1}{n_2} \text{ sec.} = \frac{1}{n_1 - n_2} \text{ sec.}$$

So there are $(n_1 - n_2)$ reinforcements per second.

During the intervals between reinforcements, the relative phases undergo a complete cycle of changes, between 0 and 2π. Thus the combined wave due to their superposition waxes and wanes $(n_1 - n_2)$ times per second, giving rise to the phenomenon of **beats.**

The diagram shows how, by the usual graphical method of adding ordinates, the displacement-distance curve for the combined wave may be constructed.

In the usual experiments to demonstrate beats, it is probably true



Superposition of two waves causing beats.

to say that the two waves do not follow identical paths through the medium, and that superposition only occurs at the receiving device. Thus, with two tuning forks of slightly different pitch, the striking characteristic throbbing heard is a beating due to the superposition of two simple harmonic forces of frequency n_1 and n_2 at the eardrum. Again, a curve such as that shown can be produced on the screen of a cathode ray tube by sounding the two nearly-unison forks close to a microphone, the amplified output of which is applied across the vertical deflection plates; this picture shows the variation with time of the motion of the microphone diaphragm caused by two waves that are superposed there, and may have arrived there by any paths. In short, beats are a phenomenon characteristic of frequency difference, and are observable whenever vibrations of slightly different frequencies are superposed, whether by the reception of such waves or not.

Analytically, if we consider two compression waves of frequencies $(n+p)$ and $(n-p)$, differing in frequency by $2p$,

$$Y_1 = A \cos 2\pi (n+p)(t - x/v),$$

and

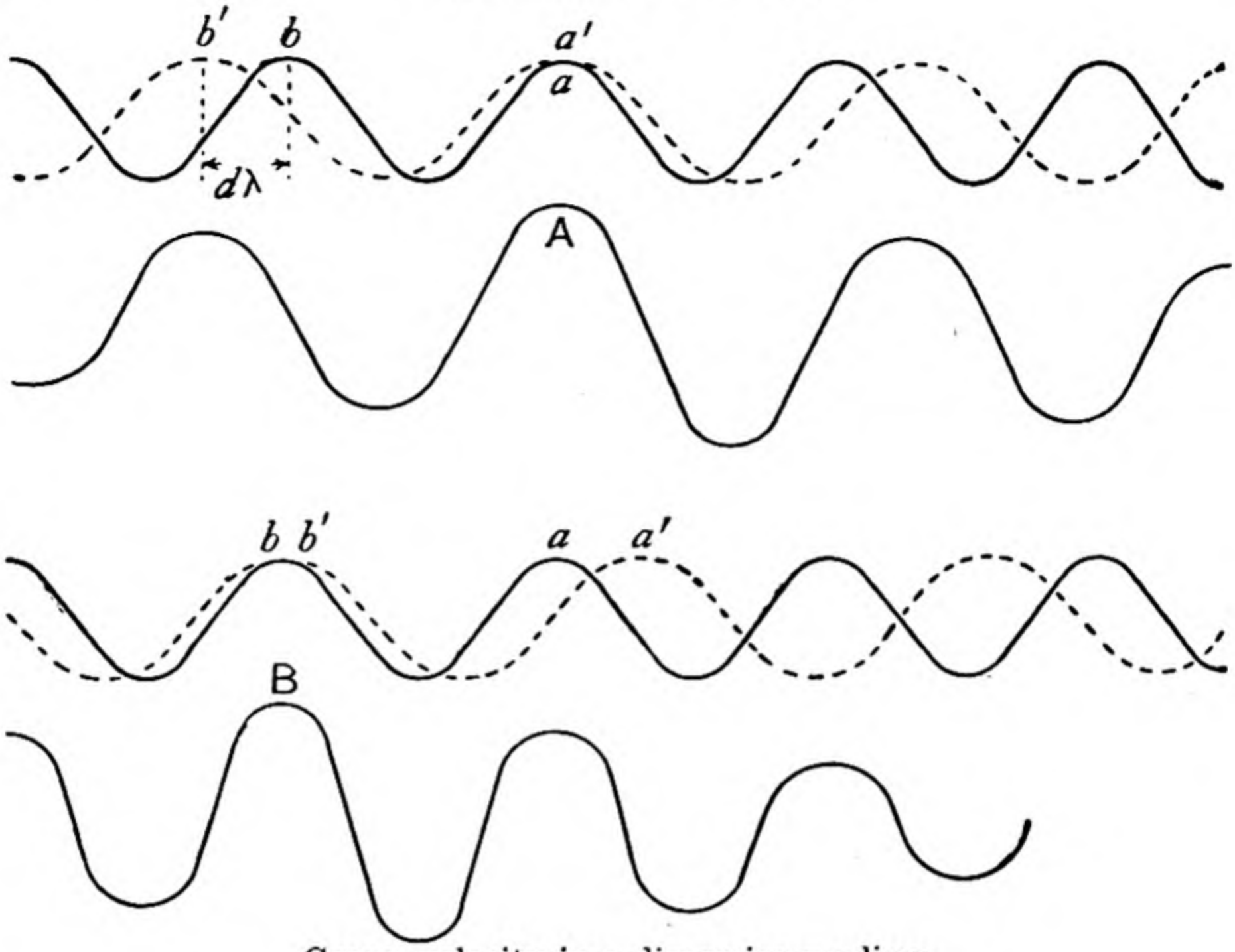
$$Y_2 = A \cos 2\pi (n-p)(t - x/v);$$

these superposed give

$$Y = Y_1 + Y_2 = 2A \cos 2\pi p (t - x/v) \cos 2\pi n (t - x/v).$$

The amplitude of this wave is itself a progressive wave of velocity v and frequency p, which gives the maximum value $2A$ for the amplitude twice per cycle, or $2p$ times per second.

Group velocity. Two waves giving rise to beats at a receiver may arrive there by any path. But the particular case when they are everywhere superposed during their journey through the medium is important. Working in terms of wave-length instead of frequency,



Group velocity in a dispersive medium.

let the wave-lengths be λ (full line) and $\lambda + d\lambda$ (dotted line) as represented in the diagram. The resulting disturbance of the medium is then a succession of " groups " or travelling beats. If both waves travel with the same velocity v, then any two particular coincident crests, as a and a' are always identified with a particular peak, as A, and the group moves along also with velocity v.

But if the medium causes dispersion (that is, if the wave velocity varies with the wave-length) matters are different. Let the velocities corresponding to wave-lengths λ and $\lambda + d\lambda$ be v and $v + dv$, so that the dotted wave is overtaking the other with the relative velocity dv. Crest a' moves beyond crest a; and crest b' moves forward to coincide with crest b, covering the distance $d\lambda$ with velocity dv in time $d\lambda/dv$. At the end of this interval, the peak of the group is at B, and has thus moved backwards relative to the individual waves by a distance λ. Relative to the waves comprising it, the peak of the group moves backwards by a distance λ in time $d\lambda/dv$, and has a retrograde velocity of $\lambda \div \frac{d\lambda}{dv}$, or $\lambda \frac{dv}{d\lambda}$.

Hence the group velocity, or the velocity with which the peak of each group travels, is

$$v - \lambda \frac{dv}{d\lambda}.$$

Doppler effect. Consider a source S emitting n waves per second, which travel outwards from S with constant velocity v cm. per sec., and succeed one another in space at intervals λ cm., where $v = n\lambda$. If S is at rest, an observer O also at rest records the reception of waves of frequency n (which we will call the true frequency) and wave-length λ (the true wave-length). How are these observations affected if there is relative motion of S and O, due to motion of S alone, of O alone, or of both together?

This is a general problem, important in sound and in optics, and in certain other examples as well. If the relative motion is in such a sense that S and O are approaching, then the observed wave-length is *less* than λ, and the observed frequency greater than n; if they are receding from one another, the observed wave-length is *greater* than λ, and the observed frequency less than n. For there to be any observable change, the relative velocity must be at least an appreciable fraction of v as a rule.

In terms of sound, this means that a relative velocity of approach *raises* a note above the true pitch, while a relative velocity of separation *lowers* it below the true pitch. In the case of light, if the source emits a *line* spectrum in which each line has a definite true wave-length, relative velocity of approach shifts each line towards the *violet* (short wave-length) end of the spectrum; while relative velocity of separation shifts each line towards the *red* (long wave-length) end. Nothing can be observed with a continuous spectrum, since a general shifting of all wave-lengths should still result in a continuous spectrum. These observations are known as the **Doppler effect, or Doppler-Fizeau principle.**

It should be noted that the velocity of sound in air at 0° C. is about 33,000 cm. per sec. (or 1100 ft. per sec., or 750 miles per hour), so that ordinary vehicles travel at speeds which are a measurable fraction of this, while modern aircraft may approach and even exceed it. The velocity of light, 3×10^{10} cm. per sec., is of a quite different order of magnitude, and nearly all the examples of velocities that can give a detectable wave-length shift are those of stars.

Also, the acoustic observer notices a difference in pitch, attributable to a *frequency ratio*, and really detects the ratio

observed frequency/true frequency.

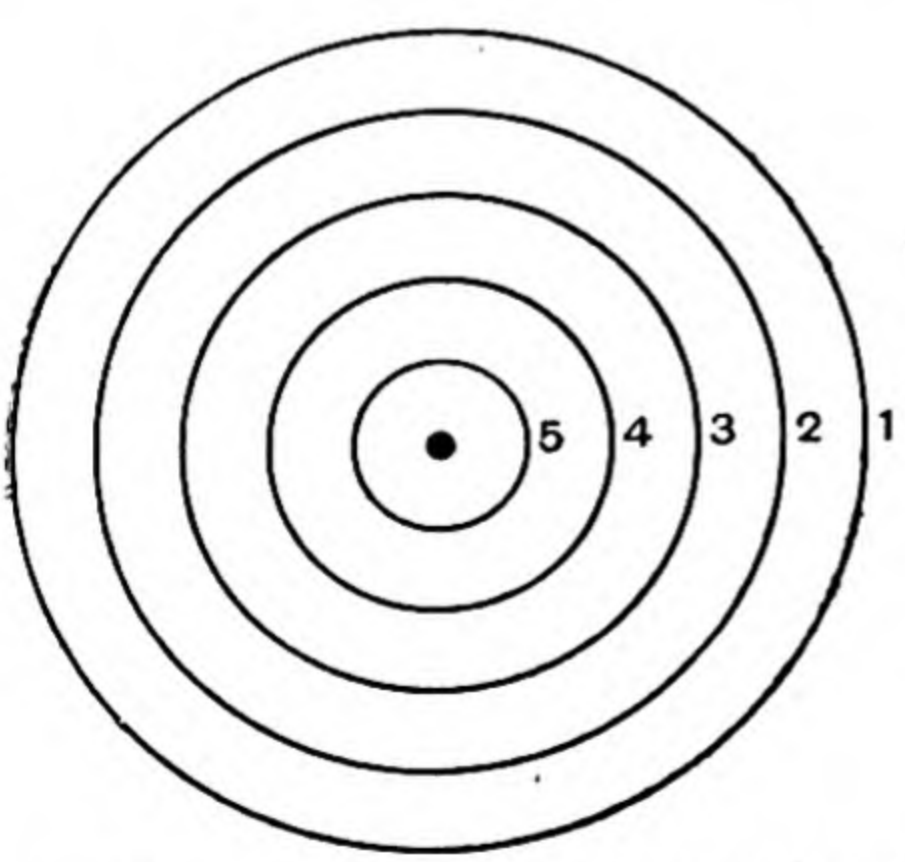
The optical observer notices a *difference in wave-length*, and detects directly the difference (observed wave-length – true wave-length).

Finally, the problem is not one of relative velocity alone. Motion of the source causes a result which is physically quite distinct from that of motion of the observer at the same rate.

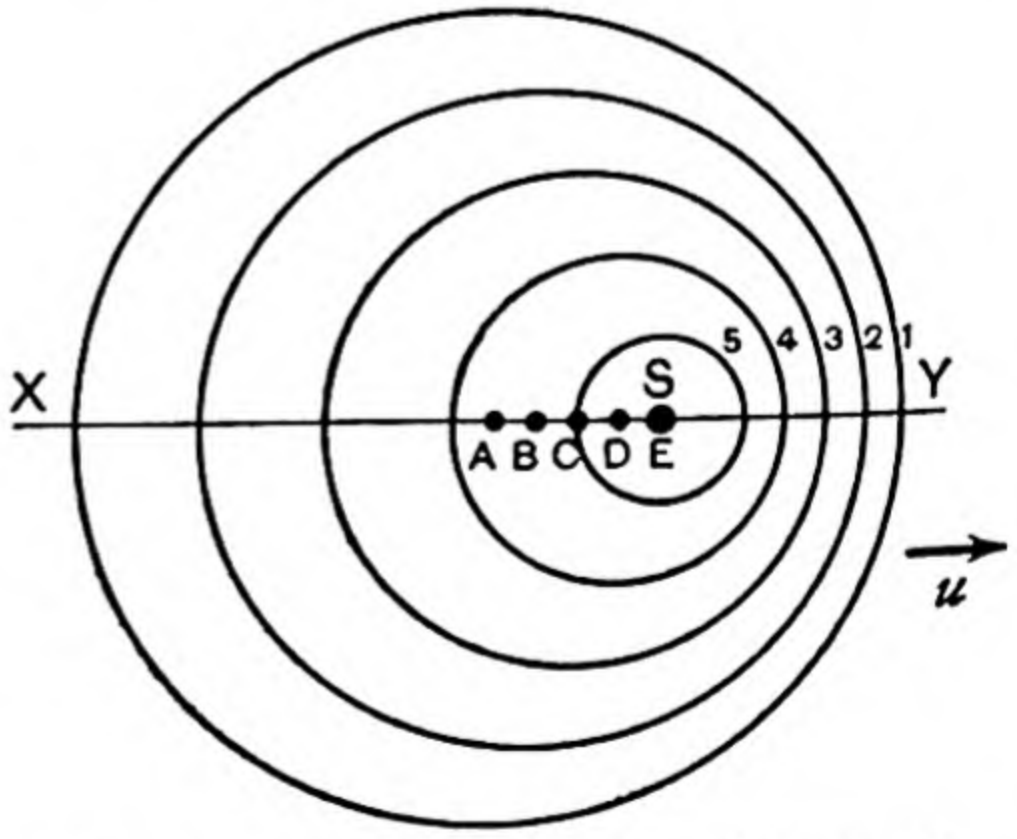
Case 1: *Fixed observer, moving source.* Suppose the observer O is at rest, and that the source S is in motion with velocity u along the line SO. Let n be the true frequency, v cm. per sec. the wave velocity. Then, whatever happens to S or to O, n is always constant and unalterable and v is fixed if the medium is at rest. But the wave-length, or the distance in space between successive waves, is actually different in different directions. It is important to realise that this is not just an hallucination of the observer, but must happen whether there is any observer to record it or not.

In diagram (a), the source S at rest is represented by the central dot, and the waves 1, 2, 3, 4, 5, emitted in that order, succeed one another in space at a distance v/n, which is the true wave-length; a stationary observer, wherever placed, finds the wave-length to be v/n.

In diagram (b), A, B, C, D, E, are the successive positions of S, which is moving along the line XY from X to Y with velocity u cm.



(a) Wave pattern from source at rest.



(b) Wave pattern from moving source.

per sec. Wave 1 has been emitted when S was at A, wave 2 from B, and so on. The distances AB, BC, etc., each covered at velocity u cm. per sec in $1/n$ sec., each measure u/n cm. If the observer O is at Y, the separation in space of the waves he receives is less than λ by the distance that S has moved towards him in between successsive emissions, and so the observed wave-length λ' is less than λ by an amount $\delta\lambda = \frac{u}{n}$. As $v = n\lambda$, $\frac{1}{n} = \frac{\lambda}{v}$, so $\delta\lambda = \frac{u}{v} . \lambda$;

or, rearranging this to show $\delta\lambda$ as a fraction of λ,

$$\frac{\delta\lambda}{\lambda} = \frac{u}{v}.$$

This is the formula applicable in the *optical case*.

As velocity = frequency × wave-length,

the apparent frequency n' corresponding to wave-length λ' is such that $v = n'\lambda'$, and as $v = n\lambda$, therefore $n'/n = \lambda/\lambda'$.

Now, $$\lambda' = \lambda - \delta\lambda = \lambda - \frac{u}{v}\lambda = \lambda\left(\frac{v-u}{v}\right),$$

so $$\frac{\lambda}{\lambda'} = \frac{v}{v-u},$$

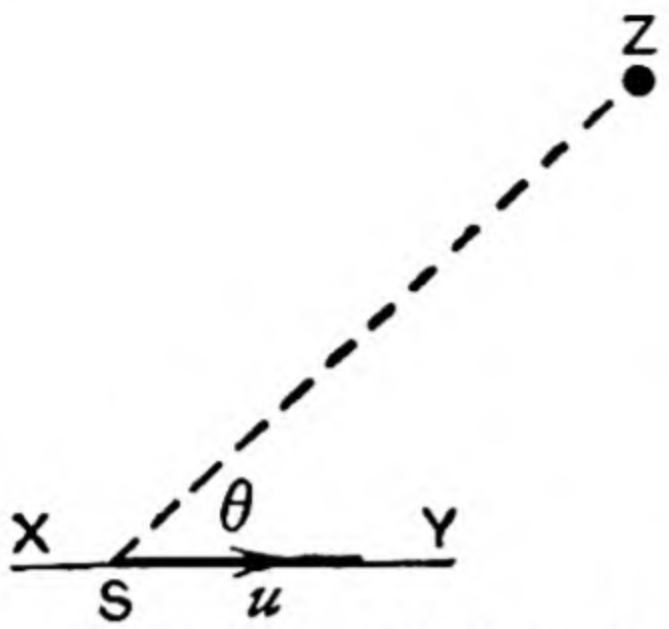
and $$\frac{n'}{n} = \frac{v}{v-u}.$$

This, giving the frequency ratio, is the formula applicable in the *acoustic case*.

It has been assumed above that S is approaching O with velocity u in the line of sight. From the figure, it is clear that if the observer were at X, so that S receded from O with the same velocity, each wave would be lengthened to the recipient by $\frac{u}{n}$ cm.; then the wave-length is apparently *increased* by $\delta\lambda = \frac{u}{v}\lambda$, and the observed wave-length, $\lambda' = \lambda + \delta\lambda = \lambda\left(\frac{v+u}{v}\right)$,

and $$\frac{n'}{n} = \frac{v}{v+u}.$$

More generally, to an observer at Z (see diagram), along a line making an acute angle with the direction XY, and so far away that



Observer not in line of motion of source.

θ does not vary appreciably during the observation, the displacement of S in the direction of Z between successive emissions is not u/n, but $u \cos \theta/n$. Reasoning as before, the proportional decrease in wave-length is

$$\frac{\delta\lambda}{\lambda} = \frac{u \cos \theta}{v},$$

whence

$$\lambda' = \lambda \left(\frac{v - u \cos \theta}{v}\right),$$

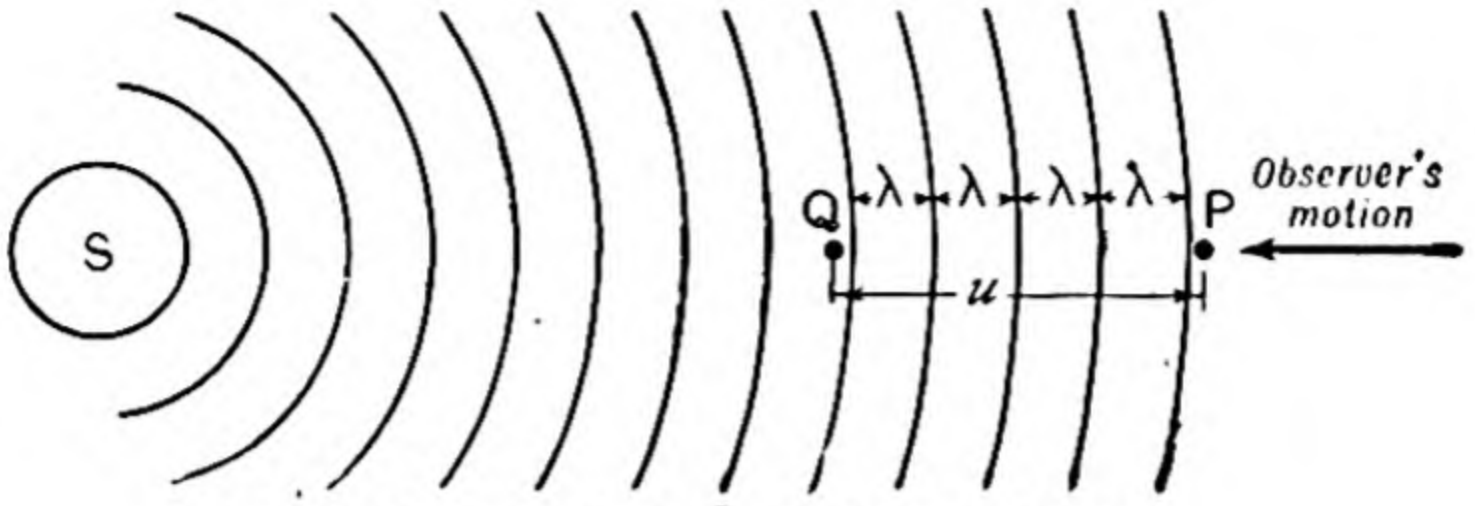
and

$$\frac{n'}{n} = \frac{v}{v - u \cos \theta}.$$

If $\theta = 0$, $\cos \theta = 1$, and we have the case already treated of the observer at X (p. 358); if $\theta = 180°$, $\cos \theta = -1$, and this is the case for the observer at Y. Or, we can say that the formulae first derived cover the general case, if we write $u \cos \theta$ instead of u. As the source moves along the line XY, θ gets steadily greater, and $u \cos \theta$ decreases; when $\theta = 90°$, $\cos \theta = 0$, and $n' = n$; when θ is greater than $90°$, $\cos \theta$ is negative and n' is less than n. It will be seen that there is a more or less gradual change from $n' = \frac{nv}{v-u}$ to $n' = \frac{nv}{v+u}$ as the source moves past Z from a very distant point beyond X to a very distant point beyond Y. But if Z lies in the line XY, the change is sudden and occurs as the source passes over Z.

The prime effect when the source moves is a change in wave-length.

Case 2: Fixed source, moving observer. Let S be fixed, emitting waves of frequency n and wave-length λ, which form a symmetrical pattern around it (see diagram). Let the observer, O, move towards S with velocity u cm. per sec., thus covering the distance from P to Q, where PQ is u cm., in 1 sec. If the observer, O, had remained at P throughout this second, he would have received only n waves; at the end of that second the space PQ would have contained a number u/λ of waves that had passed Q and not yet reached P. But as the observer, O, actually moves from P to Q, these additional u/λ waves are intercepted, so the frequency is increased by δn, where $\delta n = u/\lambda$; since $v = n\lambda$, $1/\lambda = n/v$,



Stationary source, moving observer.

$$\therefore\ \delta n = \frac{u}{v} n,$$

or,
$$\frac{\delta n}{n} = \frac{u}{v}.$$

The prime effect with a moving observer is a change in frequency.

The observed frequency $n' = n + \delta n,$

$$= n + \frac{u}{v} n, \qquad = n\left(\frac{v+u}{v}\right),$$

and
$$\frac{n'}{n} = \frac{v+u}{v}.$$

Since $n\lambda = n'\lambda' = v$, the apparent wave-length λ' for a velocity of approach u is given by

$$\lambda' = \lambda \frac{v}{v+u};$$

whence the expression for the wave-length change can be shown to be

$$\frac{\delta\lambda}{\lambda} = \frac{u}{v+u},$$

with the sign of u reversed if the observer is receding.

As before, if the velocity of O is u cm. per sec. at an angle θ to OS, the distance covered in 1 sec. in the direction of OS is $u \cos\theta$ cm.; whence, repeating the argument with $u \cos\theta$ substituted for u,

$$\frac{n'}{n} = \frac{v + u\cos\theta}{v}, \qquad \frac{\lambda'}{\lambda} = \frac{v}{v + u\cos\theta},$$

$$\frac{\delta\lambda}{\lambda} = \frac{u\cos\theta}{v + u\cos\theta},$$

all with u reckoned as a velocity of approach.

Distinction between the two cases. Whether there is any practical distinction between the effect of moving source and of moving observer for a given relative velocity depends on the magnitude of u. Thus, considering a whistle borne on a locomotive travelling at 100 ft. per sec. towards a fixed observer,

$$n'/n = v/(v-u) = 1100/(1100-100) = 11/10\ ;$$

while, if an observer travel on a similar vehicle at the same speed towards a stationary whistle, of the same pitch,

$$n'/n = (v+u)/v = (1100+100)/1100 = 12/11,$$

appreciably different from the other case.

The optical shift of spectral lines is caused by relative velocities of up to about 1/100 that of light, and so the distinction would be small. While astronomical evidence of another kind may perhaps indicate the contributions of the motions of source and observer to the relative velocity, it is always usual to treat stars as cases of "moving source, fixed observer."

General expressions. The foregoing treatment has worked out each particular case separately, but the formulae can be generalised using algebraic signs to denote direction. It is advisable *not* to rely on formulae; the only safe rule is to relate every problem to a diagram, and work from first principles.

So far, we have worked out a series of particular cases related to diagrams, taking u to be positive if source and observer are approaching one another.

For consistency, adopting the convention that the direction in which the sound travels to the observer is the positive direction, the observer's velocity in the argument of p. 361 should be reckoned negative; that is, we should write $u_0 = (-u)$. Then

$$\frac{n'}{n} = \frac{v+u}{v} \quad \text{becomes} \quad \frac{n'}{n} = \frac{v-u_0}{v}, \text{ and so on.}$$

Considering only direct approach or recession, if u_s be the velocity of the source and u_o that of the observer, both reckoned positive if in the the direction of propagation of the sound and negative if in the opposite direction, then the ratio of observed frequency n' to true

frequency n is $\frac{n'}{n} = \frac{v}{v - u_s}$ if the source alone moves, $\frac{n'}{n} = \frac{v - u_o}{v}$ if the observer alone moves, and $\frac{n'}{n} = \frac{v - u_o}{v - u_s}$ if both are in motion.

If the medium moves with velocity w along the line joining source and observer, the effective value of v is $v + w$, whence

$$\frac{n'}{n} = \frac{v + w - u_o}{v + w - u_s};$$

this expression is only relevant in the acoustic case.

EXAMPLES ON CHAPTER II

1. An explosive percussion signal on a rail is set off by a locomotive passing over it. A listener 1 km. away with one ear to the rail hears two reports. What is the time interval between them?

[Young's modulus for steel $= 2 \times 10^{12}$ dynes cm.$^{-2}$; density of steel $= 7{\cdot}8$ gm. cm.$^{-3}$; density of air $= {\cdot}0013$ gm. cm.$^{-3}$; ratio of specific heats of air $= 1{\cdot}4$; 1 atmosphere $= 10^6$ dynes cm.$^{-2}$.] (C. S.)

2. Two wires, A and B, made of the same metal, of equal lengths and with diameters in the ratio 7 : 4, when set into transverse vibration give notes whose frequencies are in the ratio 4 : 5. These notes are brought into unison when the tension in B is diminished by 1·5 kgm. wt. Find the tension in A and the original tension in B. (N.)

3. A rope 5 metres long has a total mass of 245 gm. It is stretched with a constant tension of 1 kgm. weight. If it is fixed at one end and shaken by hand at the other, what frequency of shaking will make it break up into three vibrating segments? (C. S.)

4. A stretched steel wire, 0·5 mm. in diameter, is supported at two points 30 cm. apart. What tension must be applied to make its lowest characteristic frequency of transverse vibration 1000 cycles per second? What approximations do you make?

[Density of steel $= 7{\cdot}8$ gm./c.c.] (C. S.)

5. A weight of 5 kgm. suspended from the lower end of a uniform wire of length 1 metre and diameter 0·36 mm. produces an extension of 2·5 mm. Find the frequency of the note emitted when the stretched wire is stroked by a resined cloth. The density of the material of the wire is 7·8 gm. per c.c. (N.)

6. Two strings of the same material and of the same cross-section are suspended on a sonometer. One is loaded with 12 kgm. and the other with 3 kgm. The first string is tuned to the first harmonic of the second string. If the second string is 100 cm. in length, what is the length of the first string? (C.S.)

7. Write down expressions for the velocity of propagation along a stretched wire of (*a*) transverse waves, (*b*) longitudinal waves.

Find the ratio of the fundamental frequencies of transverse and longitudinal vibrations for a steel wire, diameter 1 mm., mounted on a sonometer and stretched by a force of 10 kilograms weight.

[Young's modulus for steel $= 20 \times 10^{11}$ dynes per sq. cm.]

Describe and explain a method of finding the velocity of longitudinal waves in a long steel rod. (N.)

8. Distinguish between *free*, *forced*, and *resonant* vibrations, giving *one* example of each.

9. Describe in detail an experimental arrangement for producing stationary sound waves in air. Show how the displacement, particle velocity, pressure and density vary at a series of points $\lambda/8$ apart. (C. S.)

10. Distinguish between progressive and stationary waves. A sounding organ-pipe closed at one end is adjusted to resonance. Describe carefully the motion of the air in the pipe and the changes of pressure with time at various points along the pipe.

The second overtone of a pipe closed at one end has the same frequency as the third overtone of a pipe open at both ends. Neglecting end effects, compare the lengths of the pipes. (C. S.)

11. Describe an experiment to illustrate the formation of stationary waves (with more than one loop) on a string or wire. If a copper wire 0·193 mm. in diameter is used, what must be the tension to form loops 30 cm. long when the frequency of vibration of the wire is 64 per sec.? Give the result in gm. wt.

[The density of copper is 8·93 gm. per c.c.] (N.)

12. Distinguish between progressive and stationary waves.

Describe the motion of the air in a closed sounding organ pipe adjusted to resonance, and the changes of pressure with time at various points along the pipe. Describe some experimental evidence in support of your statements. (C. S.)

13. What are the resonance frequencies of the air in a narrow pipe one metre long open at both ends at 20° C. and 740 mm. of mercury?

[Ratio of specific heats of air $= 1{\cdot}4$
Density of air at N.T.P. $= 0{\cdot}0013$ gm./c.c.
Density of mercury $= 13{\cdot}6$ gm./c.c.
Value of g $= 981$ cm./sec.^{2}] (C. S.)

14. Explain the production of stationary waves. A train of sound waves is propagated along a wide pipe and is being reflected from an open end. If the amplitude of the waves is 0·002 cm., the frequency 1000, and the wave-length 33 cm., find the amplitude of the vibration at a point 20 cm. from the open end inside the pipe. (C. S.)

15. A brass wire and a steel wire, of the same length, diameter, and tension, give 5 beats per second when mounted on a sonometer and made

to give their fundamental tones simultaneously. Calculate the frequencies of vibration of the two tones, given that the densities of brass and steel are 8·4 and 7·8 gm. per c.c., respectively. (O. & C.)

16. Two wires vibrate transversely in unison. The tension in one wire is increased by 1 per cent. and now, when they vibrate simultaneously, three beats are heard in 2 sec. What was the original frequency of vibration of the two wires? (N.)

17. A vibrating tuning-fork is moving steadily with a velocity of 150 cm./sec. normally towards a wall from which the sound waves are reflected. If the frequency of the fork is 512 sec.$^{-1}$, what will be the frequency of the beats heard by a stationary observer who has just been passed by the fork?

[Velocity of sound = 330 m./sec.] (C. S.)

18. Give an account of the formation of beats and deduce an expression for the frequency of the beats in terms of the frequencies of the sources.

Two trains approach one another along the same line each moving with a velocity of 10 ft. per sec. A whistle of frequency 250 per sec. is sounded on each train. Find the frequency of the beats between the two sources as heard by an observer on either train. (C. S.)

19. Show that when two notes of frequencies f_1, f_2 are sounded simultaneously " beats " are produced of frequency $f_1 \sim f_2$.

An air-raid siren, situated 550 ft. from the vertical face of a sharply-rising cliff, emits a note which rises in frequency uniformly from 0 to 250 c./sec. in 5 sec. and then drops uniformly through the same range in the same time. Observers notice a beating effect which they attribute to reflection of the sound from the face of the cliff. Discuss the phenomenon, and show how you would expect the beats to vary in clearness and in frequency for observers in different situations.

[Velocity of sound = 1100 ft./sec.] (C. S.)

20. Explain why the motion of a source of sound affects its pitch as heard by a stationary observer.

What is the velocity of the source along the line joining the source to the observer if, as a result of the motion, the frequency of the note heard is (*a*) increased in the ratio 16 : 15, (*b*) decreased in the ratio 15 : 16? Assume that the velocity of sound is 1120 ft. per sec. and give the results in feet per sec. Derive any formula employed. (N.)

21. Explain what is meant by the Doppler effect and obtain an accurate expression for it when an observer is moving towards or away from a fixed source.

An object is dropped from an aeroplane travelling horizontally in a straight line with a velocity of 250 kilometres per hour at a height of 5 kilometres. Immediately the object strikes the ground a hooter near the place where it strikes is sounded and emits a note of frequency 300 vibrations per sec. Neglecting effects due to air resistance, calculate

the time that elapses between the moment the object leaves the aeroplane and that at which the pilot first hears the hooter and also the initial frequency of the note heard.

[$g = 980$ cm. sec.$^{-2}$. Velocity of sound in air $= 330$ m./sec.] (C. S.)

22. Give a general account of the Doppler effect, with examples in sound. Consider the bearing which the velocity of the wind may have on the effect.

A plane is travelling horizontally in a straight line in still air with a velocity of 400 ft. per sec. Find the percentage difference between the frequency of the sound of the plane heard by an observer on the ground at the instant when he sees the plane vertically overhead and the " true " frequency. Assume the velocity of sound in air to be 1120 ft. per sec. (N.)

23. How is it possible to learn something about the motion of the stars from their spectra? As an illustration consider the case of a star receding from the earth with a velocity of 300 miles per sec. Assume the velocity of light to be 186,000 miles per sec., and consider a spectral line of wave-length 3968×10^{-8} cm. (N.)

24. A source emitting sound of natural frequency n moves in a straight line l with constant velocity v. A stationary observer is at a point not on the line. Prove that, to him, the apparent frequency at any instant is $nc/(c - v\cos\theta)$, where c is the velocity of sound and θ the angle between l and the line joining observer and source.

Calculate the rate of change of frequency at this instant and show that if the source passes *close* to the observer he will notice a *sudden* drop of pitch. (O. S.)

25. When spectra of small areas of the sun's surface at opposite ends of the solar equator are compared, corresponding lines in the region $\lambda = 4500$A. show a relative shift of 0·058A. Deduce the solar rotation period, assuming the earth to be in the sun's equatorial plane.

[Diameter of sun $= 1{\cdot}39 \times 10^6$ km.
Velocity of light $= 3 \times 10^{10}$ cm./sec.
1A. $= 10^{-8}$ cm.] (C. S.)

26. A man is standing beside a railway line observing the whistle of a passing train. The whistle, which has a natural frequency of 1000 cycles/sec., suffers an apparent change of frequency of 100 cycles/sec. as it passes the observer. What is the speed of the train?

[Velocity of sound $= 1100$ ft./sec.] (C. S.)

27. A source of sound gives out waves of frequency n. Discuss the effect of motion (a) of the source, (b) of the observer, (c) of both the source and observer, on the frequency of the sound as heard by the observer. The motion in each case may be assumed to be along the line joining the source and observer.

A train is moving with uniform velocity v on a straight track between two bridges A and B over the track, the motion being towards A. An observer on the train hears the echo of the train's whistle reflected from

each of the bridges. If the velocity of sound is V, find the ratio of the wave-lengths of the waves reflected from A and B and the ratio of the frequencies of the echoes heard by the observer. (C. S.)

28. Give an account of the Doppler effect in sound and light.

A man directs a beam of sound of frequency 12,000 cycles/sec. at an approaching motor car. He listens to the beats between the reflected sound and that emitted. What is the frequency of these beats if the car is travelling at 30 m.p.h. (44 ft./sec.)? (C. S.)

29. Explain the effect of the motion of the source and of the observer on the apparent pitch of a note.

An express train blowing its whistle travels through a station, and an observer standing on the platform notices that as the engine passes him the drop in pitch of the whistle corresponds to an interval of 6/5. What is the speed of the train? (Speed of sound in air = 1100 ft. per sec.) (O. & C.)

30. Give an account of the Doppler effect in the propagation of sound and light. If 5×10^{-9} cm. is the smallest difference of wave-length which can be detected by a spectrometer in the neighbourhood of the wave-length 6×10^{-5} cm., what is the smallest velocity of a star relative to the earth which can be measured?

(The velocity of light is 3×10^{10} cm. per sec.) (O. & C.)

CHAPTER 12

GRAVITATION

The solar system. The earth is one of a company of nine planets which travel in orbits round the sun. The others are the planets Mercury, Venus, Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto. Numerous comets following very flattened elliptical orbits, and huge numbers of small planetary bodies called asteroids, complete the family of solar satellites. The planets are distinguishable from the " fixed stars ", some by their conspicuous brilliance and all by the fact that they describe paths of their own in the heavens. Five were known to the ancients, and are (with the possible exception of Mercury) easily located by anyone.

Three are of more recent discovery: Uranus was found by Herschel early in the nineteenth century, Neptune by Leverrier and Adams in 1846, and Pluto in 1931. Each of the later observations, made by telescope, was preceded by calculation which showed that such a body was to be expected in about the observed position.

Kepler's Laws. Kepler (1609), from the observations of Tycho Brahe, a Danish astronomer, who was the first to attempt reasonably precise records of the positions of the planets, stated three laws concerning the motions of the planets round the sun.

(1) Each planet describes an ellipse, with the sun in one of its foci.

(2) For a given planet, the radius joining the planet to the sun sweeps out equal areas of the orbit in equal times.

(3) The squares of the periodic times of the various planets are proportional to the cubes of the major axes of their orbits.

In the table on p. 370, the ratio of the major axis to the minor axis of the ellipse described by each planet is shown, and the figures are all close enough to unity for the purposes of elementary physics. The radius of the orbit of each is given in millions of miles, and the periodic time in years. The ratio r^3/T^2 has been worked out for each, using four-figure tables. The agreement is, within the limits of this arithmetic, very close indeed.

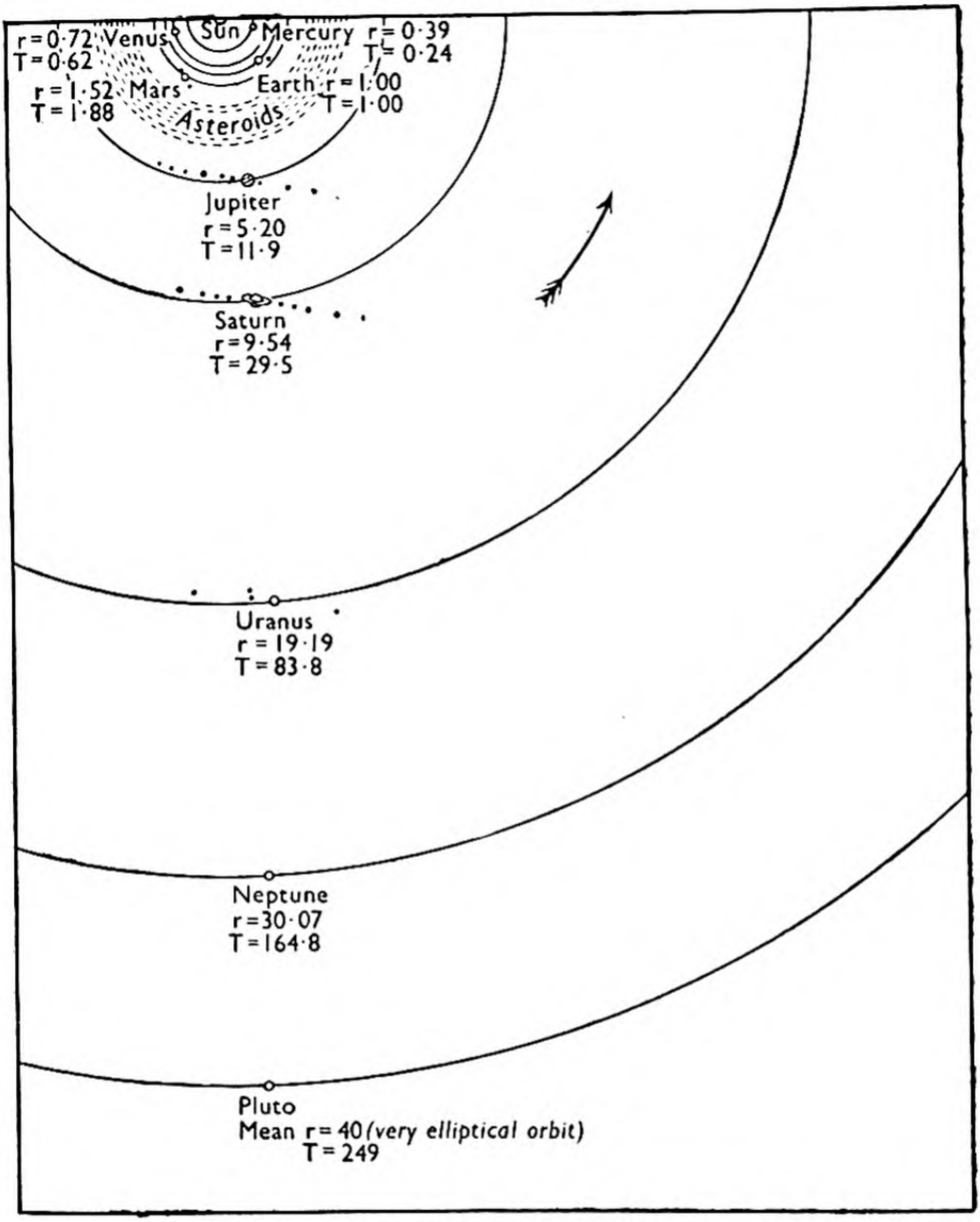


Diagram representing the solar system (not to scale) ;
r and T for the earth's orbit taken as unity.

Pluto has been excluded from this table. Its period is about 249 years and the " mean radius " of its very elliptical orbit about 3900 million miles. Present data indicate that its course will carry it within the orbit of Neptune in the near future.

Planet	Ratio of major to minor axis	Radius of orbit in millions of miles (r)	Periodic time in years (T)	$\frac{r^3}{T^2}$
Mercury	1·02	36	0·24	$8{\cdot}10 \times 10^5$
Venus	nearly 1	67·4	0·62	$7{\cdot}97 \times 10^5$
Earth	1·001	93	1·00	$8{\cdot}04 \times 10^5$
Mars	1·005	142	1·88	$8{\cdot}08 \times 10^5$
Jupiter	1·004	485	11·86	$8{\cdot}10 \times 10^5$
Saturn	1·002	885	29·46	$7{\cdot}98 \times 10^5$
Uranus	1·001	1780	83·75	$8{\cdot}06 \times 10^5$
Neptune	nearly 1	2800	164·8	$8{\cdot}10 \times 10^5$

Except in the case of Mercury, and possibly Mars, the elliptical orbits of the planets differ very little from circles—at least they are near enough to circles for our present discussion. A circle is a particular case of the ellipse, in which the major and minor axes are equal and the two foci coincide at the centre. We can thus restate Kepler's Laws in the following words for our purposes :

(1) Each planet describes a circle, with the sun at the centre.

(2) Each planet describes its circular path at uniform speed.

(3) The squares of the periodic times of the various planets are proportional to the cubes of the radii of their orbits ; that is, if r stand for the radius, T for the period, and ω ($=2\pi/T$) for the angular velocity of the line joining a planet to the sun,

$$\frac{r^3}{T^2}, \quad \text{or} \quad r^3\omega^2,$$

has the same value for each planet.

We know (p. 79) that a body moving with uniform speed in a circle experiences an acceleration $r\omega^2$, or v^2/r, towards the centre of the circle, and that this acceleration is provided by an external force $mr\omega^2$, (or mv^2/r), absolute units towards the centre of the circle. Two questions now arise : How is the force which maintains the planets in their circular paths provided, and how does it happen that r^3/T^2, and consequently $r^3\omega^2$, is the same for all the planets?

Newton showed that the explanation of the origin of the pull towards the centre is **gravitation** : that is, the planets are attracted

towards the sun in just the same manner as a body at the earth's surface is attracted towards the earth ; and that Kepler's laws followed as a consequence of the way in which the attraction between two bodies varies with their distance apart. Gravitation is a universal effect, operating according to the same law between every two particles in the universe.

Before stating the law of Universal Gravitation formally, it will be instructive to follow a very simplified version of Newton's exposition of the generality of the law. We must suppose that he believed in such a law for a very long time before he was able to establish it ; but it was no windfall from the tree of knowledge, and was reached only after great and mighty labours. He even explored in great detail an alternative view of planetary motion, that of Descartes, who supposed they were carried round in vortices in a fluid medium. Newton showed that this would be incompatible with Kepler's laws ; his work on fluid motion (p. 242) was directed to that end.

Universal Gravitation. All bodies at the same place fall freely to the earth with the same uniform acceleration. That is, the weight of a body, or the earth's gravitational pull on a body at its surface, is proportional to the mass of that body. The equal and opposite reaction to this, the pull of the body on the earth, should presumably follow the same rule, and be proportional to the mass of the attracted body—the earth. Thus, if there is a gravitational pull between any two bodies this should be proportional to the mass m_1 of the first and also to the mass m_2 of the second ; that is, proportional to the product, $m_1 . m_2$, of their masses.

Now, suppose that the force between two bodies varies inversely as the nth power of the distance between them, so that the force between two bodies of masses m_1 and m_2 at distance r apart is

$$F = k\frac{m_1 m_2}{r^n},$$

where k is a constant of proportionality, and F is in absolute units.

Applying this to the case of a planet describing a circular orbit of radius r round the sun, write m, the mass of the planet, for m_1, and S, the mass of the sun, for m_2. The force on the planet is then

$$k\frac{mS}{r^n} \quad \text{absolute units.}$$

This force maintains the planet in its circular path ; so it must be equal to $mr\omega^2$, where ω is the angular velocity of the radius joining the planet to the sun.

Hence $$mr\omega^2 = kmS/r^n, \quad \text{or} \quad r^{n+1}\omega^2 = kS.$$

This fits in with Kepler's third law if (i) $n = 2$—or the law of force is the law of inverse squares ; and (ii) k is not just a constant of proportionality for the equations for one planet, but is a *universal* constant—the same for all of them.

Thus it is concluded that the gravitational force between the sun and a planet is proportional to the product of their masses, and is inversely proportional to the square of the distance between them ; and that the constant of proportionality is the same for each planet.

It has now to be shown that the earth's gravitational attraction is of the same kind, and follows the same law. If this is so, a body projected horizontally from the top of a high enough mountain with a great enough velocity would describe a circular orbit as a satellite of the earth, like the moon, only nearer ; and if the motion of the moon itself fits a calculation of this kind, the point is proved.

Assume that the earth itself is at rest (so that the centre of the moon's orbit is at the centre of the earth and not at the common centre of gravity), and suppose the sun's attraction on both bodies is negligible ; both these simplifications affect the numerical result very little. Also assume that the earth's gravitation at points beyond it is the same as if the whole mass of the earth were concentrated at its centre ; this is a very important assumption, and in it lies the root of the whole matter.

Let R be the radius of the earth ; then the radius of the moon's orbit is known to be nearly $60R$. The moon's angular velocity, ω, is 2π radians per month, and the acceleration towards the centre of the path is $60R\omega^2$, which, when the values of R and ω are substituted in the relevant units, gives 0·0089 ft. sec.$^{-2}$

The acceleration, g, of a freely-falling body at the earth's surface is due to the earth's gravitational attraction at distance R from the centre ; assuming the inverse square law, the acceleration of the moon, at distance $60R$, towards the centre of the earth should be $g/60^2$, which works out to 0·0089 ft. sec.$^{-2}$. The agreement is convincing.

Newton first approached this problem in 1665 or 1666, using an accepted value of R which was about 15 per cent. too low, which made $60R\omega^2$ about 0·0073 ft. sec.$^{-2}$, a discrepancy too great to claim as rough agreement. It is often said that this result led him to abandon the matter for some twenty years, as the reconciled calculations appeared in the first edition (1687) of the *Principia*. The more accurate value of R there employed was that of Picard (1671), and it is also said that on hearing of this value Newton revised the arithmetic with great excitement. It is curious that Newton was not discouraged by a discrepancy of nearly the same order from publishing his calculation of the velocity of sound in air, and it is likely that he knew of determinations, in agreement with Picard's, which had been made before 1665. It seems more probable that he was engaged in wrestling with the real difficulty—that of proving that universal gravitation between all particles would require that the earth should attract external bodies just as if the whole of its mass were concentrated at the centre. For if this is established, something much more general than planetary or terrestrial attraction follows; *universal* gravitation is itself demonstrated by the earth and the planets.

In Prop. LXXIV, Cor. II, Newton states:

> " If a corpuscle placed without a homogenous sphere is attracted by a force inversely proportional to the square of its distance from the centre, and the sphere consists of attractive particles, the force of every particle will decrease as the square of the distance from each particle."

That is, universal gravitation between all particles can be deduced from the fact that a sphere such as the earth behaves in this way.

The words " corpuscle ", " homogeneous sphere ", and " particle " should be noted; nothing is said about nondescript " bodies ".

The simple formula for gravitational attraction applies to particles, which are masses of very small dimensions; to celestial bodies, since the distances involved are so great compared with their dimensions that they are treated as particles; and to homogeneous spheres (see p. 378), treated as if they were massive particles located at their centres. But the application of the formula to attractions between " bodies " in general is mathematically rather difficult.

The enunciation of the Law of Universal Gravitation given in Thomson and Tait's *Natural Philosophy* is :

Every particle of matter in the universe attracts every other particle with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses, and inversely as the square of their distance from each other.

These authorities call the law a " property of matter " ; Newton did not subscribe to the view that gravitation is an innate property of matter, and would frame no hypothesis as to its nature.

In symbols, Newton's Law of Universal Gravitation is written

$$F = \frac{Gm_1m_2}{r^2} \quad \text{absolute units,}$$

where F is the force of attraction between the two particles, m_1 and m_2 are their masses, r is the distance between them, and G is the Universal Gravitation Constant.

Analogous laws of force are found in electrostatics and in magnetism, though in these cases the medium through which the attraction acts plays an important part. There is no experimental evidence for the existence of any gravitational property corresponding to magnetic permeability or dielectric permittivity.

Value of G. If m_1 and m_2 are in grams, r in centimetres, and F in dynes, the value of G is nearly $6{\cdot}6 \times 10^{-8}$ C.G.S. units.

The dimensions of G are those of

$$\frac{(\text{force}) \times (\text{distance})^2}{(\text{mass})^2},$$

or $$MLT^{-2} \times L^2 \times M^{-2}, \quad \text{or} \quad M^{-1}L^3T^{-2}.$$

The very small numerical value of G makes its determination a matter of extreme delicacy, for the force between two " particles " each of mass 1 gm. separated by 1 cm. is of the order of 10^{-10} gm. wt.

This difficulty has been faced in several ways. The attraction due to a very large mass, such as a mountain or a deep layer of the earth's crust, has been measured. Very sensitive beam balances and torsion balances have been designed to measure the attraction between masses of reasonable size ; and there are other valuable methods. But it is only proposed to deal in detail with one method

in this chapter, and the reader is referred to Poynting and Thomson's *Properties of Matter* for a full historical survey of the experimental work.

The value of G is a constant of extreme importance. Once this is known we can, with the help of other available information, find the mass of the sun, the mass of the earth, and the mass of any planet which possesses a satellite (as do all except Mercury, Venus, and, so far as is known, Pluto).

Mass of the sun. From the previous argument, it is seen that for the orbit of any planet $r^3\omega^2 = GS$, where G is now written instead of k.

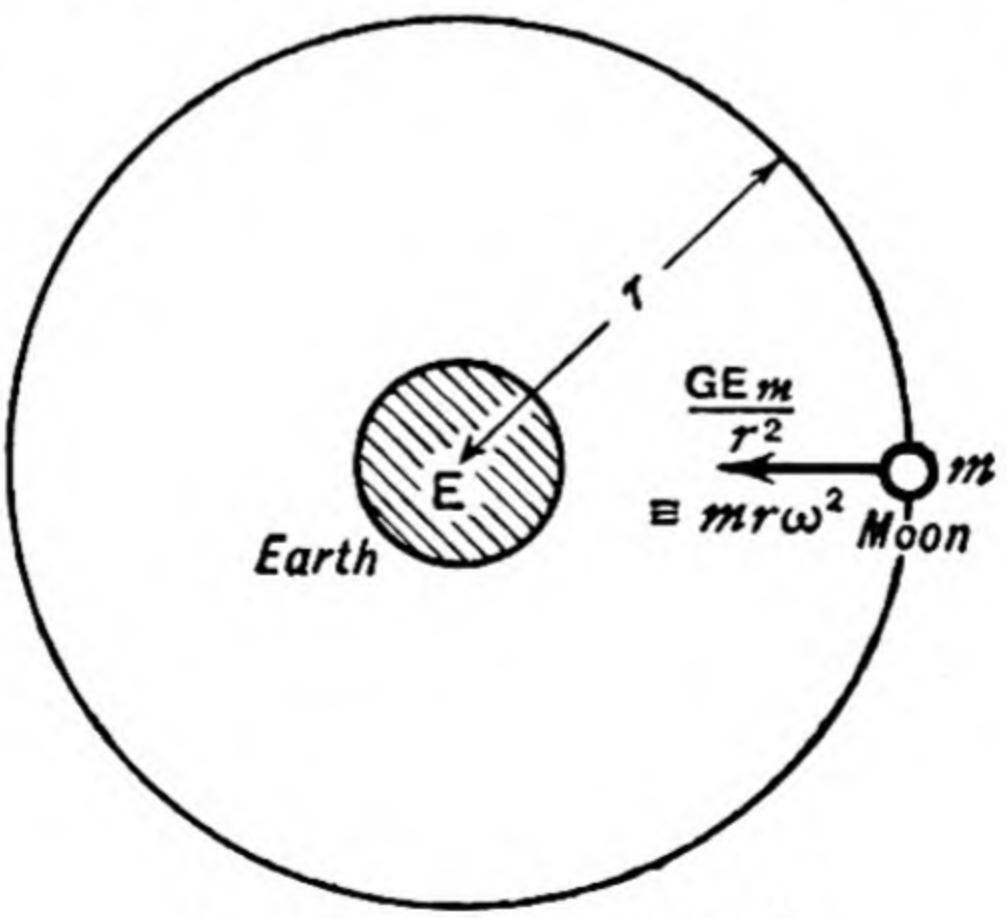
Thus $$S = r^3\omega^2/G.$$

The table of p. 370 gives a mean value of about 8×10^5 for $r^3\omega^2$, when r is in millions of miles and T in years. Bringing r to centimetres and T to seconds, and remembering that $\omega = 2\pi/T$, we get for $r^3\omega^2$ in C.G.S. units

$$\frac{8 \times 10^5 \times 10^{18} \times (30{\cdot}4 \times 5280)^3 \times 4\pi^2}{(3600 \times 24 \times 365{\cdot}3)^2} = 1{\cdot}311 \times 10^{26}.$$

So, $$S = 1{\cdot}311 \times 10^{26}/6{\cdot}6 \times 10^{-8} = 1{\cdot}99 \times 10^{33} \text{ gm.}$$

Mass of the earth. While it is probably not the most convenient way of marshalling our information to this end, the mass of the earth can be found similarly by considering the motion of the *moon* round the *earth*. If m be the mass of the moon, E the mass of the earth, r the radius of the moon's orbit, and ω the angular velocity of



Moon gravitating to earth.

the line joining moon and earth (2π radians per month), then, as before (see diagram),

$$mr\omega^2 = GmE/r^2,$$

whence $$r^3\omega^2 = GE, \quad \text{or} \quad E = r^3\omega^2/G.$$

In round figures,

$$r = 240{,}000 \text{ miles} = 240{,}000 \times 30{\cdot}4 \times 5280 \text{ cm.},$$

$$\omega = 2\pi/(27{\cdot}3 \times 24 \times 3600) \text{ radians per second}$$

and
$$E = \frac{(240{,}000 \times 30{\cdot}4 \times 5280)^3 \times 4\pi^2}{(27{\cdot}3 \times 24 \times 3600)^2 \times 6{\cdot}6 \times 10^{-8}}$$

$$= 6{\cdot}1 \times 10^{27} \text{ gm.},$$

which is about 3 per cent. greater than the accepted value.

Masses of the planets. Each of the outer planets from Mars to Neptune is accompanied by satellites of its own. Twelve satellites of Jupiter and ten of Saturn have been discovered, but for both these planets only four are given in the table opposite, with their names and the numbers by which they are usually known. The mean radius of the orbit of each satellite is given in centimetres, and the periodic time in seconds. It is seen that, for any one planet, the value of r^3/T^2 is constant for all its satellites, showing that Kepler's third law holds, and that the satellites are held in their orbits by an inverse-square law gravitational attraction. As ω is $2\pi/T$, the mass of each planet can be found by taking $4\pi^2r^3/T^2$ for any of its satellites, and dividing by G. The figures here given have been rounded off, and the resulting masses expressed in terms of E differ by a few per cent. from those obtained by more precise arithmetic.

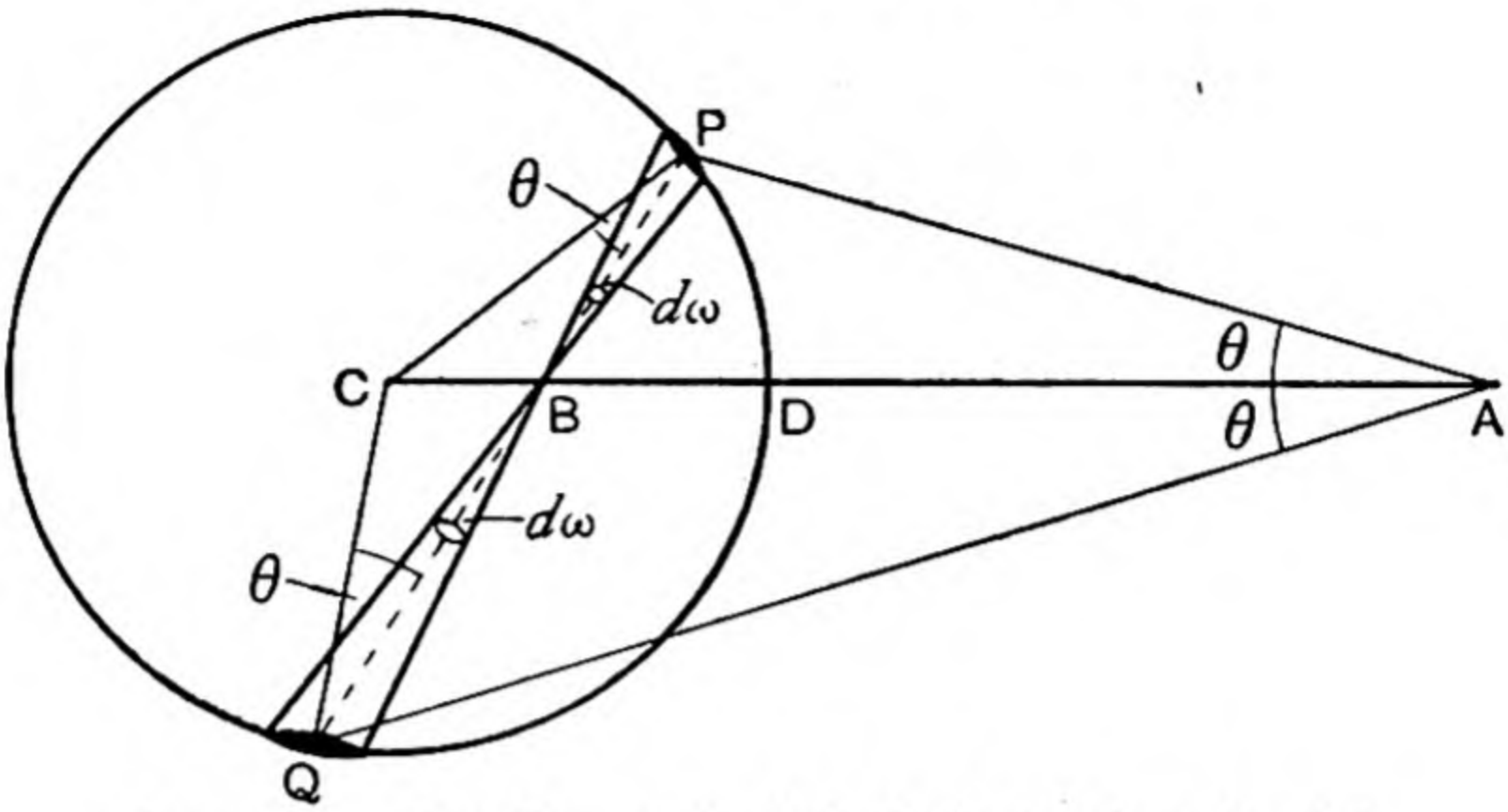
Two important attraction theorems. It is not, in general, true that for purposes of gravitational attraction a body can be supposed to act as if a particle of the same mass were placed at the centre of mass of the body. But it can be proved that :

(1) For a uniform solid sphere, or for a uniform hollow spherical shell, the attraction at points outside the sphere or shell is that which would result if a particle of the same mass were at the centre of the sphere or shell.

(2) For a uniform hollow spherical shell the attraction at points within the shell is zero.

Planet	Name of satellite	Mean Radius r of orbit in cm.	Periodic time, T sec.	$\frac{r^3}{T^2}$	Mass in grams, calculated from $\frac{4\pi^2}{G}\cdot\frac{r^3}{T^2}$; and in terms of earth's mass E
Mars	Phobos	$9{\cdot}7 \times 10^8$	$2{\cdot}75 \times 10^4$	$1{\cdot}19 \times 10^{18}$	$7{\cdot}0 \times 10^{26}$ gm.
(2 satellites)	Deimos	$2{\cdot}4 \times 10^9$	$1{\cdot}09 \times 10^5$	$1{\cdot}19 \times 10^{18}$	$=0{\cdot}12\ E$
Jupiter	Io (I)	$4{\cdot}19 \times 10^{10}$	$1{\cdot}51 \times 10^5$	$3{\cdot}22 \times 10^{21}$	
	Europa (II)	$6{\cdot}67 \times 10^{10}$	$3{\cdot}06 \times 10^5$	$3{\cdot}16 \times 10^{21}$	$1{\cdot}9 \times 10^{30}$ gm.
(12 satellites)	Ganymede (III)	$1{\cdot}06 \times 10^{11}$	$6{\cdot}16 \times 10^5$	$3{\cdot}17 \times 10^{21}$	$=320\ E$
	Callisto (IV)	$1{\cdot}87 \times 10^{11}$	$1{\cdot}38 \times 10^6$	$3{\cdot}39 \times 10^{21}$	
Saturn	Mimas (I)	$1{\cdot}87 \times 10^{10}$	$7{\cdot}9 \times 10^4$	$1{\cdot}04 \times 10^{21}$	
	Enceladus (II)	$2{\cdot}39 \times 10^{10}$	$1{\cdot}15 \times 10^5$	$1{\cdot}04 \times 10^{21}$	$5{\cdot}9 \times 10^{29}$ gm.
(10 satellites)	Iapetus (VII)	$3{\cdot}59 \times 10^{11}$	$6{\cdot}8 \times 10^6$	$0{\cdot}99 \times 10^{21}$	$=98\ E$
	Phoebe (VIII)	$1{\cdot}30 \times 10^{12}$	$4{\cdot}76 \times 10^7$	$0{\cdot}98 \times 10^{21}$	
Uranus	Ariel	$1{\cdot}90 \times 10^{10}$	$2{\cdot}16 \times 10^5$	$1{\cdot}48 \times 10^{20}$	
	Umbriel	$2{\cdot}68 \times 10^{10}$	$3{\cdot}56 \times 10^5$	$1{\cdot}52 \times 10^{20}$	$8{\cdot}9 \times 10^{28}$ gm.
(4 satellites)	Titania	$4{\cdot}36 \times 10^{10}$	$7{\cdot}49 \times 10^5$	$1{\cdot}49 \times 10^{20}$	$=14\ E$
	Oberon	$5{\cdot}82 \times 10^{10}$	$1{\cdot}12 \times 10^6$	$1{\cdot}56 \times 10^{20}$	
Neptune (1 satellite)		$3{\cdot}57 \times 10^{10}$	$5{\cdot}06 \times 10^5$	$1{\cdot}77 \times 10^{20}$	$1{\cdot}05 \times 10^{29}$ gm. $=17{\cdot}5\ E$

Consider a uniform spherical shell of mass ρ per unit area. Let A be any external point and let AC cut the surface of the shell in D (see diagram, p. 378). The usual approach in this kind of problem is to divide the surface up into elements by means of cones ; here it is important to select the right point to serve as vertex for these cones, and this point is B, the point in AC such that $\frac{BC}{CD}=\frac{CD}{CA}$. Let PQ be a double cone of small solid angle $d\omega$, cutting out small elements of area at P and Q, and having its vertex at B. It follows that the



Attraction of uniform spherical shell at external point.

triangles BCP, APC, and ABQ are similar, and that the angles CPB, CQB, PAD, and QAD, all marked θ in the diagram, are equal.

The areas of the surface elements at P and Q are :

$$BP^2 \,.\, d\omega \,.\, \sec\theta \quad \text{and} \quad BQ^2 \,.\, d\omega \,.\, \sec\theta\,;$$

their masses are

$$\rho \,.\, BP^2 \,.\, d\omega \,.\, \sec\theta \quad \text{and} \quad \rho \,.\, BQ^2 \,.\, d\omega \,.\, \sec\theta\,;$$

and their attractions on a particle of unit mass placed at A are :

$$\frac{G \,.\, \rho \,.\, BP^2 \,.\, d\omega \,.\, \sec\theta}{AP^2} \quad \text{along } AP,$$

and

$$\frac{G \,.\, \rho \,.\, BQ^2 \,.\, d\omega \,.\, \sec\theta}{AQ^2} \quad \text{along } AQ.$$

But as the triangles BCP, APC, and ABQ are similar,

$$\frac{BP}{AP} = \frac{BQ}{AQ} = \frac{CP}{AC},$$

so that the attractions due to both elements are equal, each being

$$G \,.\, \rho \,.\, \frac{CP^2}{AC^2} \,.\, d\omega \,.\, \sec\theta.$$

Their resultant is

$$2G \,.\, \rho \,.\, \frac{CP^2}{AC^2} \,.\, d\omega \,.\, \sec\theta \,.\, \cos\theta,$$

or

$$2G \,.\, \rho \,.\, \frac{CP^2}{AC^2} \,.\, d\omega, \quad \text{along } AC.$$

The sum of all the attractions for all such pairs of elements is

$$2G\rho \cdot \frac{CP^2}{AC^2} \sum_0^{2\pi} d\omega = G \cdot \frac{4\pi\rho CP^2}{AC^2}.$$

But $4\pi\, CP^2\rho$ is the mass M of the shell, so the attraction on unit mass at A is

$$\frac{G \cdot M}{AC^2},$$

just as if the whole mass M had been concentrated at C.

The same result must be true of a uniform solid sphere, which is a series of such concentric shells, and also for a body such as the earth, which can be regarded as a series of uniform concentric shells for each of which the value of ρ depends on the radius.

For a point A within a uniform spherical shell (see diagram), take A as the vertex of a double cone. Using the same notation as before, the attraction of the element at P on unit mass at A is

$$\frac{G \cdot \rho \cdot AP^2 \cdot d\omega \cdot \sec\theta}{AP^2} \quad \text{along } AP;$$

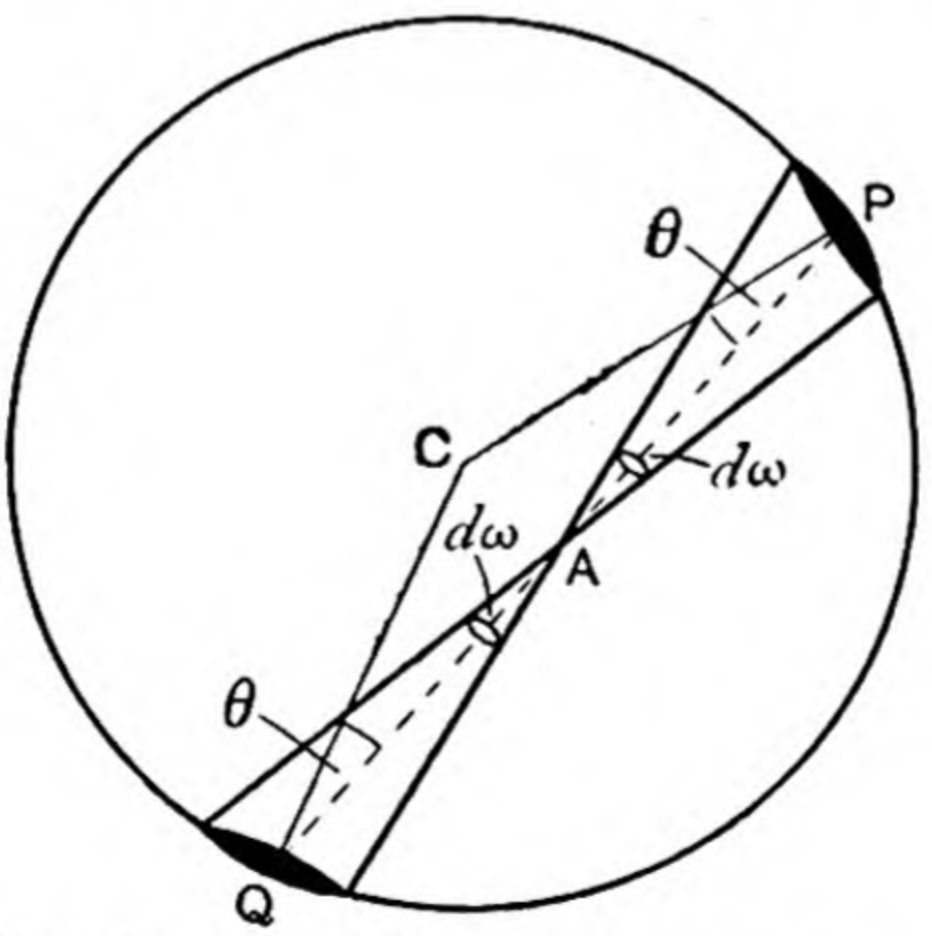
and that of the element at Q is

$$\frac{G \cdot \rho \cdot AQ^2 \cdot d\omega \cdot \sec\theta}{AQ^2} \quad \text{along } AQ.$$

These are each equal to

$$G \cdot \rho \cdot d\omega \cdot \sec\theta,$$

and are oppositely directed, so their resultant is *zero*.

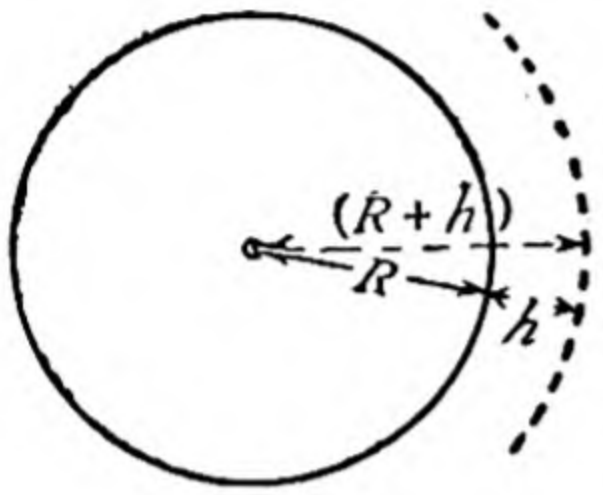


Attraction of uniform spherical shell at internal point.

The same is true for every other such cone. Hence the attraction of the shell at all points inside it is zero. This applies to any shell, however thick ; and also to the case when ρ varies with the radius in a thick shell.

Variation of g with altitude above sea-level. Let R be the radius of the earth, E its mass, and g the value of the acceleration at sea-level. Then

$$g = GE/R^2.$$



Supposing the earth is a uniform sphere, the acceleration g' at height h above sea-level is, since E acts from the centre of the earth,

$$g' = \frac{GE}{(R+h)^2} = \frac{GE}{R^2}\left(1+\frac{h}{R}\right)^{-2}$$

$$= g\left(1+\frac{h}{R}\right)^{-2}.$$

Since h must always be small compared with R,

$$g' = g\left(1-\frac{2h}{R}\right),$$

and

$$g' - g = -\frac{2h}{R} \cdot g.$$

Velocity of escape. The weight of a body of mass m gm. at a distance x cm. $(x > R)$ from the centre of the earth is GmE/x^2 dynes. The work which must be done upon it in order to raise it through a further δx cm. is

$$\frac{GmE}{x^2} \cdot \delta x \text{ ergs.}$$

The work which must be done upon it in order to remove it to an infinite distance is the sum of all the work done in all the intervening δx steps, from $x = R$ to $x = \infty$, or

$$\sum_{R}^{\infty} \frac{GmE}{x^2} \cdot \delta x \text{ ergs.}$$

This summation, evaluated by the integral calculus, gives $\frac{GmE}{R}$ ergs.

A body of kinetic energy greater than this would, if projected outwards in any direction, escape completely from the earth. The critical

velocity v above which escape is possible is given by $\frac{1}{2}mv^2 = GmE/R$, or

$$v = \sqrt{\frac{2GE}{R}} \text{ cm./sec.}$$

Using the values $G = 6{\cdot}6 \times 10^{-8}$ C.G.S., $E = 6 \times 10^{27}$ gm., $R = 6{\cdot}4 \times 10^{8}$ cm., this escape velocity is about $1{\cdot}1 \times 10^{6}$ cm./sec., about six times the root-mean-square velocity of hydrogen molecules at 0° C., and more than thirty times the velocity of sound in air at the same temperature.

Variation of g with depth below the surface. Again, supposing the earth to be a uniform sphere, an observer at depth x below the surface, is inside a spherical shell of radius R and thickness x (see diagram), so that the attraction due to this part of the earth's mass is zero. The effective attracting mass is a sphere of radius $(R - x)$.

If d be the density, supposed uniform throughout, the total mass of the whole earth, E, is $\frac{4}{3}\pi R^3 d$; while the mass of the portion effective for attraction at distance x from the surface, E', is $\frac{4}{3}\pi (R-x)^3 d$. Thus

$$g = \frac{GE}{R^2} = \frac{G\frac{4}{3}\pi R^3 d}{R^2} = \frac{4}{3}\pi R\, d\, G,$$

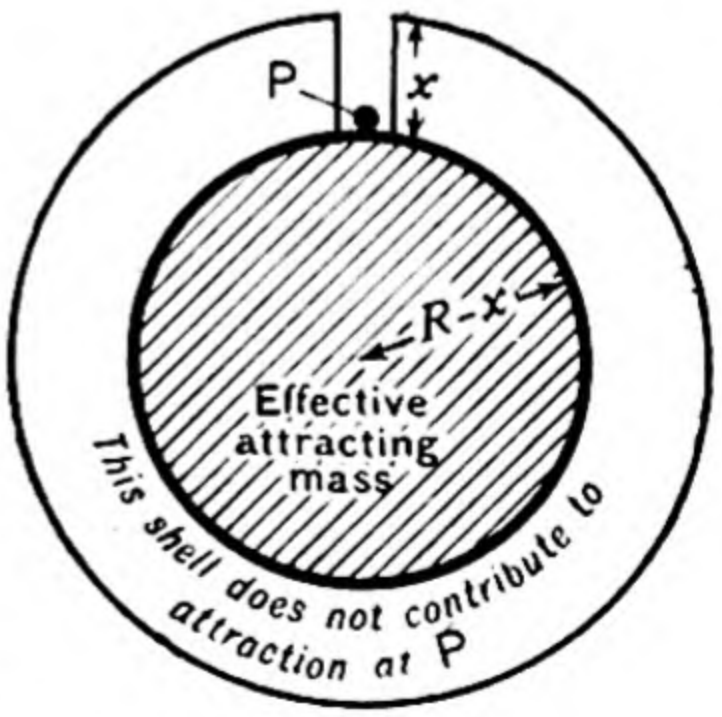
while at depth x the value of the acceleration of gravity is

$$g' = \frac{GE'}{(R-x)^2} = \frac{4}{3}\pi (R-x)\, d\, G = \frac{4}{3}\pi R\, d\, G \left(1 - \frac{x}{R}\right)$$

$$= g\left(1 - \frac{x}{R}\right), \text{ as } g = \tfrac{4}{3}\pi R\, d\, G.$$

Hence $$g' - g = -\frac{x}{R} \cdot g.$$

Thus the value of the acceleration due to gravity is greatest at the surface of the earth, and decreases both for rise above the surface and descent below the surface.



The Harton coal-pit experiment. The calculation of the last section, treating the earth as a uniform sphere, leads to

an entirely false numerical value of the rate of decrease with depth since the mean density of the surface strata of the earth's crust is about half the mean density of the earth as a whole. Let ρ be the density of the surface strata ; then the volume of the surface layer is $4\pi R^2 x$, its mass $4\pi R^2 x\rho$, and the effective attracting mass at depth x is $(E - 4\pi R^2 x\rho)$.

So, as $g = \frac{GE}{R^2}$;

$$g' = \frac{G(E - 4\pi R^2 x\rho)}{(R-x)^2} = \frac{G(E - 4\pi R^2 x\rho)}{R^2\left(1 - \frac{x}{R}\right)^2}$$

$$= \frac{GE}{R^2}\left(1 + \frac{2x}{R}\right) - 4\pi x\rho G\left(1 + \frac{2x}{R}\right)$$

$$= (g - 4\pi x\rho G)\left(1 + \frac{2x}{R}\right).$$

If then the values of g and g', at the surface and at depth x, be measured, together with R and the mean density of the surface layers above the level of the second observation, the value of G can be calculated.

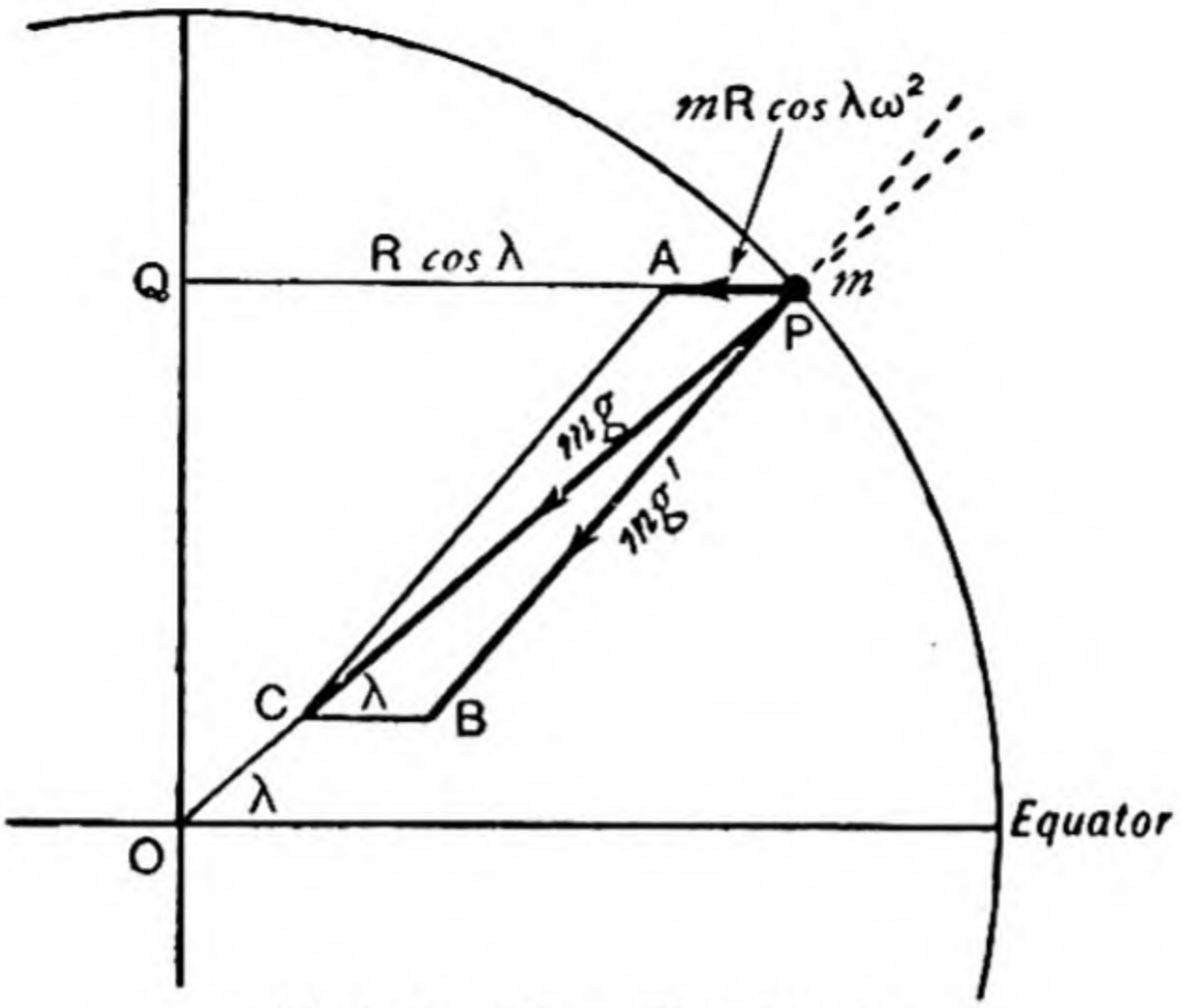
Airy, in 1854, swung pendulums at the top and bottom of the shaft of the Harton Pit, near Sunderland, and obtained a value of about 6·5 gm. cm.$^{-3}$ for the mean density of the earth.

Variation of g with latitude. Apart from minor local factors, there are two reasons why g decreases as we move from the poles towards the equator. The first is that the polar radius is slightly less than the equatorial radius ; the second is that part of the earth's gravitational pull on a body at the surface is employed in maintaining the body in its circular path, so that only the other component is available to accelerate it vertically. We shall consider only the second of these effects, and suppose the earth is spherical.

First, the gravitational pull towards the centre of the earth experienced by a body of mass m anywhere at its surface is

$$\frac{mGE}{R^2},$$

which we will call mg. For a body at the equator, moving in a path of radius R with angular velocity ω (which is 2π radians per day



Variation of g with latitude.

converted to radians per second), the force experienced towards the centre of this path is $mR\omega^2$, and if g' be the acceleration of gravity at the equator

$$mg = mg' + mR\omega^2,$$

or,
$$g = g' + R\omega^2 ;$$

whence,
$$g' = g - R\omega^2.$$

For a body at latitude λ, the circular path described has radius $R \cos \lambda$; to maintain this, a force directed towards the centre of this path, Q, of magnitude $mR \cos \lambda\omega^2$, must be provided.

The force mGE/R^2, or mg, towards the centre O of the earth, must furnish this. We must therefore resolve mg into two components, one of which is $mR \cos \lambda \omega^2$ in the direction PQ; the other component has to be determined, and this is mg'—but it is *not* directed towards O. In fact, the vertical at P, as determined by plumb-line, is not exactly normal to the surface.

In the accompanying diagram, PC represents mg, and the adjacent sides PA and PB of the parallelogram represent $mR \cos \lambda\omega^2$ and mg' respectively. Using the cosine formula,

$$(mg')^2 = (mg)^2 + (mR \cos \lambda \omega^2)^2 - 2mg \,.\, mR \cos^2\lambda \omega^2$$

or,

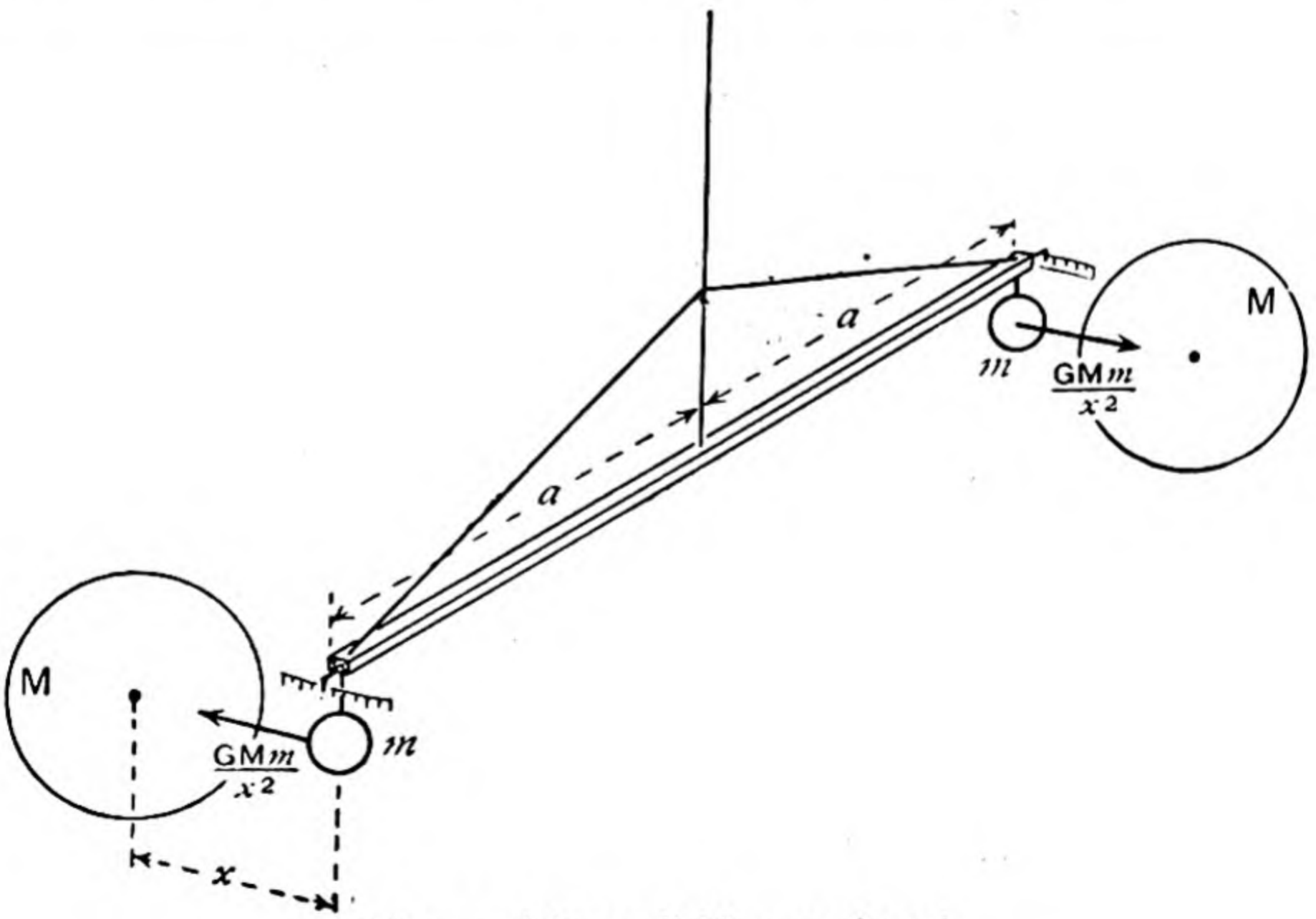
$$g'^2 = g^2 - 2gR \cos^2 \lambda \omega^2 + R^2 \cos^2 \lambda \omega^4.$$

This expression differs from that given in standard tables for the variation of g with λ, where of course the variation of R with latitude is considered and the $\cos^2\lambda$ factors are expressed in terms of $\cos 2\lambda$, but is of the same form.

Torsion-balance method of determining G. This experiment, first performed by Cavendish in 1797, was repeated in 1895 by Boys using apparatus very similar in principle, but affording greater accuracy on account of differences in design. The theory common to both experiments will first be given, after which we shall consider some of the special features of Boys' apparatus.

Theory. Consider two small equal spheres, each of mass m gm., at the ends of a long light rod of length $2a$ cm., suspended symmetrically from a fine wire which requires a couple of k dyne-cm. to turn the lower end through one radian (see diagram).

Two large spheres, each of mass M gm., are brought up in the same horizontal plane as the smaller spheres, so that the distance between the centres of gravity of each pair of spheres is x cm. The rod then swings through a small angle θ radians, which is observed.



Scheme of Cavendish's experiment.

The force between corresponding pairs of spheres is GMm/x^2 dynes, so the moment of the deflecting couple is

$$\frac{GMm}{x^2} \cdot 2a \text{ dyne-cm.},$$

and the moment of the restoring couple due to the wire is $k\theta$ dyne-cm. Thus, in equilibrium,

$$\frac{GMm}{x^2} \cdot 2a = k\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The value of k is found by measuring the time T of oscillation of the suspended system about its axis; then, if I be its moment of inertia about this axis, $T = 2\pi\sqrt{I/k}$ (p. 298), so $k = 4\pi^2 I/T^2$;

Hence

$$\frac{GMm}{x^2} \cdot 2a = \frac{4\pi^2 I}{T^2}\theta,$$

and

$$G = \frac{2\pi^2 I x^2}{T^2 Mma}\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

As G is so very small, the angle θ will always be small and difficult to measure accurately. It appears at first that the way to obtain a large deflection θ is to make the whole of the apparatus as large as possible, bearing in mind the limitation set by the fact that the supporting wire has to bear the weight of the suspended system.

Rearranging equation (1),

$$\theta = G \cdot \frac{M}{x^2} \cdot \frac{m}{k} \cdot 2a,$$

it is interesting to see the result of doubling the linear dimensions of each part of the apparatus.

(*a*) *Effect of doubling the radii of the masses M.* If the radius of each is doubled, the mass is multiplied by eight; but now the distance x will have to be approximately double its previous value, so the factor M/x^2 is only twice its previous value. Thus if all the rest of the apparatus is unchanged, and the radius of the attracting masses is doubled, θ should be doubled.

(*b*) *Effect of doubling the radii of the suspended masses.* Everything here depends on the torsion wire, which we will suppose initially to be so thin that it is carrying its maximum safe load. The maximum load is proportional to the area of cross-section, that is, to the square

of the radius of cross-section. But the couple required to produce one radian twist is proportional to the fourth power of the radius of cross-section, and inversely proportional to the length of the wire; the formula is $k = \pi n r^4/2l$, where r is the radius, l the length, and n the modulus of rigidity.

If the radius of each of the suspended spheres is doubled, the mass of each is increased eight times. Assuming that the weight of the rod is negligible, this means that the total weight to be borne by the wire is eight times greater, so, if it is not to break, the suspension must be of eight times the cross-section area, or the radius of cross-section must be $\sqrt{8}$ times greater. The couple k to produce one radian twist must now be 64 times as great, so the factor m/k is one-eighth of its previous value. Doubling the radius of each of the suspended masses, and leaving other dimensions unaltered with the exception of that of the wire, which is strengthened sufficiently to take the greater load, thus *reduces* θ to one-eighth of its previous value. It seems, then, that there is much to be gained by a light suspended system : in fact, *the smaller the suspended masses are, the better it is.*

One factor has been left out of account—the length $2a$ of the torsion arm. If this is doubled, while all other dimensions remain unaltered, θ is doubled.

Finally, if the *length* of the torsion wire is doubled, this halves k, and so, if this were the only change, would double θ.

If, then, we double the linear dimensions *of each part of the apparatus* (with the exception of the diameter of the suspension, dictated by its tensile strength and beyond our control to vary at will), M/x^2 is *doubled* ; m/k is *reduced to one-eighth* by altering the radius of each of the small masses m, but if we also double the length of the wire m/k is instead *reduced to one-fourth*, and $2a$ is doubled ; so for

$$\theta = G \cdot \frac{M}{x^2} \cdot \frac{m}{k} \cdot 2a$$

the change is $2 \times \frac{1}{4} \times 2 = 1$; that is, the sensitiveness of the apparatus is unaltered.

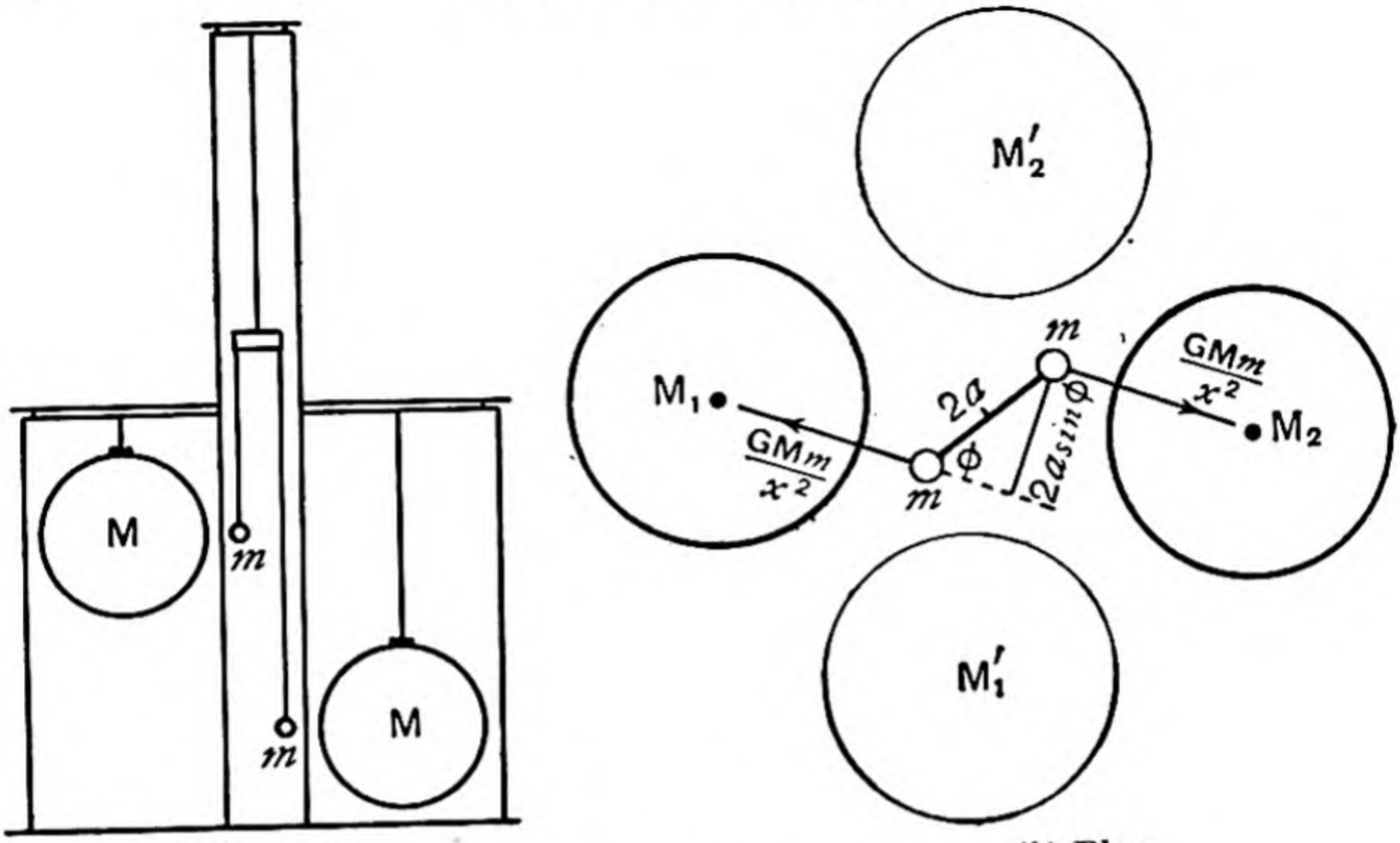
The ideal apparatus would then have very large attracting masses ; a very light suspended system, with small attracted masses ; and a long torsion arm.

Such an apparatus was used by Cavendish ; his torsion arm was a deal lath 6 ft. long. Details of the experiment will be found in most of the more advanced books ; for example, Poynting and Thomson's *Properties of Matter* (C. Griffin & Co.), or Cox's *Mechanics* (Cambridge University Press). The only point we need dwell on here is that the size of the apparatus made it very difficult to keep the whole at a uniform temperature, though it was carefully encased, and the main sources of error were internal draughts caused by convection currents.

If this difficulty is to be overcome, the only solution is to sacrifice the long torsion arm, and use instead a very short one, so that the whole apparatus can be compressed into a space that can be maintained at a uniform temperature. This necessarily means a decrease in sensitiveness—unless some material of greater tensile strength than the silvered copper used by Cavendish, with a comparable or smaller modulus of rigidity, is available. Boys found a means of drawing very fine fibres of fused quartz, a material with tensile strength about four times that of drawn copper, and slightly smaller modulus of rigidity. A fused quartz wire thus need only have a cross-section of a quarter the area of a copper wire in order to support a given load safely—that is, only half the radius—and thus will have a value of k rather less than one-sixteenth of that of a copper wire which could have carried the load. The torsion arm could thus be a few inches long instead of some six feet, and still the apparatus would be as sensitive as that of Cavendish, without being subject to the errors due to inequality of temperature.

Boys used a fused quartz suspension about 17 in. long and about 0·0005 cm. diameter, with a torsion arm slightly less than one inch long. The attracted spheres were gold balls $\frac{1}{4}$ in. in diameter, and the attracting spheres lead balls about $4\frac{1}{2}$ in. in diameter. The torsion rod was itself a mirror, and the reflection of a beam of light directed on it fell on a distant scale, so that very small deflections could be read.

Attracted and attracting masses were arranged in pairs, at levels differing by about 6 in. to reduce the effect of cross-attractions, which in any event had of course to be allowed for. The attracted balls hung in an inner protecting tube. Diagram *a* shows the apparatus in elevation, and diagram *b* in plan. Diagram *b* also indicates the



(*a*) Elevation. (*b*) Plan.

Boys' Apparatus.

way in which observations were taken ; first, the scale reading was noted when the attracting masses were in the position M_1M_2, so that the attraction in one direction was a maximum ; this is not quite the same as that stated on p. 384, for if the line joining the centres of each M and each m makes an angle ϕ with the torsion arm, the effective arm of the couple is $2a \sin \phi$ and not $2a$. Next, the attracting masses were moved into the positions $M_1'M_2'$, and the scale again read. The difference between the readings gave 2θ. The remainder of the calculation is of course as on page 384.

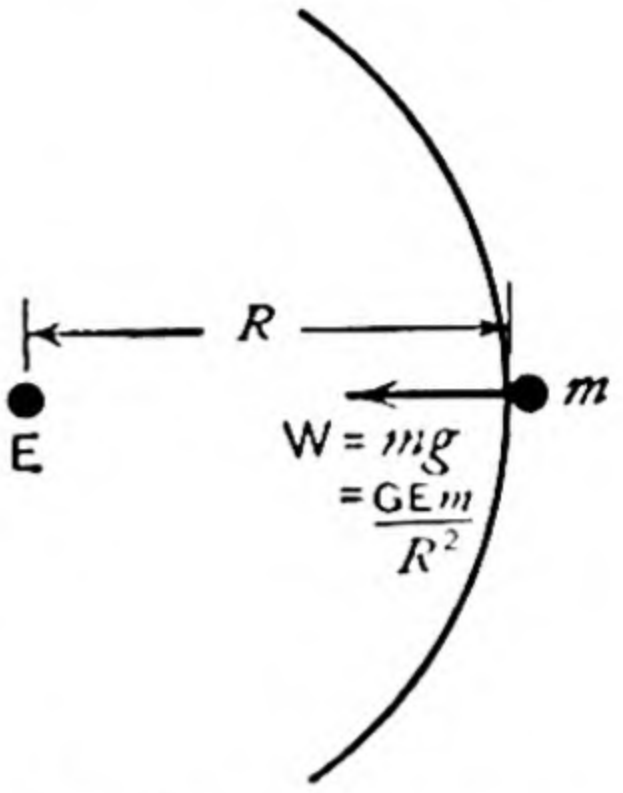
Boys' value for G was $6{\cdot}6576 \times 10^{-8}$ C.G.S., and the value for the mean density of the earth corresponding to this is 5·5270 gm. per c.c.

G and g ; the mass of the earth. Instead of considering the acceleration towards the centre of the earth of the moon in its orbit, as a factor in determining the mass of the earth, why should we not take instead the acceleration of a body at the earth's surface?

Such a body, of mass m gm., is acted on by its weight mg dynes ; if this is due to gravitational attraction between the body and the earth, of mass E gm., and radius R cm., and if the mass of the earth acts for purposes of attraction as if all concentrated at its centre, then the gravitational pull on the body is GEm/R^2 dynes.

Thus $mg = GEm/R^2$,

or, $E = gR^2/G$.



So, measurement of g and G, together with the earth's radius, enables the mass of the earth to be calculated. The measurement of R is a relatively simple surveyor's problem and need not be discussed in detail here; for if, with clock and sextant, the latitude and longitude of two stations is determined, measurement of their distance apart enables R to be calculated.

Regarding the earth as a sphere of uniform density d gm. per c.c., which thus represents the *mean* density, we can write

$$E = \tfrac{4}{3}\pi R^3 d,$$

so $$\tfrac{4}{3}\pi R^3 d = gR^2/G,$$

whence $$d = \frac{3}{4}\,\frac{g}{\pi RG}.$$

The problem of finding the mean density of the earth thus involves the same information as does finding the mass of the earth.

Finally, the last equation can be rewritten

$$G = \tfrac{3}{4}g/\pi Rd,$$

so that knowledge of g, R, and d enables G to be calculated. Newton, estimating that the mean density d was about twice that of the average density of the surface materials (between 2 and 3 gm. per c.c.), that is, d lay between about 5 and 6 gm. per c.c., was thus able to arrive at a rough value for G.

It is important to realise that, while G is an essential factor in calculating the mass and mean density of the earth (and is by far the most difficult one to measure), two other quantities must also be determined, namely the radius of the earth, R, and the acceleration of gravity, g.

Determination of g. A great number of simple dynamical experiments enable the value of g to be calculated roughly, but cannot easily be made to furnish reasonably accurate results; either because conditions are not in practice quite so simple as the elementary theory supposes, or else because the value of g is so great that at

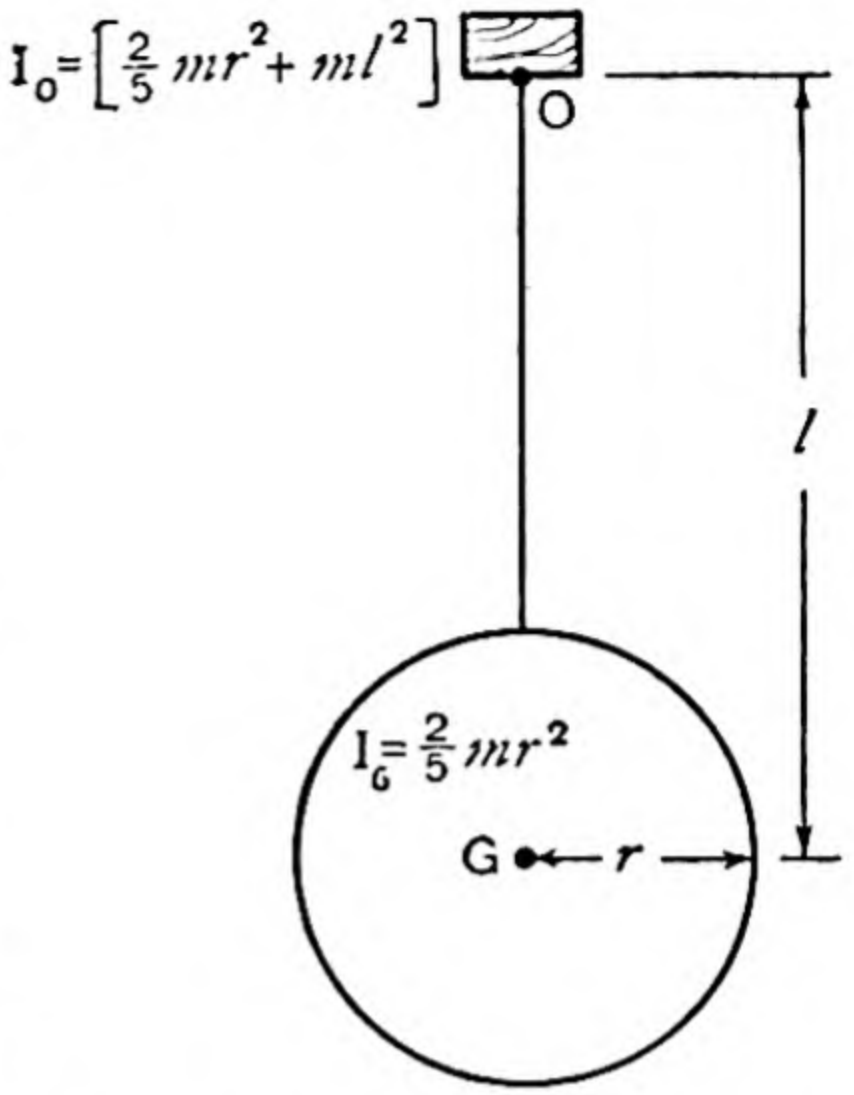
some stage in the experiment a very short interval of time has to be measured. Suppose, for example, we measure the distance s cm. fallen from rest in time t sec., and calculate g from the equation $s = \frac{1}{2}gt^2$. To begin with, the body is not falling freely, and the effect of air resistance though small has to be reckoned with ; and then, since g is approximately 1000 cm. sec.$^{-2}$, the body covers about five metres in the first second, so that for any indoor experiment time intervals of less than one second have to be measured accurately.

All accurate determinations are made by the use of a pendulum. For a simple pendulum making small oscillations, the bob describes a small arc ; and this is equivalent to running up and down an inclined plane of varying small inclination, so that the effective acceleration is as it were a small fraction of g. The time for a large number of oscillations can be determined, so that although the period of a single swing may be small, of the order of one second or so, the actual time intervals measured may be many times greater than this. The pendulum, in fact, helps to solve the difficulty raised by the size of g ; but the problem of over-simplified theory still remains to be attacked.

Consider a simple pendulum making small oscillations. If 1 cm. be the length of the string, and T sec. the time for one complete small oscillation, then (p. 297) $T = 2\pi\sqrt{l/g}$,

so $$g = 4\pi^2 l/T^2.$$

In deducing this relation, we assume that the bob is so small that its linear dimensions can be neglected ; that the string is so light that its mass can be neglected by comparison with that of the bob ; that the string is perfectly flexible and at the same time inextensible ; and that the support is unyielding. We also suppose that the effect of the surrounding air, consisting in part of viscous resistance and in part of a variation of the mass moved (since some of the air in the neighbourhood of the bob is set in motion by the bob), is so small that it need not be taken into account. Nothing in fact has been mentioned so far about any of these points, all of which have to be considered if the pendulum is to be used as an accurate instrument. Further difficulties also arise ; it is extremely difficult to measure accurately the distance from the supporting jaws to the centre of gravity of the bob, supposing that this is really the length l



required; and the theory supposes that the arc of swing is vanishingly small, though here theory can also provide the correction to be applied to the simple formula for any observed small amplitude of swing.

A further clash between theory and practice arises from the structure of the pendulum itself. The bob, of mass m gm. and radius r cm., has a moment of inertia about its own centre of gravity $\frac{2}{5}mr^2$, and about the point of support, $\frac{2}{5}mr^2+ml^2$. If we suppose for this discussion that the string is rigid, we have here really a compound pendulum, for which the period of vibration is given (p. 299) by the formula $T=2\pi\sqrt{I/mgh}$, where I is the moment of inertia about the axis of suspension, and h the distance of this axis from the centre of gravity of the pendulum. Substituting for I, and writing l for h,

$$T=2\pi\sqrt{\frac{\frac{2}{5}r^2+l^2}{lg}};$$

whence

$$g=\frac{4\pi^2}{T^2}\left(l+\frac{2}{5}\,\frac{r^2}{l}\right).$$

The proportional error in the value of g resulting from neglect of the dimensions of the bob should be $\frac{2}{5}\frac{r^2}{l}\div l$, or $0{\cdot}4r^2/l^2$. Thus, if $r=2$ cm., $l=100$ cm., the proportional error is $0{\cdot}4\times4\times10^{-4}=1{\cdot}6\times10^{-4}$, that is, 1·6 parts in ten thousand. If r is 5 cm., and l is 50 cm., the proportional error is $0{\cdot}4\times25\times4\times10^{-4}=4\times10^{-3}$, or 4 parts in one thousand.

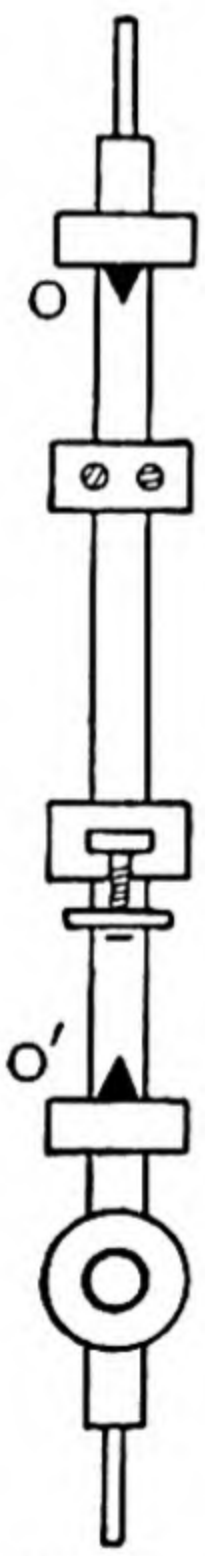
This is clearly then a point to be considered, though its effect is small for a bob of small dimensions, when presumably one would also have to take into account the mass of the supporting thread which might be an appreciable fraction of the mass of a small bob. But the obvious way of solving the problem, since any attempt to make a simple pendulum only results in a compound pendulum, is

to *start* with the intention of using a compound pendulum and apply the compound pendulum formula (p. 300).

$$T = 2\pi\sqrt{\frac{k^2+h^2}{hg}},$$

where k is the radius of gyration about an axis through the centre of gravity parallel to that in use, and h is the distance of this axis from the centre of gravity. This would involve the determination of k and the measurement of h, neither of which could usually be found with high precision.

But the compound pendulum has a remarkable property noted on page 301. If we can find two axes of suspension O and O', on opposite sides of the centre of gravity, for which the period of vibration is the same, then the distance OO' is equal to the length of a simple pendulum of exactly the same period. The distance between two knife-edge supports can be measured accurately, since both are well-defined and accessible. The reversible compound pendulum, first used for this purpose by Kater, provides the answer to most of the theoretical and practical difficulties here raised. A compound pendulum is arranged so that it may be swung either way up from two knife-edges; the mass-distribution is adjusted until the time of vibration, T, is the same for each; the distance, l, between the knife-edges is measured; then g is calculated using the *simple* pendulum expression $g = 4\pi^2 l/T^2$.



Kater's pendulum.

Kater's pendulum. The accompanying diagram shows the form of pendulum used by Kater. The knife-edges O, O' are fixed, and of the four masses indicated on the bar, the two outer ones by the knife-edges were of the same outside dimensions but of different masses; the inner weights were also externally similar. This symmetry of form reduced effects due to air buoyancy.

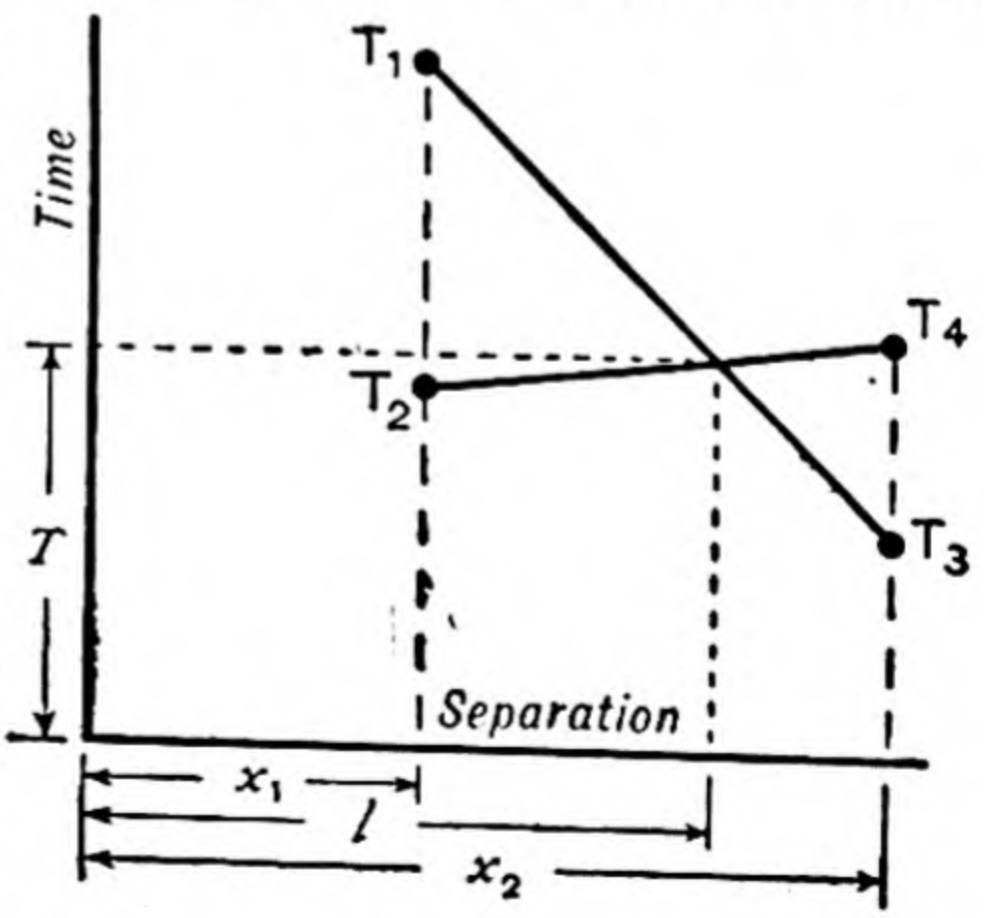
Adjustment to equality of period was made by sliding one of the inner masses along the bar, a large number of patient observations being required before this was finally achieved. The distance OO' was then carefully measured.

For timing, a pendulum clock was used, and this was first checked by astronomical observations, so that the

period of swing of the clock-pendulum was accurately known. (Incidentally, one tends to take for granted the reliability of clocks and watches, or rather to be content in daily life with a very low standard of accuracy ; it is rather alarming to realise that a clock which gains or loses only one minute a day introduces an error of one part in 1440!)

The clock pendulum bore a small reference mark ; the reversible pendulum was swung just in front of the clock pendulum, and carried a small index which just obscured the reference mark when both were in the zero position. The interval between successive "coincidences" when both were swinging was determined. Then, if coincidences occurred at intervals of n clock-pendulum swings, the period of the reversible pendulum, which must have made exactly $(n-1)$ or $(n+1)$ swings in this time, was $n/(n-1)$ or $n/(n+1)$ that of the clock pendulum—the correct alternative being obvious by inspection.

Alternative use of the reversible pendulum. The time needed for a reversible pendulum experiment is greatly reduced if instead of having fixed knife-edges and one movable weight, the pendulum is constructed so that the mass-distribution is fixed and one of the knife-edges is made movable. Suppose that O is movable while O' is fixed, and the range of travel of O is such as to include the separation that makes the two periods exactly equal. With O at its closest to O', the periods T_1 and T_2 are measured with O and O' as pivots ; O is then moved away about a centimetre, and the periods T_3 and T_4 found. The separations x_1 and x_2 of the knife-edges are measured in each case. Then, choosing a large scale, four points are plotted on graph paper. The line T_2T_4 is practically parallel to the axis of x, since the displacement of O will scarcely affect the period about O' ; the line T_1T_3 is part of the curve of the diagram on p. 300, and as the interval is so small very



little error is made in representing the variation in period with change of O by the straight line T_1T_3; the point at which the two lines intersect then gives the values l and T for the separation and the period at which the two periods are the same, whence $g = 4\pi^2 l/T^2$.

Determination of g using a metre-stick as a compound pendulum:

Method 1. A series of holes is bored at equal intervals down the centre-line of the stick, symmetrically arranged so that the position of the centre of gravity is not greatly shifted thereby. Pivoting the stick on a needle through a hole at distance h from the centre of gravity, the period T is found; this is repeated for various values of h. If k is the radius of gyration about the centre of gravity,

$$T = 2\pi\sqrt{\frac{k^2+h^2}{hg}},$$

whence

$$T^2h = \frac{4\pi^2}{g}h^2 + \frac{4\pi^2}{g}k^2.$$

The graph of T^2h against h^2 is thus a straight line, of which the gradient gives $4\pi^2/g$.

Method 2. This is really equivalent to the above procedure, except that only two sets of readings are required. If these are T_1 and T_2, for pivots at distances h_1 and h_2 from the centre of gravity,

$$T_1 = 2\pi\sqrt{\frac{k^2+h_1^2}{h_1g}}, \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{k^2+h_2^2}{h_2g}},$$

so

$$T_1^2h_1 = \frac{4\pi^2}{g}h_1^2 + \frac{4\pi^2}{g}k^2, \quad \text{and} \quad T_2^2h_2 = \frac{4\pi^2}{g}h_2^2 + \frac{4\pi^2}{g}k^2,$$

$$\therefore\ T_1^2h_1 - T_2^2h_2 = \frac{4\pi^2}{g}(h_1^2 - h_2^2),$$

giving

$$g = \frac{4\pi^2(h_1^2 - h_2^2)}{T_1^2h_1 - T_2^2h_2}.$$

Bessel's formula. Rearranging the expression, $g = \dfrac{4\pi^2(h_1^2 - h_2^2)}{T_1^2h_1 - T_2^2h_2}$, by using partial fractions,

$$\frac{8\pi^2}{g} = \frac{T_1^2 + T_2^2}{h_1 + h_2} + \frac{T_1^2 - T_2^2}{h_1 - h_2}.$$

With a compound pendulum of the reversible Kater type, it is

easy to measure the separation $(h_1 + h_2)$ of the knife-edges accurately; but the measurement of h_1 and h_2 separately means locating the centre of gravity first, and the individual measurements are less accurate. But the second term on the right-hand side is very small if the pendulum is adjusted so that the periods T'_1 and T'_2 are very nearly equal, so that error in measuring h_1 and h_2 separately is relatively unimportant.

Bessel's procedure for measuring g without going through the toil of adjusting until T'_1 and T'_2 are very closely equal is then to take two values of T'_1 and T'_2 for a separation of the knife-edges $(h_1 + h_2)$, which is measured directly; then to locate the centre of gravity and measure h_1 and h_2 separately; and finally to employ the formula above, substituting the directly measured $(h_1 + h_2)$ in the first term.

EXAMPLES ON CHAPTER 12

1. What is the distinction between the *mass* of a body and its *weight*? What evidence is there that the weights of bodies are proportional to their masses?

Describe in detail *one* method of determining the mass of the earth. (O.)

2. Explain what is meant by the Gravitation Constant (G). Describe an accurate method of measuring it.

If the acceleration due to gravity at the earth's surface is 32 ft. per sec.^{2}, find its value at a point (a) 1 mile above the surface, (b) 1 mile below the surface. The earth may be assumed to be a sphere of uniform density and of radius 4000 miles. (C. S.)

3. With what velocity would a shell have to be projected horizontally that it might (a) describe a circular orbit round the earth, (b) pass outside of the earth's gravitational field?

Gravitational Constant $= 6{\cdot}66 \times 10^{-8}$ c.g.s.
Gravitational Acceleration $= 981$ cm./sec.^{2}
Radius of earth $= 6{\cdot}4 \times 10^8$ cm. (C. S.)

4. Describe a method by which the mean density of the earth has been determined, and show how the result is calculated from the measurements made.

Assuming that the diameter of the sun is 109 times that of the earth, and that the mass of the sun is 330,000 times that of the earth, compare the values of solar gravity on the sun's surface and terrestrial gravity on the earth's surface. (O. & C.)

5. Describe a method of measuring the gravitational constant.

Assuming the gravitational constant to be $6{\cdot}6 \times 10^{-8}$ C.G.S., the radius of the earth to be $6{\cdot}37 \times 10^{8}$ cm. and g to be 981 cm. per sec.^{2}, calculate (a) the density of the earth, (b) the time of revolution of a satellite describing a circular orbit close to the earth's surface. (C. S.)

6. Describe Boys's method of measuring the gravitation constant G, showing how its value is deduced from the observations.

Two pieces of apparatus for this experiment are similar in all respects, but in one of them the linear dimensions of the attracting and attracted masses, their distance apart and the length of the torsion bar, are n times greater than in the other. If the suspensions are adjusted so that the periods of swing of the torsion bars with their attached masses are the same in the two instruments, find the ratio of the deflections of the bars. (C. S.)

7. Calculate the speed of projection necessary to send a body right out of the field of the earth's gravitational attraction.

$$G = 6{\cdot}66 \times 10^{-8} \text{ dyne cm.}^2 \text{ gm.}^2$$
$$\text{Mass of earth} = 5{\cdot}97 \times 10^{27} \text{ gm.}$$
$$\text{Radius of earth} = 6{\cdot}37 \times 10^{8} \text{ cm.}$$

(C. S.)

8. Describe and give the theory of an accurate method of determining the value of g, the acceleration due to gravity.

If the earth were a sphere of uniform density, how would the observed value of g vary (a) with height above the surface, (b) with depth below the surface, in a mine, (c) with latitude? (C. S.)

9. Derive the formula $\frac{v^2}{r}$ for the acceleration towards the centre of a particle moving with uniform velocity, v, in a circle of radius r.

If the period of revolution of the moon about the earth is 29·5 days and the distance from the earth to the moon is 240,000 miles, find the value of the gravitational constant G, given that the mass of the earth is $5{\cdot}9 \times 10^{21}$ tons. (C. S.)

10. Describe *one* way in which the mass of the earth has been estimated. You may assume that the acceleration due to gravity is known and that the earth is a sphere of known radius.

Show that if the planets are describing approximately circular orbits under the attraction of a central sun, then the square of the period of revolution is proportional to the cube of the radius of the orbit. Comment on the historical associations of this result. (N.)

11. Prove that a uniform spherical shell of matter exerts no gravitational attraction in its interior. Give an expression for the gravitational attraction due to the shell at an external point.

Assuming that the earth is built up of concentric shells, each of uniform density, show that it is possible to determine the gravitational constant (G) and the value of the earth's mean density (Δ) by pendulum experiments carried out at the top and bottom of a deep mine shaft.

A mass of 5 kilograms is weighed on a balance at the top of a tower 20 metres high. The mass is then suspended from the pan of the balance by a fine wire 20 metres long and is reweighed. Find the change in weight in milligrams. Assume that the radius of the earth is 6330 kilometres. (N.)

12. Assuming that the planets are moving in circular orbits, apply Kepler's laws to show that the acceleration of a planet is inversely proportional to the square of its distance from the sun. Explain the significance of this and show clearly how it leads to Newton's law of universal gravitation.

Obtain the value of g from the motion of the moon, assuming that its period of rotation round the earth is 27 days 8 hours and that the radius of its orbit is 60·1 times the radius of the earth.

Radius of earth $= 6{\cdot}36 \times 10^6$ metres. (N.)

13. Explain the relation between g, the acceleration due to gravity, and G the gravitational constant. How has the value of G been determined experimentally?

Assuming the earth to be a perfect sphere of radius $6{\cdot}4 \times 10^8$ cm., find the difference due to the rotation of the earth in the value of g at the poles and at the equator. (N.)

14. State *Newton's law of gravitation*, and explain what is meant by the *constant of gravitation*. Describe a laboratory method by which this constant has been determined.

If the constant is $6{\cdot}7 \times 10^{-8}$ cm.^{3} per gm. sec.^{2} and the radius of the earth is $6{\cdot}4 \times 10^8$ cm., calculate the mass of the earth. (O.)

15. Assuming the earth is at rest and the moon describes a circular orbit round it, find the radius of the orbit, given that the moon takes 28 days ($= 2{\cdot}4 \,.\, 10^6$ seconds) to complete the orbit, that the radius of the earth is 6400 kilometres and that the acceleration of gravity at the earth's surface is 1000 cm. per sec. per sec. (C. S.)

16. State Newton's Law of Gravitation and explain the principle of one method by which the constant of gravitation can be determined.

The value of $g = 980$ cm. sec.$^{-2}$, the earth's radius $= 6{\cdot}4 \times 10^3$ km., the distance of the moon from the earth's centre $= 3{\cdot}86 \times 10^5$ km. Assuming that the mass of the earth is much greater than that of the moon, calculate the time in days that the moon takes to go once round its orbit. (C. S.)

17. A particle is dropped down a deep shaft which extends to the centre of the earth. Find its velocity when it reaches the bottom.

[$g = 1000$cm./sec.^{2}. Radius of earth $= 6{\cdot}4 \times 10^8$ cm.] (C. S.)

18. Describe in some detail a method of measuring the universal gravitational constant. Assuming the moon's orbit to be circular, find the distance from the earth to the moon.

[$g = 1000$ cm./sec.^{2}. Radius of earth $= 6{\cdot}4 \times 10^8$ cm.
Period of moon's revolution $= 27$ days.] (C. S.)

19. Explain what is meant by the *gravitation constant* (G), and describe an accurate laboratory method of measuring it. Give an outline of the theory of your method.

Assuming that the earth is a sphere of radius 6370 kilometres and that $G = 6{\cdot}66 \times 10^{-8}$ c.g.s. units, calculate the mean density of the earth. (O. & C.)

20. Explain what is meant by the *gravitation constant* (G). Describe a torsion-balance method of measuring it.

Calculate the periodic time of an infinitesimal satellite which is describing a circular orbit about a spherical planet of unit density, the radius of the orbit being just greater than the radius of the planet. ($G = 6{\cdot}6 \times 10^{-8}$ c.g.s. units.) (O. & C.)

21. Describe an accurate method of measuring the value of the universal gravitational constant, and give the theory of the experiment.

The planet Neptune travels round the sun with a period of 165 years. Show that the radius of its orbit is approximately thirty times that of the earth's orbit, both being considered as circular. (O.)

TABLES

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
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85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 4 4	5 6 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155			

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

FUNCTIONS OF ANGLES AT 1° INTERVALS

Angle	Radians	Chords	Sine	Tangent	Cotangent	Cosine			
0°	0	0	0	0	∞	1	1·414	1·5708	**90°**
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	**89**
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	**88**
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	**87**
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	**86**
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	**85**
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	**84**
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	**83**
8	·1396	·139	·1392	·1405	7·1154	·9903	1·312	1·4312	**82**
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	**81**
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	**80**
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	**79**
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	**78**
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	**77**
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	**76**
15	·2618	·261	·2588	·2679	3·7321	·9659	1·217	1·3090	**75**
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	**74**
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	**73**
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	**72**
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	**71**
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	**70**
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	**69**
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	**68**
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	**67**
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	**66**
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	**65**
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	**64**
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	**63**
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	**62**
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	**61**
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	**60**
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	**59**
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	**58**
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	**57**
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	**56**
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	**55**
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	**54**
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	**53**
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	**52**
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	**51**
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	**50**
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	**49**
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	**48**
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	**47**
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	**46**
45	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	**45**
			Cosine	Cotangent	Tangent	Sine	Chords	Radians	Angle

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ANSWERS

Chapter 1 (p. 29)

2. 0·0144 cm.

3. c.g. is 0·286 cm. below knife edges which are collinear.

4. 35·5 per cent. **5.** 1·90 gm. **6.** 3·23 c.c.

10. $s = 42$ gm. ; $k = 2780$ dyne cm.$^{-1}$. **11.** $k = 30$; $g = 963$ cm. sec.$^{-2}$.

12. $a = 0{\cdot}023$; $r = 1{\cdot}26$. **13.** $a = 5 \times 10^{-6}$; $n = 3$.

14. $a = 0{\cdot}2$; $n = 5{\cdot}1$. **15.** $A = 5{\cdot}1 \times 10^{7}$; $B = -6{\cdot}0 \times 10^{7}$.

17. $A = 102$; $b = 0{\cdot}119$.

18. $A = 1{\cdot}52 \times 10^{-5}$; $b = 1930$; a poor straight line.

19. Plot $T^{1/2}\eta^{-1}$ against T^{-1} ; or $T^{3/2}\eta^{-1}$ against T ; $A = 1{\cdot}52 \times 10^{-5}$; $B = 117$.

20. Readings give $C = 72{\cdot}8$; $r = 5$; the real index in McLeod's formula should be 4.

21. 14·3 cm. sec.$^{-2}$; 865 cm. ; 365 cm. **22.** 32 ft. sec.$^{-2}$.

Chapter 2 (p. 55)

1. 309 yards ; 105°. **2.** 0·92 kgm. wt. ; 72·3° to P.

3. 250 gm. wt. ; 433·0 gm. wt. **4.** 7660 gm. cm. ; the same ; 383 gm. wt.

5. (*a*) 22 lb. wt. ; 7·5 ft. from 7 lb. wt.
(*b*) 20 kgm. wt. ; 40 cm. from 12 kgm. wt.
(*c*) 9 lb. wt. ; $2\frac{1}{9}$ inch from side remote from B.

6. 1 kgm. wt. at B ; 1·732 kgm. metre.

7. 5·83 m. sec.$^{-1}$ at 36° 54′ to bank. **8.** 28·3 m.p.h. S.W. ; 13·9 ft. sec.$^{-2}$.

9. 87·2 ft. sec.$^{-1}$ at 143° 41′ to original direction.

10. 0·0088 ft. sec.$^{-2}$. **11.** $2{\cdot}5 \times 10^{18}$ cm. sec.$^{-2}$.

12. 5 inches from 1–4 ; 3 inches from 3–4.

13. Along line of symmetry between E and BC, $\frac{8}{9}$ inch from E.

14. $66\frac{2}{3}$ lb. wt. **15.** 15 cm. from heavier end.

16. 1·18″ from centre. **17.** $9\frac{1}{4}$ lb. × ft.

18. 128·2 lb. wt. at 70° 54′ to 80 lb. wt.

20. 223·6 gm. wt. ; 26° 34′ E. of N. **21.** 4800 ft. sec.$^{-1}$; 53° 5′.

22. $\dfrac{2\sqrt{(b^2+c^2)u^2 - b^2v^2}}{u^2 - v^2}$. **23.** 46°. **24.** 20 m.p.h. at 30° E. of S.

Chapter 3 (p. 86)

1. 165 ft. **2.** 2 ft./sec. ; 3·0 sec. **3.** 148·8 lb. wt. ; 131·2 lb. wt.

4. 880 ft. **5.** 25·6 ft. sec.$^{-2}$. **6.** Uniform speed.

7. 1440 ft. **8.** 9 min. ; 1960 yards. **9.** 14·0 sec.

10. Time, 10·0 sec. ; velocity $320\sqrt{2}$ ft. sec.$^{-1}$ at 45° to ground.

11. 4 ft. sec.$^{-2}$. **12.** 23·93 cm. sec.$^{-2}$. **13.** 37·5 cm. sec.$^{-2}$.

14. 3·16 sec. ; 63·3 ft. sec.$^{-1}$. **15.** 5 sec. ; 10 sec.

16. Consider a link of length x ft. lying on the table, raised by a uniform force F poundals ; this acquires velocity $4{\cdot}5\pi$ ft. sec.$^{-1}$ while being raised x feet, so acceleration is $4{\cdot}5^2\pi^2/2x$ ft. sec.$^{-2}$; and as mass is $3x$ lb.,

$$F = \frac{4{\cdot}5^2\pi^2}{2x} \times \frac{3x}{32} = 9{\cdot}37 \text{ lb. wt.}$$

During raising of first 20 ft., pull on top is then 69·37 lb. wt. ; after this, a drop of 9·37 lb. wt. ; then diminishes at uniform rate to zero.

Graph then consists of rectangle of altitude 69·37 lb. wt. and triangle of altitude 60 lb. wt.

Total energy furnished = 1387·4 + 600 = 1987·4 ft. lb. Of this, the projecting rectangular strip at the top, representing 9·37 × 20 = 187·4 ft. lb., goes in providing the K.E. of the chain.

17. 16 ft. sec.$^{-2}$; 30°.

18. Trajectory is other half of a parabola that would be described by a body *released* from aircraft. Velocity = $\sqrt{u^2 + 2gh}$ at $\tan^{-1}(\sqrt{2gh}/u)$ to horizontal.

19. 120 sec. ; one mile past the station.

20. Height 4572 ft. ; speed 312·3 ft. sec.$^{-1}$.

Chapter 4 (p. 108)

1. 8·0 cm. **2.** 43·3 lb. wt. **3.** 0·732.

4. 12 lb. wt. ; 10 inches.

5. 43·3 lb. wt. at 30°, and 25 lb. wt. at 60° to vertical. **6.** $1 : \sqrt{3}$.

7. 4 ft. and 3 ft. ; 8·524 lb. wt. in the 4 ft., and 5·224 lb. wt. in the 3 ft.

8. 60° to vertical ; reaction 13·23 lb. wt. at 40° 54′ to vertical.

9. 10 lb. wt. **10.** 1·00 lb. wt. and 3·46 lb. wt.

11. 136·6 lb. wt. ; 36·6 lb. wt. ; 86·6 lb. wt.

12. Least, at 90° to string, is 100 lb. wt.

13. Wind, 68·4 lb. wt. ; upthrust, 187·9 lb. wt.

14. 20 inches from ground. **15.** About 8° N. of W. ; 0·78 hr.

Chapter 5 (p. 147)

1. 78·6 ft. sec.$^{-1}$.

2. Velocity of mass = 32 ft. sec.$^{-1}$ at impact. By law of conservation of momentum, velocity of system after impact = 32/3 ft. sec.$^{-1}$. So K.E. of pile and driver = $\frac{1}{2} \times 1500 \times (\frac{32}{3})^2$ ft. lbal. = $\frac{1}{2} \times 1500 \times (\frac{32}{3})^2 \times \frac{1}{32}$ ft. lb.

Resistance opposing motion is resultant of ground resistance and weight of system, (3360 − 1500) = 1860 lb. wt. ; so, as K.E. is exchanged doing work against this resistance, the distance travelled is 1·434 feet.

This is a typical practical problem ; note the sequence of steps :

(*a*) Apply the conservation of momentum to the impact, then, having found the velocity of the system,

(*b*) find the K.E. of the system, then

(*c*) equate this to the work done.

3. 62·5 per cent.

4. (*a*) $\frac{11}{45}$ ft. sec.$^{-1}$; (*b*) $\frac{3}{4}$ mile ; (*c*) 17·1 lb. wt. ; (*d*) 186,560 ft. lb.

5. 2·5 ft. **6.** 1210 ft. **7.** (*a*) 528, (*b*) 67·8 per cent., (*c*) 396 ft. lb.

8. 32 ft. ; 64 ft. **9.** 6 ; 25 lb. wt. ; 4·8. **10.** 0·4 H.P.

11. $\frac{1}{4}$. **13.** $1{\cdot}65 \times 10^6$ ft. lb. ; 10 H.P.

14. $m_A/m_B = 4$; 64 per cent. **15.** 2400 ft. sec.$^{-1}$.

16. Calling the deflexion θ, $\sin\theta = \frac{1}{20}$; $\cos\theta = 0{\cdot}99875$; so vertical height risen $= 100(1 - \cos\theta) = 0{\cdot}125$ cm.

So velocity after impact $= \sqrt{2 \times 981 \times 0{\cdot}125} = 15{\cdot}66$ cm. sec.$^{-1}$. Using law of conservation of momentum, $v = 1973$ cm. sec.$^{-1}$, whence energy lost is $3{\cdot}863 \times 10^6$ ergs.

17. Elastic, 112 cm. sec.$^{-1}$; rest.
Inelastic, 56 cm. sec.$^{-1}$.
Energy loss is $1{\cdot}57 \times 10^5$ ergs.

18. $\frac{m}{M} \cdot V\cos\theta$. **19.** Rest ; 10 cm. sec.$^{-1}$; 2500 ergs.

20. Consider limiting friction at edge about which overturning may occur ; if this can equal the force needed to cause overturning, necessary couple is available.

21. Velocities, 100 cm. sec.$^{-1}$, 25 cm. sec.$^{-1}$ in direction of the original velocity of the larger ; e is 0·5.

22. 4200 ft. lb. **23.** 5·35 lb. wt. **24.** 1135 cm. sec.$^{-1}$.

25. 15·08 H.P. ; 16·6 per cent.

26. Approximately, $v_1 = d\sqrt{g/l}$; $v_2 = \dfrac{M+m}{m}\, d\sqrt{g/l}$; ratio *lost* K.E./*initial* K.E. is $M/(M+m)$.

27. 34·6 ft. ; stops in 1·571 sec., needing 235 H.P.

28. Length of escalator is immaterial ; 158·4 H.P. ; 52·8 H.P.

29. 20 ft. sec.$^{-1}$; 750 lb. wt. **30.** $(\mu_1 - \mu_2)s/\mu_1(\mu_1 + 1)$.

32. Best attempted after reading pp. 153 and 162. Let applied force F turn system through a small angle ϕ giving it angular velocity ω about O. Linear velocities of m_1 and m_2 are $r_1\omega$ and $r_2\omega$; work done by $F = F\cos\theta \times r_1\phi$; K.E. gained is $\frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2$.

So $F\cos\theta \,.\, r_1\phi = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2$, and $\omega^2 = 2\alpha\phi$, where α is angular acceleration. Hence $\alpha = F\cos\theta r_1/(m_1r_1^2 + m_2r_2^2)$ and is thus proportional to the moment $F\cos\theta r_1$.

34. $2{\cdot}98 \times 10^9$ ergs. **36.** $m = \pi r^2\rho\sqrt{gh}$. where ρ is density of liquid.

Chapter 6 (p. 171)

1. 7290 gm. cm.^{2}. **3.** 104·2 ft. sec.$^{-1}$.

5. Angular momentum is conserved, so if x is change in angular velocity per day, then $\frac{2}{5}mr^2\omega = \left(\frac{2}{5}mr^2 + \frac{2}{3}\,\frac{m}{5 \times 10^{19}}\,r^2\right)(\omega + x)$, so angular acceleration is $-3{\cdot}86 \times 10^{-24}$ rad. sec.$^{-2}$.

6. 17° ; 0·306 ; 61·2 lb. wt.

7. $\cos^{-1}(\frac{1}{3}) = 70°\ 27'$.

8. (*a*) 89·1 cm. sec.$^{-2}$; (*b*) 90·8 gm. wt. ; (*c*) 29·7 rad. sec.$^{-2}$.

9. 0·782 sec. **11.** 447 rad. sec.$^{-1}$. **12.** 12·62 r.p.m.
13. 0·19 ft. lb. ; 4·41 ft. lb. **14.** 69·3 ft. sec.$^{-1}$.
15. 385·5 ft. sec.$^{-1}$. **16.** 0·072. **17.** 3·098 ft.
19. 18·98 ft. sec.$^{-1}$; 3·125 tons wt. ; 0·625. **20.** $6{\cdot}13 \times 10^6$ gm. cm.^{2}.
21. 34° 44′ to vertical. **22.** 0·804.
23. 934 cm. sec.$^{-2}$; 980·7 cm. sec.$^{-2}$.

Chapter 7 (p. 190)

1. 72 kgm. wt. **2.** 1200 gm. wt.
3. 1000 gm. wt. ; 25 gm. wt. **4.** 7200 lb. wt.
5. 1033·6 gm. wt. cm.$^{-2}$; 913·6 gm. wt. cm.$^{-2}$; 8·82 cm.
6. 21·4 inches. **7.** 3·95 c.c.
8. (*a*) 1·895, (*b*) 7·94 gm., (*c*) 4·3 gm. wt.
9. 7·35 gm. ; 20 c.c. **10.** 0·278 ; 26·0 gm. wt.
11. $\frac{5}{8}$ inch. **12.** 6·5 cm. **13.** 48 gm. wt. ; 4 cm.
16. 52·23 lb. **17.** 144·3 kgm. wt. **18.** 93·7 per cent.
19. This problem is harder than it appears ; if we suppose impossible " constant lift " conditions, and equate this lift to the weight of cable, a height of 4800 metres is obtained ; in this case the stress in the cable due to its own weight would in any case slightly exceed the breaking stress! But assuming a closed balloon of constant mass and volume, and an isothermal atmosphere with pressure varying according to the law $p = p_0 \exp(-1{\cdot}25 \times 10^{-3}h)$, which is approximately $p = p_0(1 - 1{\cdot}25 \times 10^{-3}h)$, where h is the altitude in metres, so that the Archimedes upthrust at height h is $(1 - 1{\cdot}25 \times 10^{-3}h)$ of its value at ground level, the value calculated is 2852 metres. See Example 10, Chapter 8.
20. 2·897 gm. ; 2·50 gm. wt.
21. Calculus is needed for the solution of this problem.

When base of cylinder is distance x cm. below free surface, upthrust is $Ax\rho$ gm. wt. and net upward force $(Ax\rho - Al\sigma)g$ dynes. So work done on cylinder in depressing a further dx is $(Ax\rho - Al\sigma)g\,dx$ ergs. Integration from initial depth $l\sigma/\rho$ to final depth l gives for the work done $\frac{1}{2}\dfrac{Al^2}{\rho}(\rho - \sigma)^2 g$ ergs.

Similar argument in the second case gives

$$\tfrac{1}{2}\frac{Al^2(\rho - \sigma)g}{\rho}\left[\frac{B(\rho - \sigma) + 2A\sigma}{B - A}\right] \text{ ergs.}$$

22. (*a*) $\frac{11}{12}$; (*b*) 14·29 gm. **23.** 24,000 kgm. wt.
24. 2·00 cm. mercury.

Chapter 8 (p. 236)

1. $1{\cdot}993 \times 10^{12}$ dyne cm.$^{-2}$.
$4{\cdot}71 \times 10^5$ ergs.
2. $2{\cdot}94 \times 10^6$ ergs. **3.** Nearly 0·5 mm. ; $1{\cdot}96 \times 10^5$ ergs.
5. Minimum velocity at top is 313·3 cm. sec.$^{-1}$; velocity at bottom, 700 cm. sec.$^{-1}$; extension 0·15 mm.
6. $Y = 3{\cdot}18 \times 10^7$ lb. wt./sq. in. ; 0·125 ft. lb. **8.** 481 ergs.

9. 10^{12} dyne cm.$^{-2}$; 2×10^8 dyne cm.$^{-2}$.

10. A difficult problem. With obvious symbols, as

$$-v\,dp/dv = \rho\,dp/d\rho = 2{\cdot}2 \times 10^7,$$

$$dp = 2{\cdot}2 \times 10^7 \,.\, d\rho/\rho.$$

If $d\rho/\rho$ is to be 10^{-2}, $dp = 2{\cdot}2 \times 10^5$ gm. wt. cm.$^{-2}$. Assuming variation of pressure with depth to be linear, as $p = p_0 + h\rho$, $dp = \rho\,dh$ to first order of small quantities, whence $dh = 2{\cdot}2 \times 10^5$ cm.

In the isothermal atmosphere, the pressure in absolute units being $p = g\rho h$ dyne cm.$^{-2}$, where ρ is the density of the air at height h cm. and pressure p, the change in pressure with altitude is $dp = -g\rho\,dh$.

Now ρ is proportional to p if the temperature is constant ; and actually $\rho = \dfrac{pM}{RT}$, where M is the effective molecular weight, R the gas constant for one gram molecule, and T the absolute temperature—supposing the air to behave as an ideal gas.

Then $$dp/p = \frac{-Mg}{RT}\,dh.$$

Whence $p = p_0 \exp\{(-Mg/RT)h\}$, or $\log_e(p_0/p) = \dfrac{Mg}{RT}h.$

As exp x is approximately $(1 + x)$ if x is small, the formula of Example 19, Chapter 7, has been derived from this by substituting numerical values, putting h in metres, and using this approximation.

11. Extension 1·783 mm.

Tensions $3{\cdot}57 \times 10^7$ dynes in steel, $2{\cdot}67 \times 10^7$ dynes in copper.

12. 1·35 cm. **13.** $8{\cdot}87 \times 10^4$ kgm. wt. **14.** $\frac{1}{6}\dfrac{A\rho^2g^2}{q}l^3$.

15. $1{\cdot}32 \times 10^9$ dyne cm.$^{-2}$; length is immaterial.

16. Stress $= \dfrac{mgl}{\pi r^2 y}$; strain $= \dfrac{y^2}{2l^2}$.

Young's modulus $= \dfrac{2mgl^3}{\pi r^2 y^3}$.

18. 7·97 kgm. wt. ; $3{\cdot}97 \times 10^7$ ergs.

19. $p_n = p\left(\dfrac{V}{V+v}\right)^n$; 8 complete strokes.

20, 21, 24 and **25.** All 75·0 cm. **22.** 1·4 atmospheres ; 50 strokes.

23. 2·42 cm. mercury. **26.** 749 mm.

27. With the theory of question 10 in a little simpler form, let

$$dp = g\rho\,dh.$$

Now if $\rho = Kp$, as it must under isothermal conditions,

$$K = 1{\cdot}3 \times 10^{-3} \div (76 \times 13{\cdot}6 \times 981)$$

from the data.

Integrating

$$dp = -Kgp\,dh,\ p = p_0 \exp -Kgh,\ \text{so } \log_e(p_0/p) = Kgh.$$

Substituting, $h = \log_e\left(\dfrac{76}{75}\right)\left(\dfrac{76 \times 13{\cdot}6}{1{\cdot}3 \times 10^{-3}}\right) = 104{\cdot}6$ metres.

28. 758·8 mm. **29.** 26 strokes. **30.** 1·213.

Chapter 9 (p. 279)

4. $v = kr p^{\frac{1}{2}} \rho^{-\frac{1}{2}}$.

5. $\omega = kR^{-1} S^{\frac{1}{2}} \rho^{-\frac{1}{2}}$.

6. $p = k\rho c^2$; $l = k\rho m^{-1} \sigma^{-2}$.

8. $v = kT^{\frac{1}{2}} \rho^{-\frac{1}{2}} l^{-\frac{1}{2}}$.

9. $t = kr^{\frac{3}{2}} \rho^{\frac{1}{2}} T^{-\frac{1}{2}}$.

10. $n = kl^{-1} s^{\frac{1}{2}} \rho^{-\frac{1}{2}}$.

11. $C = kr^4 \phi n l^{-1}$.

12. $C = kR^{-1} Y r^{-4}$.

13. $E = kY^x s^y$, where $x + y = 1$.

14. $h = kr\left(\frac{r\rho q}{Y}\right)^n$, where n is any number.

15. $V/t = kpr^4 l^{-1} \eta^{-1}$.

16. See text.

17. 15·9 cm. sec.$^{-1}$.

20. $1{\cdot}142 \times 10^{-2}$ cm. sec.$^{-1}$.

21. $v = k\omega\eta^{-1} r^{-1}$.

23. (a) stable, (b) unstable.

24. 4·0 cm., concave to the smaller bubble.

26. Object distance u is given by $u = f - f^2 p/4T$, where f is the focal length of the lens and the pressure excess p is positive for convex bubble.

27. 2·4 cm.

28. 8·81 mm.

29. 43·4 dyne cm.$^{-1}$.

30. If separation of plates is $2y$ cm., radii r cm., anticlastic liquid surface exposed to air has pressure difference $p = T\left(\frac{1}{y} - \frac{1}{r}\right)$, and effective force is $\pi r^2 T\left(\frac{1}{y} - \frac{1}{r}\right)$ holding them together.

31. 26·9 dyne cm.$^{-1}$; 699 dynes.

32. $m'g = mg + 2\pi rT - \pi r^2 h\rho g$.

33. 2·67 cm. ; convex.

34. $h = \frac{2T}{dg}\left(\frac{1}{a} - \frac{1}{b}\right)$ cm.

35. $1{\cdot}034 \times 10^6$ dyne cm.$^{-2}$.

36. 400π ergs plus external work done by expansion of gas within the bubble ; that actually expended on creating free surface energy is of course 400π ergs.

37. 0·583 cm.

38. $1{\cdot}38 \times 10^{-8}$ cm. ; this, incidentally, is of molecular dimensions.

39. 0·11° C.

40. 2520 dynes.

41. $T = \frac{mg(x - y)}{x^2 - y^2 + h^2}$ dyne cm.$^{-1}$.

43. s.g. is 0·8.

44. 24·5 dyne cm.$^{-2}$.

Chapter 10 (p. 314)

2. $a = 13{\cdot}07$. $T = 2{\cdot}03$ sec.

3. 4·88 cm. The P.E. lost by mass falling $(10 + x)$ cm. is $80 \times 981 \times (10 + x)$ ergs ; this equals $\frac{1}{2}kx^2$, where $k = 500 \times 981/5$ dyne cm.$^{-1}$.

4. 6201 gm. cm.$^{-2}$. [Moments of inertia are I and $(I + 2 \times 10 \times 10^2)$ in the two cases.]

5. $T = 2\pi\sqrt{\frac{7}{5}\frac{(R - r)}{g}}$; $F = \frac{2}{7}mg \sin\theta$ dynes ; $\mu = \frac{2}{7}\tan\theta$.

6. Max. velocity of surface $= 200\pi a$ cm. sec.$^{-1}$, where a is amplitude. Energy condition for particle to separate from surface is

$$\tfrac{1}{2}mv^2 > mga, \quad \text{or} \quad v^2 > 2ga,$$

which gives $a = 4{\cdot}98 \times 10^{-3}$ cm. for limiting case.

7. Treat as bifilar problem (number of cords immaterial) ; $T = 4\pi\sqrt{l/g}$.

8. 8·66 gm. cm.$^{-3}$.

9. 0·635 sec.

10. $9{\cdot}08 \times 10^{13}$ sec.$^{-1}$.

11. $T = 2\pi\sqrt{L/g}$; $\frac{1}{2}Mgd^2/L$ ergs.

12. $T = 0{\cdot}28$ sec.

13. $T = 2\pi\sqrt{\dfrac{MV}{PA^2}}$ sec.

14. $a = 5$ cm. ; $t = 2$ sec.

15. As qn. 11 ; $T = 2\pi\sqrt{L/g}$.

16. 0·453 sec.

17. $a = 5$ cm. ; $v = 5$ cm. sec.$^{-1}$.

18. $T = 2\pi\left(\dfrac{2}{g}\right)^{\frac{1}{2}}\left(\dfrac{I}{M}\right)^{\frac{1}{4}}$.

20. 99·42 cm. ; K.E. is 35520 ergs.

22. $\delta = 1{\cdot}108$; $\log_e \delta = 0{\cdot}1018$.

Chapter 11 (p. 363)

1. 2·85 sec.

2. $4\frac{1}{6}$ kgm. wt. ; $8\frac{1}{6}$ kgm. wt.

3. 4·25 sec.$^{-1}$.

4. 56·2 kgm. wt.

5. 4961 sec.$^{-1}$. Young's modulus, the elastic constant involved, is calculated from the data.

6. Also 100 cm.

7. $n_T/n_L = \sqrt{T/m}/\sqrt{Y/\rho} = \frac{1}{40}$.

10. Open pipe twice length of closed one.

11. 39·2 gm. wt.

13. 171 sec.$^{-1}$, $\times 1, 2, 3 \ldots$.

14. See p. 341. Superpose $y_1 = a \sin \dfrac{2\pi}{\lambda}(vt - x)$ and $y_2 = a \sin \dfrac{2\pi}{\lambda}(vt + x)$ to give $y = 2a \cos \dfrac{2\pi x}{\lambda} \sin 2\pi nt$. Amplitude at distance x is $2a \cos 2\pi x/\lambda$, giving 0·00317 cm.

15. Brass, 131·6 sec.$^{-1}$; steel, 136·6 sec.$^{-1}$.

16. 300 sec.$^{-1}$.

17. 4·4 sec.$^{-1}$.

18. 4·6 sec.$^{-1}$.

19. Taking the simplest possible situation, with observer directly between siren and cliff at distance x from cliff, $x/11$ beats/sec. during waxing and waning, and any number between 0 and $x/11$ during change from one condition to another. Maximum frequency when x as large as possible, and maximum clearness when x as small as possible, since amplitudes then most nearly equal.

20. $74\frac{2}{3}$ ft. sec.$^{-1}$.

21. 50·63 sec. ; 262·5 sec.$^{-1}$.

22. 14·64 sec.$^{-1}$; altitude is not given and is not needed.

23. $4{\cdot}27 \times 10^{-11}$ cm. to red.

24. For rate of change of observed frequency with time,

$$\frac{dn}{dt} = -\frac{ncv \sin\theta}{(c - v\cos\theta)^2} \cdot \frac{d\theta}{dt}.$$

If close to line, $\sin\theta \simeq 0$ and $\dfrac{d\theta}{dt}$ very small ; except while passing, when $\sin\theta$ changes very rapidly from 0 to 1 to 0, and $\cos\theta$ from 1 to 0 to -1.

25. $2{\cdot}34 \times 10^6$ sec., or 27·07 days.

26. 55 ft. sec.$^{-1}$.

27. $\lambda_A/\lambda_B = \dfrac{V - v}{V + v}$; $n_A/n_B = \dfrac{(V + v)^2}{(V - v)^2}$.

28. 960 sec.$^{-1}$. (Change in frequency from a *moving reflector* is as if due to a source moving at *twice* the rate of the reflector, since " virtual image "

of source moves twice as fast as reflector; hence $\frac{\delta\lambda}{\lambda} = -\frac{2v}{V}$, or $\frac{\delta n}{n} = \frac{2v}{V}$).

29. 100 ft. sec.$^{-1}$. **30.** 2·5 km. sec.$^{-1}$.

Chapter 12 (p. 395)

2. (*a*) 31·984; (*b*) 31·992 ft. sec.$^{-2}$.

3. (*a*) $7{\cdot}9 \times 10^5$ cm. sec.$^{-2}$, as $v^2/R = g$;

(*b*) $\sqrt{\frac{2GE}{R}} = \sqrt{2Rg} = 1{\cdot}12 \times 10^6$ cm. sec.$^{-2}$.

4. $g_{\text{sun}} : g_{\text{earth}} = 330{,}000 : 109^2 = 27{\cdot}8 : 1$.

5. (*a*) 5·57; (*b*) 5062 sec.

6. Both numerator and denominator affected by factor n^7, so θ same.

7. $1{\cdot}12 \times 10^6$ cm. sec.$^{-2}$. **9.** $2{\cdot}42 \times 10^{-10}$ lb. ft. sec. units.

11. 31·6 milligrams weight. **12.** 1000 cm. sec.$^{-2}$.

13. $R\omega^2 = 3{\cdot}38$ cm. sec.$^{-2}$. **14.** 6×10^{27} gm.

15. $3{\cdot}91 \times 10^{10}$ cm. **16.** $2{\cdot}378 \times 10^6$ sec.; or 27·52 days.

17. Acceleration at distance x is gx/R, whence S.H.M. with max. velocity $\sqrt{gR} = 8 \times 10^5$ cm. sec.$^{-1}$.

18. $3{\cdot}83 \times 10^{10}$ cm. **19.** (Take g as 981 cm. sec.$^{-2}$); 5·52 gm. cm.$^{-3}$.

20. 6898 sec. **21.** $r^3\omega^2$ constant.

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